[2018-03-21]

Let ζ be a parameterization of $\mathcal{R}$

$$\zeta(z_1, z_2, z_3) \in \mathcal{R} \quad \text{(the current shape)}$$

defined by

$$\zeta(z_1, z_2, z_3) = o + z_1 e_1 + z_2 e_2 + z_3 e_3$$

A curve c can be defined by

$$c(h) = \zeta(c_{\zeta 1}(h), c_{\zeta 2}(h), c_{\zeta 3}(h))$$

$$= o + c_{\zeta 1}(h) e_1 + c_{\zeta 2}(h) e_2 + c_{\zeta 3}(h) e_3$$

The tangent vector at $c(0)$ will be

$$c'(0) = c'_{\zeta 1}(0) e_1 + c'_{\zeta 2}(0) e_2 + c'_{\zeta 3}(0) e_3$$

A vector field on $\mathcal{R}$ can be defined as

$$v(\zeta(z_1, z_2, z_3)) = v_\zeta(z_1, z_2, z_3)$$

$$= v_{\zeta 1}(z_1, z_2, z_3) e_1$$

$$+ v_{\zeta 2}(z_1, z_2, z_3) e_2$$

$$+ v_{\zeta 3}(z_1, z_2, z_3) e_3$$

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The vector field along the curve c is

$$v(c(h)) = v_{\xi}\big(c_{\xi 1}(h), c_{\xi 2}(h), c_{\xi 3}(h)\big)$$

$$= v_{\xi 1}\big(c_{\xi 1}(h), c_{\xi 2}(h), c_{\xi 3}(h)\big)\, e_1$$

$$+ v_{\xi 2}\big(c_{\xi 1}(h), c_{\xi 2}(h), c_{\xi 3}(h)\big)\, e_2$$

$$+ v_{\xi 3}\big(c_{\xi 1}(h), c_{\xi 2}(h), c_{\xi 3}(h)\big)\, e_3$$

Derivative of the vector field v along c

$$\lim_{h\to 0} \frac{v(c(h)) - v(c(0))}{h} =$$

$$= \big(\partial_1 v_{\xi 1}\, c'_{\xi 1} + \partial_2 v_{\xi 1}\, c'_{\xi 2} + \partial_3 v_{\xi 1}\, c'_{\xi 3}\big)\, e_1$$

$$+ \big(\partial_1 v_{\xi 2}\, c'_{\xi 1} + \partial_2 v_{\xi 2}\, c'_{\xi 2} + \partial_3 v_{\xi 2}\, c'_{\xi 3}\big)\, e_2$$

$$+ \big(\partial_1 v_{\xi 3}\, c'_{\xi 1} + \partial_2 v_{\xi 3}\, c'_{\xi 3} + \partial_3 v_{\xi 3}\, c'_{\xi 3}\big)\, e_3$$

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$$= \left(v_{1,1}\, c'_{31} + v_{1,2}\, c'_{32} + v_{1,3}\, c'_{33} \right) e_1$$

$$+ \left(v_{2,1}\, c'_{31} + v_{2,2}\, c'_{32} + v_{2,3}\, c'_{33} \right) e_2$$

$$+ \left(v_{3,1}\, c'_{31} + v_{3,2}\, c'_{32} + v_{3,3}\, c'_{33} \right) e_3$$

$$= c'_{31} \left(v_{1,1}\, e_1 + v_{2,1}\, e_2 + v_{3,1}\, e_3 \right)$$

$$+ c'_{32} \left(v_{1,2}\, e_1 + v_{2,2}\, e_2 + v_{3,2}\, e_3 \right)$$

$$+ c'_{33} \left(v_{1,3}\, e_1 + v_{2,3}\, e_2 + v_{3,3}\, e_3 \right)$$

$$= c'_{31}\, L e_1 + c'_{32}\, L e_2 + c'_{33}\, L e_3$$

$$= L \left(c'_{31}\, e_1 + c'_{32}\, e_2 + c'_{33}\, e_3 \right) = L c'$$

$$\lim_{h \to 0} \frac{v(c(h)) - v(c(0))}{h} = \underbrace{\nabla v}_{} \, c'$$

gradient of the vector field

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$$o(h) := v(c(h)) - \Big(v(c(0)) + \nabla v(p_0)\big(c(h) - c(0)\big)\Big)$$

