Cauchy stress tensor and Piola stress tensor

$T \cdot \nabla v$ stress power density per unit current volume

$\nabla v\, F = \nabla \bar{v}$ velocity gradient in the reference shape

$$(T \cdot \nabla v)\, V_{\mathcal{R}} = (T \cdot \nabla v)(\det F)\, V_{\bar{\mathcal{R}}}$$

CAUCHY STRESS

$$= (\det F)\,(T \cdot \nabla \bar{v}\, F^{-1})\, V_{\bar{\mathcal{R}}}$$

$$= \underbrace{(\det F)\,(T\, F^{-T})}_{S} \cdot \nabla \bar{v}\; V_{\bar{\mathcal{R}}} = S \cdot \nabla \bar{v}\; V_{\bar{\mathcal{R}}}$$

PIOLA STRESS

$\operatorname{cof} F := (\det F)\, F^{-T}$ cofactor of F

$$\mathcal{F}^{int}(v) = -\int_{\mathcal{R}} T \cdot \nabla v \, dV = -\int_{\bar{\mathcal{R}}} (T \cdot \nabla v) \det F \, dV$$

$$= -\int_{\bar{\mathcal{R}}} S \cdot \nabla \bar{v} \, dV$$

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Force balance (power balance)

$$\mathcal{F}^{ext}(v) + \mathcal{F}^{int}(v) = 0 \quad \forall v$$

Energy imbalance principle

$$\mathcal{F}^{ext}(v) - \frac{d}{dt}\int_{\bar{\mathcal{R}}} \varphi(F)\, dV \geq 0$$

in any motion.

$\varphi(F)$ strain energy density per unit reference volume

Since $$\mathcal{F}^{ext}(v) = -\mathcal{F}^{int}(v) = \int_{\mathcal{R}} T \cdot \dot{F} F^{-1} dV = \int_{\bar{\mathcal{R}}} S \cdot \dot{F}\, dV$$

we get $$\int_{\bar{\mathcal{R}}} S \cdot \dot{F}\, dV - \frac{d}{dt}\int_{\bar{\mathcal{R}}} \varphi(F) dV \geq 0$$

We extend this inequality to any subset of $\bar{\mathcal{R}}$ and get

$$S \cdot \dot{F} - \frac{d}{dt}\varphi(F) \geq 0$$

If $S = \hat{S}(F)$, and correspondingly $T = \hat{T}(F)$, the material is an elastic material with response function $\hat{S}(F)$, and correspondingly $\hat{T}(F)$.

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[2018-04-26]

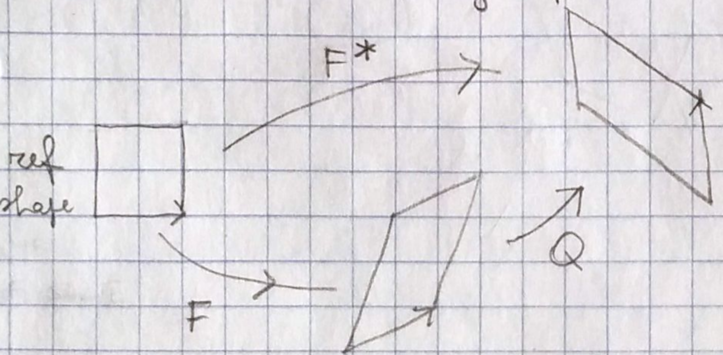
Strain energy

If

$$\hat{S}(F)\cdot\dot{F} = \frac{d}{dt}\varphi(F)$$

the material is called a hyperelastic material.

We require φ to be frame indifferent (objective, invariant under a change of observer)



$$Q^T Q = I$$

$$\varphi(F) = \varphi(F^*) \qquad F^* = QF$$

$$\Rightarrow \quad \varphi(F) = \varphi(QF) \qquad \forall Q,\ \forall F$$

$$\varphi(F) = \varphi(QRU)$$

$$\varphi(F) = \varphi(RU) = \varphi(R^* U) \qquad R^* = QR$$

$\Rightarrow$ φ is indipendent of the rotation tensor

$$\varphi(F) = \varphi(U)$$

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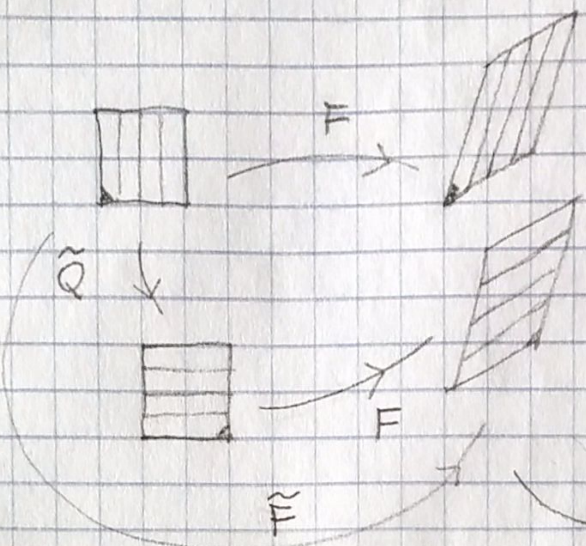
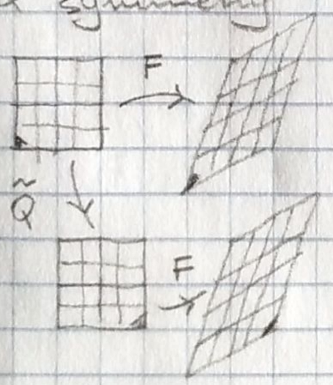
Material symmetry



$$\tilde{F} = F\tilde{Q}$$

$$\longrightarrow \varphi(F\tilde{Q}) \neq \varphi(F)$$

$\tilde{Q}$ symmetry $\qquad \varphi(F\tilde{Q}) = \varphi(F)$ $\qquad$ symmetry invariance



$$\varphi(F\tilde{Q}) = \varphi(F)$$

$$\downarrow \qquad\qquad \downarrow \text{obj}$$

$$\varphi(RU\tilde{Q}) = \varphi(U)$$

$$\downarrow$$

$$\varphi(R\tilde{Q}\tilde{Q}^T U\tilde{Q})$$

$$\text{obj} \downarrow$$

$$\varphi(\tilde{Q}^T U\tilde{Q}) = \varphi(U)$$

Symmetry group $\quad \mathcal{G} = \{\tilde{Q}_1, \tilde{Q}_2, \dots, I\} \subset Orth^+$

for a material; it characterizes both a material and its reference shapes with a zero stress.

If $\quad \mathcal{G} = Orth^+$ (the rotation group)

the material is called isotropic material.

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