[2018-04-10]

Tensor product properties

$$(u \otimes v)a = (v \cdot a)u$$

$$(u \otimes v)a \cdot b = (v \cdot a)(u \cdot b)$$

$$= (v \otimes u)b \cdot a = a \cdot (v \otimes u)b$$

$$\Rightarrow \quad \boxed{v \otimes u = (u \otimes v)^T}$$

$$(Au \otimes Bv)a = Au\,(Bv \cdot a) = Au\,(v \cdot B^T a)$$

$$= A(u \otimes v)B^T a$$

$$\boxed{Au \otimes Bv = A(u \otimes v)B^T}$$

[Notebook page scanned on 2018/04/30]

$$\operatorname{tr}(u\otimes v) = \Big(\operatorname{vol}\big((u\otimes v)e_1, e_2, e_3\big) + \operatorname{vol}\big(e_1, (u\otimes v)e_2, e_3\big) + \operatorname{vol}\big(e_1, e_2, (u\otimes v)e_3\big)\Big)\frac{1}{\operatorname{vol}(e_1, e_2, e_3)}$$

$$u = u_1 e_1 + u_2 e_2 + u_3 e_3$$

$$\operatorname{tr}(u\otimes v) = \Big((v\cdot e_1)\operatorname{vol}(u, e_2, e_3) + (v\cdot e_2)\operatorname{vol}(e_1, u, e_3) + (v\cdot e_3)\operatorname{vol}(e_1, e_2, u)\Big)\frac{1}{\operatorname{vol}(e_1, e_2, e_3)}$$

$$\operatorname{vol}(u, e_2, e_3) = u_1 \operatorname{vol}(e_1, e_2, e_3)$$
$$\operatorname{vol}(e_1, u, e_3) = u_2 \operatorname{vol}(e_1, e_2, e_3)$$
$$\operatorname{vol}(e_1, e_2, u) = u_3 \operatorname{vol}(e_1, e_2, e_3)$$

$$\operatorname{tr}(u\otimes v) = (v\cdot e_1)u_1 + (v\cdot e_2)u_2 + (v\cdot e_3)u_3 = v\cdot(u_1 e_1 + u_2 e_2 + u_3 e_3) = v\cdot u$$

[Notebook page scanned on 2018/04/30]

[2018-04-10]

Tensor product and scalar product of tensors

$$(u\otimes v)a = (v\cdot a)u$$

$$\underbrace{(u\otimes v)a}\cdot b = (v\cdot a)(u\cdot b)$$

$$\downarrow$$

$$\mathrm{tr}\big(\underbrace{((u\otimes v)a)}\otimes b\big) = \mathrm{tr}\big((u\otimes v)(a\otimes b)\big)$$
$$= \mathrm{tr}\big((u\otimes v)(b\otimes a)^T\big)$$

Let us define the "scalar product between tensors" by the formula

$$(u\otimes v)\cdot(b\otimes a) = \mathrm{tr}\big((u\otimes v)(b\otimes a)^T\big) = (u\cdot b)(v\cdot a)$$

and extend this definition to general tensors

$$A\cdot B = \mathrm{tr}(AB^T) = \mathrm{tr}\big((AB^T)^T\big) = B\cdot A$$

Since $(u\otimes v)^T\cdot(b\otimes a)^T = \mathrm{tr}\big((v\otimes u)(b\otimes a)\big) = (u\cdot b)(v\cdot a)$

we get by extention to general tensors

$$A^T\cdot B^T = \mathrm{tr}(A^T B) = \mathrm{tr}(AB^T) = A\cdot B$$

As a consequence of the properties above

$$f\cdot Lu = \mathrm{tr}(f\otimes Lu) = \mathrm{tr}\big((f\otimes u)L^T\big) = f\otimes u\cdot L$$

[Notebook page scanned on 2018/04/30]

The 9 tensors

$$e_i \otimes e_j\,,$$

with $\{e_1, e_2, e_3\}$ an orthonormal basis, turn out to be linearly independent tensors and their linear combinations generate the whole linear tensor space

$$Lin\mathcal{V}$$

$$(e_i \otimes e_j)\cdot(e_h \otimes e_k) = (e_i \cdot e_h)(e_j \cdot e_k) = \delta_{ih}\,\delta_{jk}$$

$$(e_1 \otimes e_1)\cdot(e_1 \otimes e_2) = \underset{1}{\delta_{11}}\,\underset{0}{\delta_{12}} = 0$$

$$(e_2 \otimes e_1)\cdot(e_2 \otimes e_1) = \underset{1}{\delta_{22}}\,\underset{1}{\delta_{11}} = 1$$

$$A\cdot(e_i \otimes e_j) = tr\left(A\,(e_j \otimes e_i)\right) = Ae_j \cdot e_i = a_{ij}$$

since $Ae_j = a_{ij}\,e_i$

[Notebook page scanned on 2018/04/30]