Piola stress (nominal stress)

$$\mathcal{J}^{int}(v) = -T \cdot \nabla v \; V_{\mathcal{R}}$$

$$T \cdot \nabla v \; V_{\mathcal{R}} = T \cdot \nabla v \; V_{\bar{\mathcal{R}}} \det F$$

$$\nabla v \; c' = \nabla v \; F \bar{c}'$$

$$\nabla \bar{v} = \nabla v \, F \quad \Rightarrow \quad \nabla v = \nabla \bar{v} \, F^{-1}$$

$$T \cdot \nabla v \; V_{\mathcal{R}} = \left(T \cdot \nabla \bar{v} \, F^{-1} \right) V_{\bar{\mathcal{R}}} \det F$$

$$= \left(T F^{-T} \cdot \nabla \bar{v} \right) \det F \; V_{\bar{\mathcal{R}}} \qquad (*)$$

$$= S \cdot \nabla \bar{v} \; V_{\bar{\mathcal{R}}}$$

$$\boxed{S := (\det F) \, F^{-T}}$$

(*) $$u_1 \cdot u_2 = F \bar{u}_1 \cdot u_2 = \bar{u}_1 \cdot F^T u_2 = F^{-1} u_1 \cdot F^T u_2$$

$$= u_1 \cdot (F^{-1})^T F^T u_2 \qquad \forall u_1 \quad \forall u_2$$

$$\Rightarrow \quad (F^{-1})^T F^T = I \quad \Rightarrow \quad (F^{-1})^T = (F^T)^{-1}$$

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Cofactor of F

$$A_{\mathcal{F}_1} = vol(n_1, u_2, u_3) \qquad A_{\bar{\mathcal{F}}_1} = vol(\bar{n}_1, \bar{u}_2, \bar{u}_3)$$

$$w_1 = u_1 - \underbrace{(u_1 \cdot n_1)}_{h_1} n_1 \quad \Rightarrow \quad w_1 \cdot n_1 = 0$$

$$w_1 \in span\{u_2, u_3\} \; ? \quad \Rightarrow \quad vol(u_1, u_2, u_3) = h_1 \, vol(n_1, u_2, u_3)$$

$$V_{\mathcal{R}} = h_1 A_{\mathcal{F}_1} \qquad V_{\bar{\mathcal{R}}} = \bar{h}_1 A_{\bar{\mathcal{F}}_1}$$

$$h_1 := u_1 \cdot n_1 \qquad u_2 \cdot n_1 = 0 \qquad u_3 \cdot n_1 = 0$$

$$\bar{h}_1 := \bar{u}_1 \cdot \bar{n}_1 \qquad \bar{u}_2 \cdot \bar{n}_1 = 0 \qquad \bar{u}_3 \cdot \bar{n}_1 = 0$$

$$\Rightarrow \quad h_1 = F\bar{u}_1 \cdot n_1 = \bar{u}_1 \cdot F^T n_1 \,, \quad F\bar{u}_2 \cdot n_1 = 0 \,, \quad F\bar{u}_3 \cdot n_1 = 0$$

$$\Rightarrow \quad \bar{u}_2 \cdot F^T n_1 = 0 \,, \quad \bar{u}_3 \cdot F^T n_1 = 0$$

$$\Rightarrow \quad k_1 F^T n_1 = \bar{n}_1$$

$$\Rightarrow \quad F^{-T} \bar{n}_1 = k_1 n_1$$

$$\bar{h}_1 = \bar{u}_1 \cdot \bar{n}_1 = \bar{u}_1 \cdot (k_1 F^T n_1) = k_1 \bar{u}_1 \cdot F^T n_1 = k_1 \, u_1 \cdot n_1 = k_1 h_1$$

$$\Rightarrow \quad k_1 = \frac{\bar{h}_1}{h_1} = \frac{V_{\bar{\mathcal{R}}}}{A_{\bar{\mathcal{F}}_1}} \frac{A_{\mathcal{F}_1}}{V_{\mathcal{R}}} = \frac{A_{\mathcal{F}_1}}{A_{\bar{\mathcal{F}}_1}} \frac{1}{\det F}$$

$$(\det F) F^{-T} \bar{n}_1 = \frac{A_{\mathcal{F}_1}}{A_{\bar{\mathcal{F}}_1}} n_1$$

