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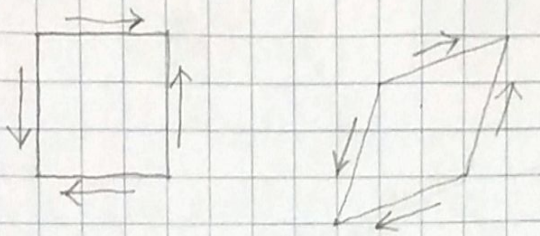
Shear deformation



$$t_1 = \tau \frac{u_2}{\|u_2\|}$$

$$t_2 = \tau \frac{u_1}{\|u_1\|}$$

$$[F] = \underbrace{\begin{pmatrix} 1 & \gamma/\lambda & \\ \gamma/\lambda & 1 & \\ & & \dfrac{\lambda^2}{\lambda^2-\gamma^2} \end{pmatrix}}_{\text{shear deformation}} \underbrace{\begin{pmatrix} \lambda & & \\ & \lambda & \\ & & \dfrac{1}{\lambda^2} \end{pmatrix}}_{\text{biaxial deformation}} = \begin{pmatrix} \lambda & \gamma & \\ \gamma & \lambda & \\ & & \dfrac{1}{\lambda^2-\gamma^2} \end{pmatrix}$$

$$[C] = \begin{pmatrix} \gamma^2+\lambda^2 & 2\gamma\lambda & \\ 2\gamma\lambda & \gamma^2+\lambda^2 & \\ & & \dfrac{1}{(\lambda^2-\gamma^2)^2} \end{pmatrix}$$

$$F^T = F \quad \Downarrow \quad = [B]$$

eigenvalues $\left\{ (\lambda+\gamma)^2, (\lambda-\gamma)^2, \dfrac{1}{(\lambda^2-\gamma^2)^2} \right\}$

principal stretches $\left\{ (\lambda+\gamma), (\lambda-\gamma), \dfrac{1}{\lambda^2-\gamma^2} \right\}$

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$\lambda_1 = \lambda + \gamma$ principal stretches

$\lambda_2 = \lambda - \gamma$ $\qquad \lambda_2 > 0 \Rightarrow \lambda > \gamma$

$\lambda_3 = 1/(\lambda^2 - \gamma^2)$

eigenvectors
$$\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$a_1 = \frac{1}{\sqrt{2}}(e_1 + e_2), \quad a_2 = \frac{1}{\sqrt{2}}(-e_1 + e_2)$$

$$U = (\lambda + \gamma)\, a_1 \otimes a_1 + (\lambda - \gamma)\, a_2 \otimes a_2 \qquad (\lambda > \gamma)$$

$\Rightarrow U = F$

$$t_1 = \tau(\gamma e_1 + \lambda e_2)\frac{1}{\sqrt{\gamma^2 + \lambda^2}}; \quad t_2 = \tau(\lambda e_1 + \gamma e_2)\frac{1}{\sqrt{\lambda^2 + \gamma^2}}$$

$$M = A_{\mathcal{F}_1}\, t_1 \otimes u_1 + A_{\mathcal{F}_2}\, t_2 \otimes u_2$$

$$M_1 = A_{\mathcal{F}_1}\, t_1 \otimes u_1 = A_{\mathcal{F}_1}\, \tau \frac{u_2}{\|u_2\|} \otimes u_1 = A_{\mathcal{F}_1}\, \tau \frac{F e_2}{\|F e_2\|} \otimes (\ell_1 F e_1)$$

$$= \tau A_{\mathcal{F}_1} \frac{\ell_1}{\sqrt{\gamma^2 + \lambda^2}} (\gamma e_1 + \lambda e_2) \otimes (\lambda e_1 + \gamma e_2)$$

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$$A_{\mathcal{F}_1} = \ell_2 \ell_3 \frac{\sqrt{\lambda^2+\gamma^2}}{\lambda^2-\gamma^2}$$

$$[\operatorname{cof} F] = \begin{pmatrix} \dfrac{\lambda}{\lambda^2-\gamma^2} & \dfrac{-\gamma}{\lambda^2-\gamma^2} & 0 \\ \dfrac{-\gamma}{\lambda^2-\gamma^2} & \dfrac{\lambda}{\lambda^2-\gamma^2} & 0 \\ 0 & 0 & (\lambda^2-\gamma^2) \end{pmatrix}$$

$$A_{\mathcal{F}_2} = \ell_3 \ell_1 \frac{\sqrt{\lambda^2+\gamma^2}}{\lambda^2-\gamma^2}$$

$$V_{\mathcal{R}} = \ell_1 \ell_2 \ell_3 \det F = \ell_1 \ell_2 \ell_3$$

$$u_1 = \ell_1 (\lambda e_1 + \gamma e_2)$$

$$u_2 = \ell_2 (\gamma e_1 + \lambda e_2)$$

$$M_1 = \tau \frac{\ell_2 \ell_3 \ell_1}{\lambda^2-\gamma^2} (\gamma e_1 + \lambda e_2) \otimes (\lambda e_1 + \gamma e_2)$$

$$M_2 = \tau \frac{\ell_3 \ell_1 \ell_2}{\lambda^2-\gamma^2} (\lambda e_1 + \gamma e_2) \otimes (\gamma e_1 + \lambda e_2)$$

$$\left[\frac{M}{V_{\mathcal{R}}}\right] = \frac{\tau}{\lambda^2-\gamma^2} \begin{pmatrix} 2\gamma\lambda & \gamma^2+\lambda^2 & 0 \\ \lambda^2+\gamma^2 & 2\lambda\gamma & 0 \\ 0 & & 0 \end{pmatrix}$$

