1

Linear elasticity (small deformations)

$$F(\varepsilon) = I + E(\varepsilon) + \Theta(\varepsilon) + o(\varepsilon)$$

$$F(\varepsilon) = R(\varepsilon)\,U(\varepsilon)$$

$$R(\varepsilon) = R(0) + \frac{d}{d\varepsilon}R\Big|_{\varepsilon=0}\varepsilon = I + \Theta + o(\varepsilon)$$

$$U(\varepsilon) = U(0) + \frac{d}{d\varepsilon}U\Big|_{\varepsilon=0}\varepsilon = I + E + o(\varepsilon)$$

$$R(\varepsilon)^T R(\varepsilon) = I \Rightarrow (I + \Theta^T)(I + \Theta) = I$$

$$I + \Theta + \Theta^T + o(\varepsilon) = I$$

$$\Rightarrow \Theta + \Theta^T = 0$$

$$U(\varepsilon) = U(\varepsilon)^T \Rightarrow I + E = I + E^T$$

$$\Rightarrow E = E^T$$

Θ infinitesimal rotation

E infinitesimal strain

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$$F^T F = (I + E^T + \Theta^T)(I + E + \Theta)$$

$$= I + E + \Theta + E^T + \Theta^T = I + 2E + o(\varepsilon)$$

$$\iota_1 = 3 + 2\,\mathrm{tr}\,E + o(\varepsilon)$$

$$\det F = 1 + \left(\frac{d}{d\varepsilon}\det F(\varepsilon)\right)\varepsilon + o(\varepsilon) \qquad (*)$$

$$= 1 + \mathrm{tr}\,E + \mathrm{tr}\,\Theta + o(\varepsilon)$$

$$= 1 + \mathrm{tr}\,E + o(\varepsilon)$$

$$(*)\ \varepsilon \frac{d}{d\varepsilon}\det F(\varepsilon) = \det F(0)\,\mathrm{tr}\left(\left(\frac{d}{d\varepsilon}F(\varepsilon)\right)F(\varepsilon)^{-1}\right)\varepsilon\Big|_{\varepsilon=0}$$

$$\simeq 1\ \mathrm{tr}\,(\Theta + E)(I - \Theta - E) = \mathrm{tr}\,E$$

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2

$$\hat{T}(F) = 2k_I J^{-\frac{5}{3}}\left(B - \frac{1}{3}\iota_1 I\right) + 2k_V (J-1) I$$

"isotropic hyperelastic (compressible neo-Hookean) material"

$$B - \frac{1}{3}\iota_1 I = I + 2E - \frac{1}{3}\left(3 + 2\,\mathrm{tr}E\right)I + o(\varepsilon)$$

$$= I + 2E - I - \frac{2}{3}(\mathrm{tr}E)I + o(\varepsilon)$$

$$= 2\left(E - \frac{1}{3}(\mathrm{tr}E)I\right) + o(\varepsilon)$$

$$J^{-\frac{5}{3}} = 1 - \frac{5}{3}\left(\frac{d}{d\varepsilon}\det F(\varepsilon)\right)\varepsilon + o(\varepsilon) = 1 - \frac{5}{3}\,\mathrm{tr}E + o(\varepsilon)$$

$$J^{-\frac{5}{3}}\left(B - \frac{1}{3}\iota_1 I\right) = 2\left(E - \frac{1}{3}(\mathrm{tr}E)I\right) + o(\varepsilon)$$

$$\hat{T}(F) = 4k_I\left(E - \frac{1}{3}(\mathrm{tr}E)I\right) + 2k_V(\mathrm{tr}E)I$$

$$= \left(2k_V - \frac{4}{3}k_I\right)(\mathrm{tr}E)I + 4k_I E$$

$$= \underbrace{\left(k - \frac{2}{3}\mu_L\right)}(\mathrm{tr}E)I + 2\mu_L E$$

$$= \lambda_L(\mathrm{tr}E)I + 2\mu_L E \qquad \text{(Lamé)}$$

$$k - \frac{2}{3}\mu_L = \lambda_L \Rightarrow k = \lambda_L + \frac{2}{3}\mu_L = \frac{1}{3}\left(3\lambda_L + 2\mu_L\right)$$