$$\lim_{h\to 0} \frac{o(h)}{h} = \lim_{h\to 0} \frac{v(c(h)) - v(c(0))}{h} - \nabla v(p_0) \lim_{h\to 0} \frac{c(h)-c(0)}{h}$$

$$\lim_{h\to 0} \frac{o(h)}{h} = \nabla v(p_0)\, c'(0) - \nabla v(p_0)\, c'(0)$$

$$\lim_{h\to 0} \frac{o(h)}{h} = 0$$

$$(\nabla v)e_1,\ (\nabla v)e_2,\ (\nabla v)e_3$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$[\nabla v] = \begin{pmatrix} v_{1,1} & v_{1,2} & v_{1,3} \\ v_{2,1} & v_{2,2} & v_{2,3} \\ v_{3,1} & v_{3,2} & v_{3,3} \end{pmatrix}$$

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[2018-03-27]

For any deformation $\bar{\mathcal{R}} \to \mathcal{R}_t$ we can write

$$c_t(h) = c_t(0) + F_t(\bar{c}(0))(\bar{c}(h) - \bar{c}(0)) + o(h)$$

and

$$c_t'(0) = \lim_{h \to 0} \frac{c_t(h) - c_t(0)}{h} = F_t(\bar{c}(0))\, \bar{c}'(0)$$

For the corresponding velocity field we can write

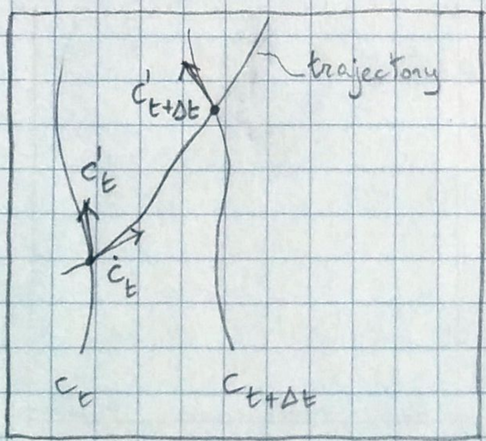
$$v(c_t(h)) = v(c_t(0)) + \nabla v(c_t(0))\,(c_t(h) - c_t(0)) + o(h)$$

and

$$\lim_{h \to 0} \frac{v(c_t(h)) - v(c_t(0))}{h} = \nabla v(c_t(0))\, c_t'(0)$$

with $v(c_t(h)) = \dot{c}_t(h)$

Let us assume $\displaystyle \lim_{h \to 0} \frac{\dot{c}_t(h) - \dot{c}_t(0)}{h} = \lim_{\Delta t \to 0} \frac{c_{t+\Delta t}'(0) - c_t'(0)}{\Delta t}$



$$\Downarrow$$

$$\nabla v(c_t(0))\, c_t'(0) = \dot{F}_t(\bar{c}(0))\, \bar{c}'(0)$$

in short form

$$\nabla v\, c' = \dot{F}\, \bar{c}'$$

$$\nabla v\, c' = \dot{F} F^{-1} c'$$

$$\boxed{\nabla v = \dot{F} F^{-1}}$$

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Mass conservation

In an affine motion, let us assume the mass to be a time independent quantity

$$\rho_0 V_{\bar{\mathcal{R}}} = \rho V_{\mathcal{R}} = \rho \det F \, V_{\bar{\mathcal{R}}}$$

with ρ the current mass density per unit current volume.

$$\frac{d}{dt}(\rho V_{\mathcal{R}}) = 0 \quad \Rightarrow \quad V_{\bar{\mathcal{R}}} \frac{d}{dt}(\rho \det F) = 0$$

$$\frac{d}{dt}(\rho \det F) = \dot{\rho} \det F + \rho \frac{d}{dt} \det F$$

$$= \dot{\rho} \det F + \rho \det F \operatorname{tr}(\dot{F} F^{-1})$$

$$\dot{F} F^{-1} = \nabla v \quad \Rightarrow \quad \operatorname{tr}(\dot{F} F^{-1}) = \operatorname{tr} \nabla v = \operatorname{div} v$$

$$\frac{d}{dt}(\rho V_{\mathcal{R}}) = 0 \quad \Rightarrow \quad \dot{\rho} + \rho \operatorname{div} v = 0$$

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