Compressible materials

$$\varphi(F) = k_I(\check{\iota}_1 - 3) + k_v(J-1)^2$$

Deformation gradient decomposition

$$F = F_v F_I$$

$$F_v := (\det F)^{1/3} I \qquad F_I = (\det F)^{-1/3} F$$

$$\Rightarrow \begin{cases} \det F_v = \det F \\ \det F_I = 1 \end{cases} \qquad J := \det F$$

$$\dot{F} F^{-1} = \dots = \dot{F}_v F_v^{-1} + \dot{F}_I F_I^{-1}$$

$$F^T F = F_I^T F_v^T F_v F_I = J^{\frac{2}{3}} F_I^T F_I$$

$$F F^T = F_v F_I F_I^T F_v^T = J^{\frac{2}{3}} F_I F_I^T$$

$$\operatorname{tr} F F^T = J^{\frac{2}{3}} \operatorname{tr} F_I F_I^T$$

$$\iota_1 = J^{\frac{2}{3}} \check{\iota}_1 \qquad \check{\iota}_1 = J^{-\frac{2}{3}} \iota_1$$

$$\operatorname{cof} F = (\det F) F^{-T} = J F_v^{-T} F_I^{-T} = J J^{-\frac{1}{3}} F_I^{-T} = J^{\frac{2}{3}} \operatorname{cof} F_I$$

$$\dot{F}_v F_v^{-1} = \frac{1}{3} J^{-\frac{2}{3}}$$

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$$\frac{d}{dt}\varphi(F) = k_I \frac{d}{dt}\check{\iota}_1 + 2k_v (J-1)\frac{d}{dt}J$$

$$\frac{d}{dt}\check{\iota}_1 = -\frac{2}{3}J^{-\frac{5}{3}}\left(\frac{d}{dt}J\right)\iota_1 + J^{-\frac{2}{3}}\frac{d}{dt}\iota_1$$

$$= -\frac{2}{3}J^{-\frac{5}{3}}J\,\mathrm{tr}(\dot{F}F^{-1})\,\iota_1 + 2J^{-\frac{2}{3}}F\cdot\dot{F}$$

$$= -\frac{2}{3}J^{-\frac{2}{3}}\iota_1\,F^{-T}\cdot\dot{F} + 2J^{-\frac{2}{3}}F\cdot\dot{F}$$

$$= 2J^{-\frac{2}{3}}\left(F - \frac{1}{3}\iota_1 F^{-T}\right)\cdot\dot{F}$$

$$\frac{d}{dt}J = J\,F^{-T}\cdot\dot{F}$$

$$\hat{S}(F)\cdot\dot{F} = \frac{d}{dt}\varphi(F)$$

Piola stress on the reference shape

$$\hat{S}(F) = 2k_I J^{-\frac{2}{3}}\left(F - \frac{1}{3}\iota_1 F^{-T}\right) + 2k_v J(J-1)F^{-T}$$

$$S = J\,T\,F^{-T} \Rightarrow T = J^{-1}S\,F^{T}$$

$$\hat{T}(F) = \underbrace{2k_I J^{-\frac{5}{3}}\left(FF^{T} - \frac{1}{3}\iota_1 I\right)}_{\text{deviatoric}} + \underbrace{\overbrace{2k_v}^{\text{bulk modulus}}(J-1)I}_{\text{spherical}}$$

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Uniaxial deformation (compressible material)

$$-\tau e_1 \leftarrow \square \rightarrow \tau e_1$$

$$\frac{M}{V_R} = \hat{T}_e(F)$$ balance & material response

(notice that there is no inner pressure)

$$[F] = \begin{pmatrix} \lambda_a & & \\ & \lambda_r & \\ & & \lambda_r \end{pmatrix} = [F_v][F_I] = J^{\frac{1}{3}} \begin{pmatrix} \lambda & & \\ & \frac{1}{\sqrt{\lambda}} & \\ & & \frac{1}{\sqrt{\lambda}} \end{pmatrix}$$

$$J := \det F = \lambda_a \lambda_r^2$$

$$[C] = \begin{pmatrix} \lambda_a^2 & & \\ & \lambda_r^2 & \\ & & \lambda_r^2 \end{pmatrix} = J^{\frac{2}{3}} \begin{pmatrix} \lambda^2 & & \\ & \frac{1}{\lambda} & \\ & & \frac{1}{\lambda} \end{pmatrix}$$

$$\iota_1 = \operatorname{tr} C = \lambda_a^2 + 2\lambda_r^2 = J^{\frac{2}{3}}\left(\lambda^2 + \frac{2}{\lambda}\right)$$

