[2018-04-26]

Isotropic hyperelastic material

$$\varphi(F) = \varphi(U) \qquad \text{objectivity}$$

$$\varphi(\tilde{Q}^T U \tilde{Q}) = \varphi(U) \quad \forall \tilde{Q} \qquad \text{isotropy}$$

$$U = \lambda_1\, a_1 \otimes a_1 + \lambda_2\, a_2 \otimes a_2 + \lambda_3\, a_3 \otimes a_3$$

$$\tilde{Q}^T (a_1 \otimes a_1) \tilde{Q} = (\tilde{Q}^T a_1) \otimes (\tilde{Q}^T a_1)$$

$$A u \otimes B v = A (u \otimes v) B^T$$

$\{a_1, a_2, a_3\}$ orthonormal basis

$\{\tilde{Q}^T a_1, \tilde{Q}^T a_2, \tilde{Q}^T a_3\}$ orthonormal basis $(\tilde{Q} a_i \cdot \tilde{Q} a_j = a_i \cdot a_j)$

Since $\tilde{Q}^T$ is an arbitrary rotation an isotropic strain energy turns out to be independent of the eigenvectors of U. Hence it depends only on the eigenvalues of U, the principal stretches $\lambda_1, \lambda_2, \lambda_3$.

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In turn, since the principal stretches are the square roots of the eigenvalues of the Cauchy-Green tensor C, we can look at φ as depending on

$$\eta_1 = \lambda_1^2, \; \eta_2 = \lambda_2^2, \; \eta_3 = \lambda_3^2$$

which are the roots of the characteristic polynomial

$$\det(C - \eta I)$$

whose normal form is

$$\eta^3 - \iota_1 \eta^2 + \iota_2 \eta - \iota_3$$

with

$$\iota_1 = \operatorname{tr} C$$

$$\iota_2 = \frac{1}{2}\left(\iota_1^2 - \operatorname{tr} C^2\right)$$

principal invariants of C

$$\iota_3 = \det C$$

Because the roots of the characteristic polynomial depend solely on $\iota_1, \iota_2, \iota_3$ we can describe an isotropic strain energy as a function

$$\varphi(F) = \varphi(U) = \hat{\varphi}(\iota_1, \iota_2, \iota_3)$$

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Isotropic hyperelastic material

$$\varphi(F) = \hat{\varphi}(\iota_1, \iota_2, \iota_3)$$

$$\hat{S}_e(F) = 2F\left((\varphi_{,1} + \varphi_{,2}\,\iota_1)I - \varphi_{,2}\,C + \varphi_{,3}\,\iota_3\,C^{-1}\right)$$

$$\hat{T}_e(F) = \frac{2}{\sqrt{\iota_3}}\left((\varphi_{,1} + \varphi_{,2}\,\iota_1)B - \varphi_{,2}\,B^2 + \varphi_{,3}\,\iota_3\,I\right)$$

$B = FF^T$ left Cauchy-Green tensor

$$\frac{d}{dt}\iota_1 = \frac{d}{dt}(F\cdot F) = 2F\cdot\dot{F}$$

$$\frac{d}{dt}\iota_2 = \frac{d}{dt}\frac{1}{2}\left((\mathrm{tr}\,C)^2 - \mathrm{tr}\,C^2\right) = \frac{d}{dt}\frac{1}{2}\left(\iota_1^2 - C\cdot C\right)$$

$$= 2\iota_1 F\cdot\dot{F} - C\cdot\dot{C} = 2\iota_1 F\cdot\dot{F} - C\cdot(\dot{F}^T F + F^T\dot{F})$$

$$= 2\iota_1 F\cdot\dot{F} - 2C\cdot F^T\dot{F} = (2\iota_1 F - 2FC)\cdot\dot{F}$$

$$\frac{d}{dt}\iota_3 = \frac{d}{dt}(\det F)^2 = 2(\det F)\frac{d}{dt}(\det F)$$

$$= 2(\det F)(\det F)\,\mathrm{tr}(\dot{F}F^{-1}) = 2\iota_3 F^{-T}\cdot\dot{F}$$

$$= 2\iota_3\, FC^{-1}\cdot\dot{F}$$

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Isochoric motion

$$\det F(t) = 1$$

$$\frac{d}{dt}(\det F) = (\det F)\, \mathrm{tr}\,(\dot{F}F^{-1}) = \overset{1}{(\det F)}\, \mathrm{tr}\, \nabla v = \mathrm{div}\, v = 0$$

Velocity gradient decomposition

$$\nabla v = \underbrace{\nabla v - \frac{1}{3}(\mathrm{tr}\,\nabla v)I}_{\text{deviatoric}} + \underbrace{\frac{1}{3}(\mathrm{tr}\,\nabla v)I}_{\text{spherical}}$$

Deformation gradient decomposition

$$F = F_v F_I \qquad F_v = (\det F)^{1/3} I$$

$$\dot{F}F^{-1} = (\dot{F}_v F_I + F_v \dot{F}_I) F_I^{-1} F_v^{-1}$$

$$= \dot{F}_v F_v^{-1} + F_v (\dot{F}_I F_I^{-1}) F_v^{-1}$$

$$= \dot{F}_v F_v^{-1} + \dot{F}_I F_I^{-1}$$

$$\mathrm{tr}\, \dot{F}_I F_I^{-1} = \left(\frac{d}{dt}(\det F_I)\right) \frac{1}{\det F_I} = 0$$

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