

Energy imbalance principle

Force balance

$$\int_{\mathcal{R}} \mathbf{f}^{\text{ext}}(\mathbf{r}) + \int_{\partial \mathcal{R}} \mathbf{t}^{\text{int}}(\mathbf{r}) = 0 \quad \forall \mathcal{R} \text{ test}$$

Power expended in any motion

$$\int_{\mathcal{R}} \mathbf{b} \cdot \mathbf{r} \, dV + \int_{\partial \mathcal{R}} \mathbf{t} \cdot \mathbf{r} \, dA = \int_{\mathcal{R}} \mathbf{T} \cdot \dot{\mathbf{F}} \mathbf{F}^{-T} \, dV$$

$$\int_{\mathcal{R}} \mathbf{T} \cdot \dot{\mathbf{F}} \mathbf{F}^{-T} \, dV = \int_{\mathcal{R}} \mathbf{S} \cdot \dot{\mathbf{F}} \, dV$$

energy imbalance principle

$$\mathbf{S} \cdot \dot{\mathbf{F}} - \frac{d}{dt} \varphi(\mathbf{F}) \geq 0 \quad \text{dissipation inequality}$$

\uparrow
 $\hat{\mathbf{S}}(\mathbf{F}) \cdot \dot{\mathbf{F}}$

$$(\mathbf{S} - \hat{\mathbf{S}}(\mathbf{F})) \cdot \dot{\mathbf{F}} \geq 0 \quad \forall \text{ motion}$$

$$\underbrace{\frac{d}{dt} \varphi(\mathbf{F})}_{\geq 0} (\mathbf{T} - \hat{\mathbf{T}}(\mathbf{F})) \cdot \dot{\mathbf{F}} \mathbf{F}^{-T} \geq 0 \quad \forall \text{ motion}$$

\mathbf{T}^+ ← dissipative part of the stress

$$\mathbf{T} = \hat{\mathbf{T}}(\mathbf{F}) + \mathbf{T}^+$$

[Notebook page scanned on 2018/09/19]

Characterizing the dissipation mechanism

T^+ cannot be a constant tensor

It should depend on the velocity gradient
in order to fulfill the inequality

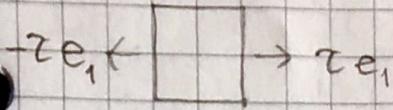
$$T^+ \cdot \dot{F}F^{-1} \geq 0$$

The simplest choice is

$$T^+ = 2 \mu \operatorname{sym} \dot{F}F^{-1}$$

$$\mu \dot{F}F^{-1} \cdot \dot{F}F^{-1} \geq 0 \Leftrightarrow \mu \geq 0$$

[Notebook page scanned on 2018/09/19]



$$T = \hat{T}_e(F) - pI + 2\mu \text{sym}(\dot{F}F^{-1})$$

$$\sigma_1 = \hat{\sigma}_1(A) - p + 2\mu \frac{\dot{\lambda}}{\lambda}$$

$$\sigma_1 = \tau$$

$$\sigma_2 = \hat{\sigma}_2(A) - p - \mu \frac{\dot{\lambda}}{\lambda}$$

$$\sigma_2 = 0$$

$$\sigma_3 = \hat{\sigma}_3(A) - p - \mu \frac{\dot{\lambda}}{\lambda}$$

$$\sigma_3 = 0$$

(material response)

(balance equations)

$$\text{tr} T_e(F) = 0$$

$$\hat{\sigma}_1(A) - p + 2\mu \frac{\dot{\lambda}}{\lambda} = \tau$$

$$\hat{\sigma}_2(A) - p - \mu \frac{\dot{\lambda}}{\lambda} = 0$$

$$\hat{\sigma}_3(A) - p - \mu \frac{\dot{\lambda}}{\lambda} = 0$$

taking the trace

$$0 - 3p + 0 = \tau$$

$$\Rightarrow p = -\frac{1}{3}\tau$$

[Notebook page scanned on 2018/09/19]

$$\hat{\sigma}_1(\lambda) = \frac{4}{3} \kappa \left(\lambda^2 - \frac{1}{\lambda} \right)$$

$$\frac{4}{3} \kappa \left(\lambda^2 - \frac{1}{\lambda} \right) + \frac{1}{3} \tau + 2\mu \frac{\dot{\lambda}}{\lambda} = \tau$$

$$\hat{\sigma}_2(\lambda) - \hat{\sigma}_3(\lambda) = 0$$

$$\hat{\sigma}_2(\lambda) + \hat{\sigma}_3(\lambda) - 2\rho - 2\mu \frac{\dot{\lambda}}{\lambda} = 0$$

$$\hat{\sigma}_2(\lambda) + \hat{\sigma}_3(\lambda) = -\hat{\sigma}_1(\lambda) = -\frac{4}{3} \kappa \left(\lambda^2 - \frac{1}{\lambda} \right)$$

$$\Rightarrow \hat{\sigma}_2(\lambda) = -\frac{2}{3} \kappa \left(\lambda^2 - \frac{1}{\lambda} \right)$$

$$\hat{\sigma}_3(\lambda) = -\frac{2}{3} \kappa \left(\lambda^2 - \frac{1}{\lambda} \right)$$

$$\frac{4}{3} \kappa \left(\lambda^2 - \frac{1}{\lambda} \right) + 2\mu \frac{\dot{\lambda}}{\lambda} = \frac{2}{3} \tau$$

$$2\kappa \left(\lambda^2 - \frac{1}{\lambda} \right) + 3\mu \frac{\dot{\lambda}}{\lambda} = \tau$$