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Linear elasticity

isotropic material

$$\hat{T}(F) = \lambda_L (\operatorname{tr} E) I + 2\mu_L E$$

λ_L, μ_L Lamé moduli

$\mu_L = 2k_I$ shear modulus

$\kappa = 2k_v$ bulk modulus

$\lambda_L = \kappa - \frac{2}{3}\mu_L$

From uniaxial deformation (linearized)

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Uniaxial deformation

$$[F] = J^{\frac{1}{3}} \begin{pmatrix} \lambda & & \\ & \frac{1}{\sqrt{\lambda}} & \\ & & \frac{1}{\sqrt{\lambda}} \end{pmatrix} = \begin{pmatrix} \lambda_a & & \\ & \lambda_r & \\ & & \lambda_r \end{pmatrix}$$

$$\begin{cases} \lambda_a = \lambda\, J^{\frac{1}{3}} \\ \lambda_r = \frac{1}{\sqrt{\lambda}}\, J^{\frac{1}{3}} \end{cases} \Rightarrow \begin{cases} \frac{\lambda_a}{\lambda_r} = \lambda\sqrt{\lambda} \Rightarrow \lambda = \left(\frac{\lambda_a}{\lambda_r}\right)^{\frac{2}{3}} \\ J = \lambda_a \lambda_r^2 \end{cases}$$

[...]

$$\begin{cases} \frac{\tau}{k_I} = 2J\left(\lambda^2 - \frac{1}{\lambda}\right) \\ \frac{\tau}{k_V} = 6(J-1) \end{cases}$$

$$\lambda_a = 1 + \varepsilon_a \;,\; \lambda_r = 1 + \varepsilon_r$$

$$J = 1 + \varepsilon_a + 2\varepsilon_r \qquad \text{(small strains)}$$

$$\lambda = 1 + \frac{2}{3}(\varepsilon_a - \varepsilon_r)$$

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$$2J\left(\lambda^2-\frac{1}{\lambda}\right)=0+2(2+1)\frac{2}{3}(\varepsilon_a-\varepsilon_r)$$
$$=4(\varepsilon_a-\varepsilon_r)$$

$$\begin{cases}\dfrac{\tau}{2k_I}=2(\varepsilon_a-\varepsilon_r)\\[2ex]\dfrac{\tau}{2k_V}=3(2\varepsilon_r+\varepsilon_a)\end{cases}$$

$$\begin{cases}\dfrac{\tau}{\mu_L}=2(\varepsilon_a-\varepsilon_r)\\[2ex]\dfrac{\tau}{k}=3(2\varepsilon_r+\varepsilon_a)\end{cases}$$

$$\begin{cases}\dfrac{\tau}{2\mu_L}=\varepsilon_a-\varepsilon_r\\[2ex]\dfrac{\tau}{3k}=\varepsilon_a+2\varepsilon_r\end{cases}$$

$$\tau\left(\frac{1}{\mu_L}+\frac{1}{3k}\right)=3\varepsilon_a$$

$$\tau\left(-\frac{1}{2\mu_L}+\frac{1}{3k}\right)=3\varepsilon_r$$

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$$k = \lambda_L + \frac{2}{3}\mu_L = \frac{1}{3}\left(3\lambda_L + 2\mu_L\right)$$

$$\tau = 3\,\frac{3\mu_L k}{3k + \mu_L}\,\varepsilon_a$$

$$= 3\,\frac{\mu_L\left(3\lambda_L + 2\mu_L\right)}{3\lambda_L + 2\mu_L + \mu_L}\,\varepsilon_a$$

$$= \underbrace{\frac{\mu_L\left(3\lambda_L + 2\mu_L\right)}{\lambda_L + \mu_L}}_{Y}\,\varepsilon_a \qquad \text{Young's modulus}$$

$$\frac{\varepsilon_r}{\varepsilon_a} = \frac{-3k + 2\mu_L}{6\mu_L k}\,\frac{3\mu_L k}{3k + \mu_L}$$

$$= \frac{-3\lambda_L - 2\mu_L + 2\mu_L}{2\left(3\lambda_L + 2\mu_L + \mu_L\right)}$$

$$= \underbrace{-\frac{\lambda_L}{2\left(\lambda_L + \mu_L\right)}}_{\text{Poisson's ratio}}$$

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