[2018-04-18]

Cauchy continuum

$$\mathcal{F}^{ext}(v) = \int_{\mathcal{R}} b \cdot v \, dV + \int_{\partial\mathcal{R}} t \cdot v \, dA$$

$$\mathcal{F}^{int}(v) = -\int_{\mathcal{R}} z \cdot v \, dV - \int_{\mathcal{R}} T \cdot \nabla v \, dV$$

material frame indifference (objectivity)

$$z \cdot v + T \cdot \nabla v = 0$$

for any rigid test velocity field

$$v(x) = v_o + W(x - p_o)$$

$$z \cdot v_o + z \otimes (x - p_o) \cdot W + T \cdot W = 0 \quad \forall v_o, \forall W$$

$$\Rightarrow \begin{cases} z = 0 \\ \operatorname{skw} T = 0 \end{cases}$$

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Balance law

$$\mathcal{F}^{ext}(v) + \mathcal{F}^{int}(v) = 0 \qquad \forall v$$

$$\int_{\mathcal{R}} b \cdot v \, dV + \int_{\partial\mathcal{R}} t \cdot v \, dA = \int_{\mathcal{R}} T \cdot \nabla v \, dV$$

Defining the divergence of a tensor field T by

$$\operatorname{div} T \cdot v = \operatorname{div}(T^T v) - T \cdot \nabla v \qquad \forall v$$

$$\int_{\mathcal{R}} b \cdot v \, dV + \int_{\partial\mathcal{R}} t \cdot v \, dA + \int_{\mathcal{R}} \operatorname{div} T \cdot v \, dV - \int_{\mathcal{R}} \operatorname{div}(T^T v) \, dV = 0$$

divergence theorem

with n outward unit normal vector field

$$\int_{\mathcal{R}} \operatorname{div}(T^T v) \, dV = \int_{\partial\mathcal{R}} T^T v \cdot n \, dA$$

$$\int_{\mathcal{R}} (b + \operatorname{div} T) \cdot v \, dV + \int_{\partial\mathcal{R}} (t - Tn) \cdot v \, dA = 0 \qquad \forall v$$

$$\Rightarrow \qquad \begin{cases} \operatorname{div} T + b = 0 \\ Tn = t \end{cases} \qquad \text{Cauchy balance equations}$$

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$$[T] = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$ matrix of T in an orthonormal basis e_i

$$\operatorname{div} T \cdot e_1 = \operatorname{div}(T^T e_1) = \operatorname{tr} \nabla (T^T e_1)$$

$$T^T e_1 = \sigma_{11}\, e_1 + \sigma_{12}\, e_2 + \sigma_{13}\, \sigma_3$$

$$[\nabla (T^T e_1)] = \begin{pmatrix} \sigma_{11,1} & \sigma_{11,2} & \sigma_{11,3} \\ \sigma_{12,1} & \sigma_{12,2} & \sigma_{12,3} \\ \sigma_{13,1} & \sigma_{13,2} & \sigma_{13,3} \end{pmatrix}$$

$$\operatorname{div} T \cdot e_1 = \sigma_{11,1} + \sigma_{12,2} + \sigma_{13,3}$$

$$T^T e_2 = \sigma_{21}\, e_1 + \sigma_{22}\, e_2 + \sigma_{23}\, e_3$$

$$T^T e_3 = \sigma_{31}\, e_1 + \sigma_{32}\, e_2 + \sigma_{33}\, e_3$$

$$\operatorname{div} T \cdot e_2 = \sigma_{21,1} + \sigma_{22,2} + \sigma_{23,3}$$

$$\operatorname{div} T \cdot e_3 = \sigma_{31,1} + \sigma_{32,2} + \sigma_{33,3}$$

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Deviatoric and spherical part of a tensor

For any tensor L, if we define

$$\operatorname{sph} L := \frac{1}{3}(\operatorname{tr} L) I \qquad \text{SPHERICAL PART}$$

and

$$\operatorname{dev} L := L - \frac{1}{3}(\operatorname{tr} L) I \qquad \text{DEVIATORIC PART}$$

we get

$$\operatorname{dev} L + \operatorname{sph} L = L \quad , \quad \operatorname{tr}(\operatorname{dev} L) = 0$$

In general

$$\begin{aligned} T \cdot L &= (\operatorname{dev} T + \operatorname{sph} T) \cdot (\operatorname{dev} L + \operatorname{sph} L) \\ &= \operatorname{dev} T \cdot \operatorname{dev} L + \operatorname{sph} T \cdot \operatorname{sph} L \\ &\quad + \operatorname{dev} T \cdot \operatorname{sph} L + \operatorname{sph} T \cdot \operatorname{dev} L \end{aligned}$$

Since

$$\begin{aligned} \operatorname{dev} T \cdot \operatorname{sph} L &= \left(T - \frac{1}{3}(\operatorname{tr} T) I\right) \cdot \left(\frac{1}{3}(\operatorname{tr} L) I\right) \\ &= \frac{1}{3}(\operatorname{tr} L)\, T \cdot I - \frac{1}{9}(\operatorname{tr} T)(\operatorname{tr} L)\, I \cdot I \\ &= \frac{1}{3}(\operatorname{tr} L)(\operatorname{tr} T) - \frac{1}{9}(\operatorname{tr} T)(\operatorname{tr} L) \operatorname{tr}(I) = 0 \end{aligned}$$

then

$$T \cdot L = \operatorname{dev} T \cdot \operatorname{dev} L + \operatorname{sph} T \cdot \operatorname{sph} L$$

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