Balance law for rigid bodies

We assume that whatever the force distribution $\mathcal{F}$ it must comply with the balance law

$$\mathcal{F}(v) = 0 \quad \text{for any test velocity field } v \qquad (*)$$

Since $v(x) = v_o + W(x - p_o)$ (rigid velocity fields)

for

$$\mathcal{F}(v) = f_A \cdot v(p_A) + f_B \cdot v(p_B) + f_C \cdot v(p_C)$$

we get the representation

$$\mathcal{F}(v) = (f_A + f_B + f_C) \cdot v_o + \big(f_A \otimes (p_A - p_o) + \dots\big) \cdot W$$
$$= f \cdot v_o + M \cdot W$$

Hence

$$\mathcal{F}(v) = 0 \ \forall v \iff f = 0, \ \text{skw}\, M = 0$$

(*) $\mathcal{F}(v) = 0 \ \forall v$ balance law is also called

virtual power balance law

balance law weak form

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Balance law for affine bodies

$$v(x) = v_0 + L(x - p_0)$$

We assume again

$$\mathcal{F}(v) = 0 \quad \forall \text{test velocity field } v$$

while giving a characterization as regard as the nature of the force distribution

$$\mathcal{F}(v) = \overset{ext}{\mathcal{F}}(v) + \overset{int}{\mathcal{F}}(v)$$

with $\overset{int}{\mathcal{F}}(v) = 0 \quad \forall$ rigid test velocity field (objectivity)

Using the representations

$$\mathcal{F}^{ext}(v) = f^{ext} \cdot v_0 + M^{ext} \cdot L$$

$$\mathcal{F}^{int}(v) = -(z \cdot v_0 + T \cdot L) V_{\mathcal{R}}$$

we arrive at

$$z = 0, \; \mathrm{skw}\, T = 0 \quad \text{(from objectivity principle)}$$

and

$$\begin{cases} f^{ext} = 0 \\ M^{ext} = T\, V_{\mathcal{R}} \end{cases}$$

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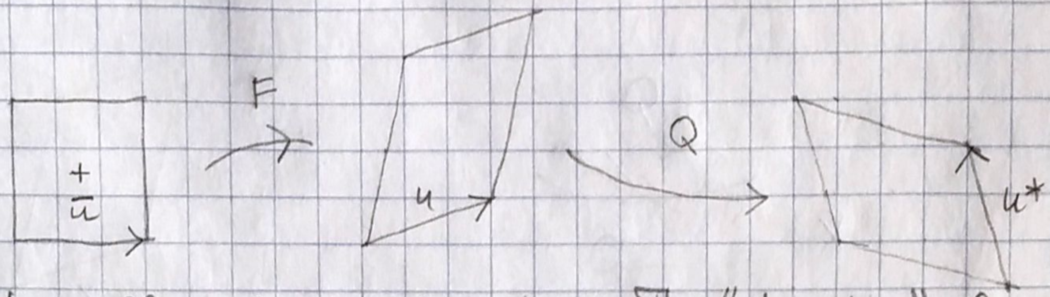
Objectivity of the internal force distribution



We call a force distribution $\mathcal{F}$ "objective" if under a change of observer it changes into $\mathcal{F}^*$ such that

$\mathcal{F}(v) = \mathcal{F}^*(v^*)$ for any affine test velocity fields v, v^* related by $v_o^* = Q v_o + \dot q$, $F^* = QF$, whatever the change of observer (defining $Q, \dot Q, \dot q$).

Hence $$z\cdot v_o + T\cdot L = z^*\cdot v_o^* + T^*\cdot L^*$$

$\forall v_o, \forall L$ and related v_o^*, L^* $(\forall \dot q, \forall Q, \forall \dot Q)$

Since $$L^* = \dot F^*(F^*)^{-1} = (\dot Q F + Q \dot F)F^{-1}Q^T) = \dot Q Q^T + Q L Q^T$$

we get

$$z\cdot v_o + T\cdot L = z^*\cdot (Q v_o + \dot q) + T^*\cdot \dot Q Q^T + T^*\cdot Q L Q^T$$

By selecting test velocity fields with $L=0$ and any change of observer characterized by $\dot Q = 0, \dot q = 0$

we get $$z\cdot v_o = z^*\cdot Q v_o \quad \forall v_o \Rightarrow z = Q^T z^*$$

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Allowing $\dot{q}$ to take any value (still with $L=0$, $\dot{Q}=0$)
we arrive at the necessary condition for objectivity

$$z^* \cdot \dot{q} = 0 \quad \forall \dot{q} \;\Rightarrow\; z^* = 0 \;\Rightarrow\; z = Q^T z^* = 0$$

with the original condition reduced to

$$T \cdot L = T^* \cdot \dot{Q} Q^T + T^* \cdot Q L Q^T$$

Considering now any test velocity gradient and any change of observer characterized by $\dot{Q} = 0$ we get

$$T \cdot L = T^* \cdot Q L Q^T \quad \forall Q, \text{ as a necessary condition}$$

$$\Rightarrow T = Q^T T^* Q \;\Rightarrow\; Q T Q^T = T^*$$

Finally we are left with the original condition reduced to

$$T^* \cdot \dot{Q} Q^T = 0 \qquad \forall \dot{Q} Q^T$$

Since $\dot{Q} Q^T$ is a skew symmetric tensor
this leads to the additional necessary condition

$$\operatorname{skw} T^* = 0$$

or, equivalently,

$$\operatorname{skw} T = \operatorname{skw} Q^T T^* Q = Q^T (\operatorname{skw} T^*) Q = 0$$

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