[Notebook page scanned on 2018/09/19]

$$\hat{T}_e(F) = 2c_1 \left(FF^T - \frac{1}{3} b_2 (FF^T) I \right) \text{ neo-Hookean}$$

$$[FF^T] = \begin{pmatrix} \lambda^2 & & \\ & 1/\lambda & \\ & & 1/\lambda \end{pmatrix} \quad \dot{F}F^{-1} = \begin{pmatrix} 1 & & \\ & -\frac{1}{2} & \\ & & -\frac{1}{2} \end{pmatrix} \frac{\dot{\lambda}}{\lambda}$$

$$[\hat{T}_e(F)] = 2c_1 \begin{pmatrix} \frac{2}{3} & & \\ & -\frac{1}{3} & \\ & & -\frac{1}{3} \end{pmatrix} \left(\lambda^2 - \frac{1}{\lambda} \right)$$

$$\hat{T}_e(F) - pI + 2\mu \text{sym} \dot{F}F^{-1} = \tau e_1 \otimes e_1$$

(balance & material characterization)

sph $-p3 = \tau$

dev $\hat{T}_e(F) + 2\mu \text{sym} \dot{F}F^{-1} = \tau (e_1 \otimes e_1 - \frac{1}{3} I)$

$$\frac{4}{3} c_1 \left(\lambda^2 - \frac{1}{\lambda} \right) + 2\mu \frac{\dot{\lambda}}{\lambda} = \frac{2}{3} \tau$$

$$2c_1 \left(\lambda^2 - \frac{1}{\lambda} \right) + 3\mu \frac{\dot{\lambda}}{\lambda} = \tau$$

[Notebook page scanned on 2018/09/19]

linearization around λ_0 (stationary solution)

such that

$$2c\left(\lambda_0^2 - \frac{1}{\lambda_0}\right) = \tau$$

$$2c\left(2\lambda_0 + \frac{1}{\lambda_0^2}\right) \underbrace{(\lambda - \lambda_0)}_{\tilde{\varepsilon}} + 3\mu \frac{\dot{\lambda}}{\lambda_0} = 0$$

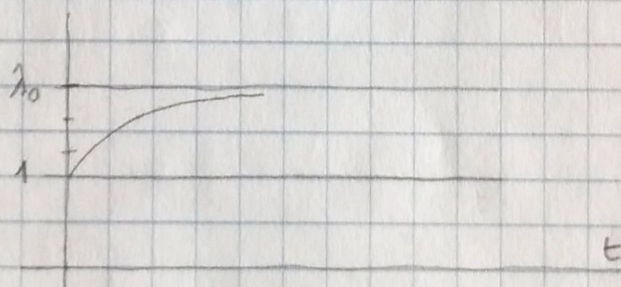
$$\frac{2}{3}c\left(2\lambda_0^2 + \frac{1}{\lambda_0}\right)(\lambda - \lambda_0) = -\mu \dot{\lambda}$$

$$\lambda(t) = \lambda_0 + \tilde{\varepsilon}_0 e^{-\frac{\alpha}{\mu}t}; \quad \tilde{\varepsilon}(t) = \tilde{\varepsilon}_0 e^{-\frac{\alpha}{\mu}t}$$

$$\frac{2}{3}c\left(2\lambda_0^2 + \frac{1}{\lambda_0}\right)\tilde{\varepsilon}_0 e^{-\frac{\alpha}{\mu}t} = +\mu \frac{\alpha}{\mu} \tilde{\varepsilon}_0 e^{-\frac{\alpha}{\mu}t}$$

$$\Rightarrow \frac{2}{3}c\left(\lambda_0^2 + \frac{1}{\lambda_0}\right) = \alpha \quad \Rightarrow \alpha = 0$$

$$t=0 \quad \lambda(0) = \lambda_0 + \tilde{\varepsilon}_0$$



$$\lambda(0) = 1 \Rightarrow 1 - \lambda_0 = \tilde{\varepsilon}_0$$

$$\Rightarrow \lambda(t) = \lambda_0 + \tilde{\varepsilon}_0 e^{-\frac{\alpha}{\mu}t}$$

[Notebook page scanned on 2018/09/19]