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balance &
material response
$$\frac{M}{V_\mathcal{B}} = \hat{T}_e(F) - pI$$

$$\begin{cases} \operatorname{tr}\dfrac{M}{V_\mathcal{B}} = -3p \\[2ex] \operatorname{dev}\dfrac{M}{V_\mathcal{B}} = \hat{T}_e(F) \end{cases}$$

$$\boxed{\tau\,\frac{4\gamma\lambda}{\lambda^2-\gamma^2} = -3p}$$

[1,1] $$\tau\,\frac{1}{\lambda^2-\gamma^2}\left(2\gamma\lambda - \frac{1}{3}4\gamma\lambda\right) = $$
$$= 2c\left((\gamma^2+\lambda^2) - \frac{1}{3}\left(2(\lambda^2+\gamma^2) + \frac{1}{(\lambda^2-\gamma^2)^2}\right)\right)$$

[1,1] $$\frac{2}{3}\tau\,\frac{\gamma\lambda}{\lambda^2-\gamma^2} = \frac{2}{3}c\left((\lambda^2+\gamma^2) - \frac{1}{(\lambda^2-\gamma^2)^2}\right)$$ ✓

[1,1] $$\tau\gamma\lambda = c\,\frac{(\lambda^2+\gamma^2)(\lambda^2-\gamma^2)^2-1}{\lambda^2-\gamma^2}$$ ✓

[1,2] $$\tau\,\frac{\lambda^2+\gamma^2}{\lambda^2-\gamma^2} = 4c\lambda\gamma$$

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We get two equations

$$\begin{cases} \tau \gamma \lambda (\lambda^2 - \gamma^2) = c\left((\lambda^2 + \gamma^2)(\lambda^2 - \gamma^2)^2 - 1 \right) \\ \tau (\lambda^2 + \gamma^2) = 4 c \lambda \gamma (\lambda^2 - \gamma^2) \end{cases}$$

Let's look for an asymptotic solution.

We assume that there exist two smooth functions

$$\begin{cases} \lambda = \hat{\lambda}(\tau) \\ \gamma = \hat{\gamma}(\tau) \end{cases} \quad \text{with} \quad \begin{cases} \hat{\lambda}(0) = 1 \\ \hat{\gamma}(0) = 0 \end{cases}$$

and replace their series expansions up to, say, the fourth order in the two equations above.

By equating coefficients of τ, τ^2, τ^3 evaluated at $\tau = 0$, we get in turn 3 equations.

The final result (skipping the lengthy computation) turns out to be

$$\hat{\lambda}(\tau) = 1 + \frac{5}{96}\frac{\tau^2}{c^2} + o(\tau^3)$$

$$\hat{\gamma}(\tau) = \frac{1}{4}\frac{\tau}{c} + \frac{7}{384}\frac{\tau^3}{c^3} + o(\tau^3)$$

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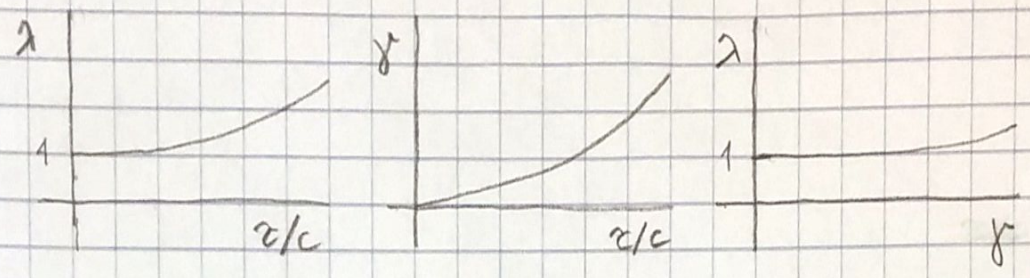

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Shear deformation (viscoelastic material)

$$T = \hat{T}_e(F) - pI + 2\mu \,\mathrm{sym}\nabla v$$

$$\mathrm{sym}\nabla v = \begin{pmatrix} \lambda & -\gamma & 0 \\ -\gamma & \lambda & 0 \\ 0 & 0 & -2\lambda \end{pmatrix} \dot{\lambda}/(\lambda^2-\gamma^2)$$

$$+ \begin{pmatrix} -\gamma & \lambda & 0 \\ \lambda & -\gamma & 0 \\ 0 & 0 & 2\gamma \end{pmatrix} \dot{\gamma}/(\lambda^2-\gamma^2)$$

We arrive at

$$\begin{cases} \tau\gamma\lambda(\lambda^2-\gamma^2) = c\left((\lambda^2+\gamma^2)(\lambda^2-\gamma^2)^2 - 1\right) \\ \qquad\qquad - 3\mu(\lambda^2-\gamma^2)(\gamma\dot{\gamma} - \lambda\dot{\lambda}) \\ \tau(\lambda^2+\gamma^2) = 4c\lambda\gamma(\lambda^2-\gamma^2) + 2\mu(\lambda\dot{\gamma} - \gamma\dot{\lambda}) \end{cases}$$

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