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$$
[\operatorname{cof} F] = \begin{pmatrix} \lambda_r^2 & & \\ & \lambda_a \lambda_r & \\ & & \lambda_a \lambda_r \end{pmatrix} = J^{\frac{2}{3}} \begin{pmatrix} \frac{1}{\lambda} & & \\ & \sqrt{\lambda} & \\ & & \sqrt{\lambda} \end{pmatrix}
$$

$$
M = A_{\mathcal{F}_1} \tau\, e_1 \otimes (\lambda_a \ell_1 e_1) = \ell_2 \ell_3 \lambda_r^2 \ell_1 \lambda_a \tau\, (e_1 \otimes e_1)
$$

$$
\frac{M}{V_{\mathcal{R}}} = \tau\, e_1 \otimes e_1
$$

$$
\frac{M}{V_{\mathcal{R}}} = \hat{T}_e(F) \qquad \text{neo-Hookean}
$$

balance & material response

$$
\begin{cases} \operatorname{dev} \dfrac{M}{V_{\mathcal{R}}} = 2 k_I J^{-\frac{5}{3}} \left(F F^T - \dfrac{1}{3} \iota_1 I \right) \\[2ex] \operatorname{tr} \dfrac{M}{V_{\mathcal{R}}} = 6 k_v (J - 1) \end{cases}
$$

$$
\Rightarrow \quad \tau = 6 k_v (J - 1)
$$

$$
\Rightarrow \quad J = 1 + \frac{1}{6} \frac{\tau}{k_v}
$$

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$$\begin{cases} \left(1-\frac{1}{3}\right)\tau = 2k_I J^{-\frac{5}{3}}\left(\lambda_a^2 - \frac{1}{3}\left(\lambda_a^2 + 2\lambda_r^2\right)\right) \\ \lambda_a \lambda_r^2 = 1 + \frac{1}{6}\frac{\tau}{k_V} \end{cases}$$

$$\begin{cases} \left(1-\frac{1}{3}\right)\tau = 2k_I J^{-\frac{5}{3}} J^{\frac{2}{3}}\left(\lambda^2 - \frac{1}{3}\left(\lambda^2 + \frac{2}{\lambda}\right)\right) \\ J = 1 + \frac{1}{6}\frac{\tau}{k_V} \end{cases}$$

$$\begin{cases} \frac{2}{3}\tau = \frac{4}{3}k_I J^{-1}\left(\lambda^2 - \frac{1}{\lambda}\right) \\ J = 1 + \frac{1}{6}\frac{\tau}{k_V} \end{cases}$$

$$\begin{cases} \frac{\tau}{k_I} = 2 J^{-1}\left(\lambda^2 - \frac{1}{\lambda}\right) \\ J = 1 + \frac{1}{6}\frac{\tau}{k_V} \end{cases}$$

$k_V \to \infty$ we recover the incompressible case

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Triaxial deformation (compressible material)

$t_1 = \tau\, e_1$

$t_2 = \tau\, e_2$

$t_3 = \tau\, e_3$

$$[F] = \begin{pmatrix} \lambda & & \\ & \lambda & \\ & & \lambda \end{pmatrix} = J^{1/3} \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$F = F_V F_I$

$F_V = (\det F)^{1/3}\, I \qquad F_I = I$

$F^T F = (\det F)^{\frac{2}{3}}\, I \qquad J := \det F$

$\operatorname{tr} F^T F = 3\, J^{\frac{2}{3}}$

$\iota_1 = 3\lambda^2 = 3\, J^{\frac{2}{3}}$

$\tilde{\iota}_1 = \operatorname{tr} F_I^T F_I = 3$

$F F^T = J^{\frac{2}{3}}\, I$

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$$\hat{T}_e(F) = 2k_I J^{-\frac{5}{3}} \underbrace{\left(J^{\frac{2}{3}} I - \frac{1}{3} 3 J^{\frac{2}{3}}\right)}_{0} + 2k_v (J-1) I$$

$$\begin{aligned} M = {} & A_{\mathcal{F}_1} \tau\, e_1 \otimes (\lambda \ell_1 e_1) \\ & + A_{\mathcal{F}_2} \tau\, e_2 \otimes (\lambda \ell_2 e_2) \\ & + A_{\mathcal{F}_3} \tau\, e_3 \otimes (\lambda \ell_3 e_3) \end{aligned}$$

$$[\operatorname{cof} F] = \begin{pmatrix} J^{2/3} & & \\ & J^{2/3} & \\ & & J^{2/3} \end{pmatrix}$$

$$A_{\mathcal{F}_1} = \ell_2 \ell_3 \lambda J^{2/3} \ell_1 = \ell_1 \ell_2 \ell_3 J^{1/3} J^{2/3} = \underbrace{\ell_1 \ell_2 \ell_3 J}_{V_\mathcal{R}}$$

$$\frac{M}{V_\mathcal{R}} = \tau (e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3) = \tau I$$

$$\frac{M}{V_\mathcal{R}} = \hat{T}_e(F)$$ neo-Hookean

$$\operatorname{tr} \frac{M}{V_\mathcal{R}} = 6 k_v (J-1)$$

$$3\tau = 6 k_v (J-1)$$

$$\tau = \underbrace{2k_v}_{\text{bulk modulus}} (J-1)$$

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