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Cofactor of F and boundary traction

$$\operatorname{cof} F := (\det F) F^{-T}$$

$$(\operatorname{cof} F)\, \bar{n}_1 = n_1 \frac{A_{\mathcal{F}_1}}{A_{\bar{\mathcal{F}}_1}}$$

Setting

$$\bar{t}_1 := S \bar{n}_1$$

we get

$$\bar{t}_1 = S \bar{n}_1 = T (\operatorname{cof} F)\, \bar{n}_1 = T n_1 \frac{A_{\mathcal{F}_1}}{A_{\bar{\mathcal{F}}_1}}$$

$$\Rightarrow \quad \bar{t}_1 = t_1 \frac{A_{\mathcal{F}_1}}{A_{\bar{\mathcal{F}}_1}} \quad , \qquad A_{\bar{\mathcal{F}}_1} \bar{t}_1 = A_{\mathcal{F}_1} t_1$$

Setting

$$a_1 := \frac{A_{\mathcal{F}_1}}{A_{\bar{\mathcal{F}}_1}}$$

we get

$$a_1 = \| (\operatorname{cof} F)\, \bar{n}_1 \|$$

$$\bar{t}_1 = t_1 a_1 \quad , \qquad \int_{\mathcal{F}_1} t_1 \, dA = \int_{\bar{\mathcal{F}}_1} t_1 a_1 \, dA = \int_{\bar{\mathcal{F}}_1} \bar{t}_1 \, dA$$

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Cofactor matrix

$$(\mathrm{cof}\,F)\,e_1 = (\det F)\,F^{-T}e_1 = n_1\,\frac{A_{\mathcal{F}_1}}{A_{\bar{\mathcal{F}}_1}}$$

choosing the orthonormal basis $\{e_1, e_2, e_3\}$ we get

$$A_{\bar{\mathcal{F}}_1} = 1,\quad A_{\bar{\mathcal{F}}_2} = 1,\quad A_{\bar{\mathcal{F}}_3} = 1,\quad V_{\bar{\mathcal{R}}} = 1$$

$$(\mathrm{cof}\,F)\,e_1 = n_1\,A_{\mathcal{F}_1} \qquad n_1\cdot Fe_2 = 0,\ n_1\cdot Fe_3 = 0,\ n_1\cdot n_1 = 1$$

$$(\mathrm{cof}\,F)\,e_1\cdot e_i = (n_1\cdot e_i)\,A_{\mathcal{F}_1} = (n_1\cdot e_i)\,\mathrm{vol}(n_1, Fe_2, Fe_3)$$

$$w_{1i} := e_i - (e_i\cdot n_1)\,n_1 \quad\Rightarrow\quad w_{1i}\cdot n_1 = 0$$

$$w_{1i}\in \mathrm{span}\{Fe_2, Fe_3\}$$

$$(n_1\cdot e_i)\,n_1 = e_i - w_{1i}$$

$$(\mathrm{cof}\,F)\,e_1\cdot e_i = \mathrm{vol}(e_i, Fe_2, Fe_3)$$

$\leftarrow$ $(i,1)$ minor

[...]

$$(\mathrm{cof}\,F)\,e_2\cdot e_i = \mathrm{vol}(Fe_1, e_i, Fe_3) \quad \leftarrow (i,2) \text{ minor}$$

$$(\mathrm{cof}\,F)\,e_3\cdot e_i = \mathrm{vol}(Fe_1, Fe_2, e_i) \quad \leftarrow (i,3) \text{ minor}$$

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