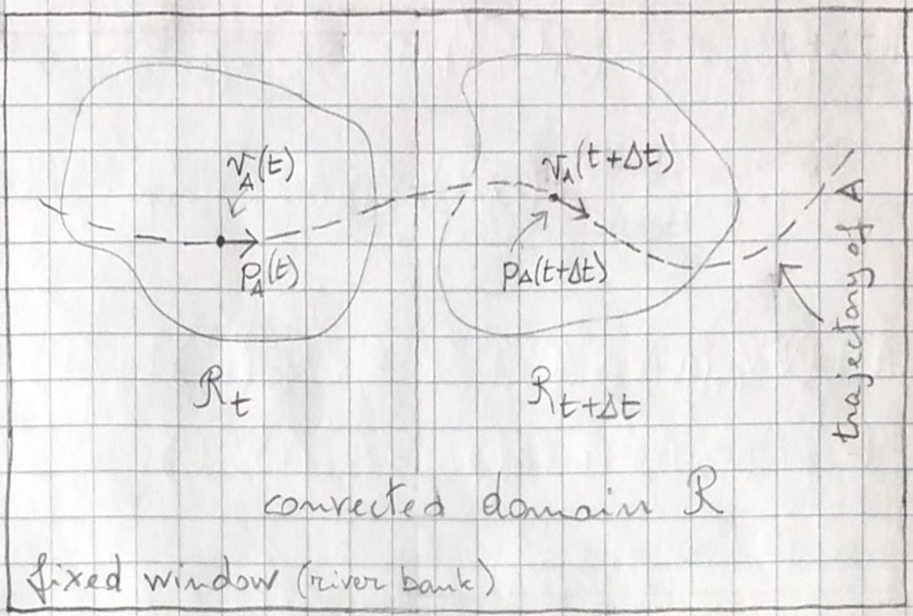
Material description and spatial description



$p_A(t)$, $v_A(t) = \dot{p}_A(t)$, $a_A(t) = \ddot{p}_A(t)$ acceleration

Spatial description (over the fixed window)

$$v_t(x) = \dot{p}_A(t) \quad \text{with} \quad x = p_A(t)$$

$$a_t(x) = \ddot{p}_A(t) \quad \text{with} \quad x = p_A(t)$$

$$\ddot{p}_A(t) = \lim_{\Delta t \to 0} \frac{\dot{p}_A(t+\Delta t) - \dot{p}_A(t)}{\Delta t}$$

$$\dot{p}_A(t+\Delta t) = v_{t+\Delta t}(y), \qquad \underline{y} = p_A(t+\Delta t)$$

$$\lim_{\Delta t \to 0} \frac{v_{t+\Delta t}(p_A(t+\Delta t)) - v_t(p_A(t))}{\Delta t} = a(p_A(t))$$

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$$a(p_A(t)) = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\Big(v_{t+\Delta t}\big(p_A(t+\Delta t)\big) - v_t\big(p_A(t)\big) \Big)$$

$$- v_{t+\Delta t}\big(p_A(t)\big) + v_{t+\Delta t}\big(p_A(t)\big)$$

$$a(p_A(t)) = \big(\nabla v_t(x)\big)\, \dot{p}_A(t) + \frac{\partial}{\partial t} v_t(x), \qquad x = p_A(t)$$

tangent vector to the trajectory

As a time dependent vector field over a fixed window it can be given the more descriptive form

$$a(x,t) = \nabla v\Big|_{(x,t)} v(x,t) + \frac{\partial}{\partial t} v(x,t)$$

In short

$$a = (\nabla v)\, v + v'$$

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Newtonian fluids

$\mathrm{div}\, v = 0$ (incompressibility)

$\hat{T}(F) = 0$ (no strain energy)

$T^{+} = 2\mu\, \mathrm{sym}\nabla v$ (viscous dissipation)

stress

$$T = -pI + 2\mu\, \mathrm{sym}\nabla v$$

force balance

$$\mathrm{div}T + b = 0$$

$$\mathrm{div}T = -\mathrm{div}(pI) + \mu(\mathrm{div}\nabla v + \mathrm{div}\nabla v^{T})$$

$$\mathrm{div}(pI)e = \mathrm{div}(pe) = \mathrm{tr}(\nabla(pe))$$

$$= \mathrm{tr}(e \otimes \nabla p) = \nabla p \cdot e \Rightarrow \mathrm{div}(pI) = \nabla p$$

$\mathrm{div}\nabla v = \Delta v$ (Laplacian)

$$\mathrm{div}\nabla v^{T} = \nabla(\underbrace{\mathrm{div}\, v}_{\to 0}) = 0$$

$$\mathrm{div}T = -\nabla p + \mu \Delta v$$

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$b = b^{in} + b_o$ (bulk force density)

b^{in} inertial force density

$b^{in} = -\rho\, a$

ρ mass density

a acceleration field

with

$$a = (\nabla v)v + v' \quad \text{(spatial description)}$$

Replacing the expressions for $\operatorname{div} T$ and b into the balance equation we get the Navier-Stokes equation

$$-\nabla p + \mu \Delta v + \rho (\nabla v) v + \rho v' + b_o = 0$$

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