A PRIMER IN (CONTINUUM) MECHANICS [2015-10-02]

Friday

(1-2) 9:00-11:00 (1.7)

positions taken by a small object (a particle)

The set of positions $\mathcal{E}$ together with a vector space $\mathcal{V}$ closed under the operation (translation)

$$+ : \mathcal{E} \times \mathcal{V} \to \mathcal{E}$$

$$x + u = y \qquad x, y \in \mathcal{E},\ u \in \mathcal{V}$$

tranforming a position x into a position y with the following properties

i) for any couple (x, y) there is only one vector, denoted by $y - x$ translating x to y

ii) for any position x and any two vectors u and v

$$(x + u) + v = x + (u + v)$$

iii) for any position x

$$x + o = x$$

where o is the null vector of $\mathcal{V}$

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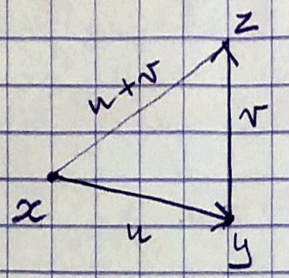
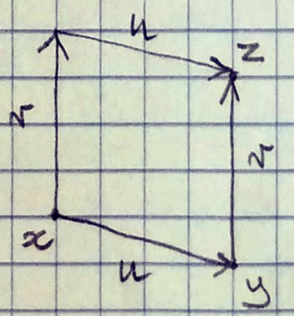
If we use dots and arrows for describing positions and vectors we can describe the second property by drawing a triangle



Since $(x+u)+v = (x+v)+u$

we can describe this equality by drawing a parallelogram



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Distance between two positions x, y

$$d(x,y) > 0 \qquad d(x,y) = d(y,x)$$

$$d(x,y) = 0 \iff x = y$$

$$d(x,y) = \| y - x \|$$

$$\| u \| = (u \cdot u)^{1/2}$$

with $\quad u \cdot v \in \mathbb{R} \qquad \forall u, v \in \mathcal{V}$

$$(\alpha u) \cdot v = \alpha (u \cdot v)$$

$$v \cdot u = u \cdot v$$

$$u \cdot u > 0 \qquad u \cdot u = 0 \iff u = 0$$

orthogonality $\quad u \cdot v = 0$

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body $\mathcal{B}$ collection of particles

Collective "behaviour" description

motion

$$p : \mathcal{B} \times \mathcal{I} \to \mathcal{E}$$

time interval $\mathcal{I} \subset \mathbb{R}$

placement at time t

$$p_t \qquad p : \mathcal{B} \times \{t\} \to \mathcal{E}$$

shape of body $\mathcal{B}$ at time t

$$\mathcal{R} = \operatorname{im} p_t \qquad \mathcal{R} \subset \mathcal{E}$$

trajectory of particle A

$$p_A \qquad p : \{A\} \times \mathcal{I} \to \mathcal{E}$$

deformation

$$\phi : \underset{\substack{\uparrow \\ \text{reference} \\ \text{shape}}}{\bar{\mathcal{R}}} \to \underset{\substack{\uparrow \\ \text{current} \\ \text{shape}}}{\mathcal{R}}$$

reference placement

$$\bar{p} : \mathcal{B} \to \bar{\mathcal{R}} \subset \mathcal{E}$$

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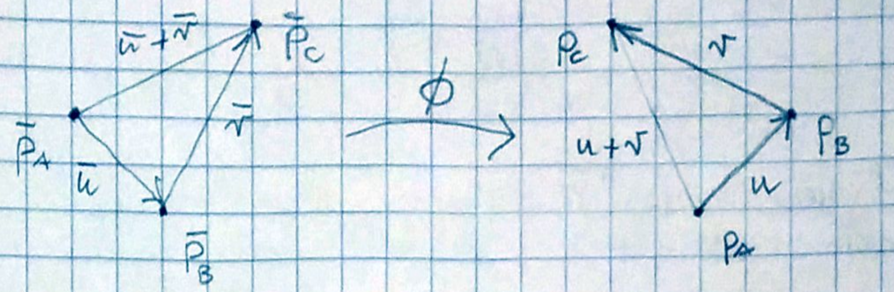
(3-4) [2015-10-06]
Tuesday 15:00 17:00

RIGID DEFORMATIONS



$\bar u = \bar p_B - \bar p_A \qquad u = \phi(\bar p_B) - \phi(\bar p_A)$

$\bar v = \bar p_C - \bar p_B \qquad v = \phi(\bar p_C) - \phi(\bar p_B)$

$\bar u + \bar v = \bar p_C - \bar p_A \qquad u + v = \phi(\bar p_C) - \phi(\bar p_A)$

$\|u\| = \|\bar u\|, \quad \|v\| = \|\bar v\|, \quad \|u+v\| = \|\bar u + \bar v\|$

$(u+v)\cdot(u+v) = (\bar u + \bar v)\cdot(\bar u + \bar v)$

$u\cdot u + 2\,u\cdot v + v\cdot v = \bar u\cdot \bar u + 2\,\bar u\cdot\bar v + \bar v\cdot\bar v$

$\|u\|^2 + 2\,u\cdot v + \|v\|^2 = \|\bar u\|^2 + 2\,\bar u\cdot\bar v + \|\bar v\|^2 \Rightarrow u\cdot v = \bar u\cdot\bar v$

In general (linearity)

$\bar u \mapsto u, \quad \bar v \mapsto v; \quad \{\bar a_1, \bar a_2, \bar a_3\} \mapsto \{a_1, a_2, a_3\}$

bases (*) →

$(\alpha\bar u + \beta\bar v) \mapsto \alpha u + \beta v + \varepsilon$

$(\alpha\bar u + \beta\bar v)\cdot\bar a_i = (\alpha u + \beta v + \varepsilon)\cdot a_i$

$\alpha(\bar u\cdot\bar a_i) + \beta(\bar v\cdot\bar a_i) = \alpha(u\cdot a_i) + \beta(v\cdot a_i) + \varepsilon\cdot a_i$

$\Rightarrow \quad \varepsilon\cdot a_i = 0 \quad i = 1, 2, 3 \quad \Rightarrow \quad \varepsilon = 0$

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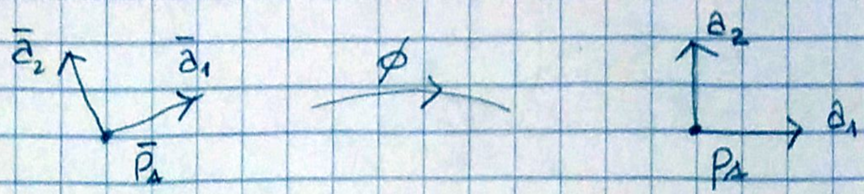
$$\{\bar{a}_1, \bar{a}_2, \bar{a}_3\} \mapsto \{a_1, a_2, a_3\}$$



Since norms and scalar products are left unchanged, an orthonormal basis is transformed into an orthonormal basis.

(*) In general if $\{\bar{a}_1, \bar{a}_2, \bar{a}_3\}$ are linearly independent vectors then the corresponding vectors $\{a_1, a_2, a_3\}$ are linearly independent vectors.

For if

$$\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3 = 0$$

then

$$(\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3) \cdot a_i = 0 \qquad i = 1, 2, 3$$

$$\alpha_1 a_1 \cdot a_i + \alpha_2 a_2 \cdot a_i + \alpha_3 a_3 \cdot a_i = 0$$

$$\Rightarrow \quad \alpha_1 \bar{a}_1 \cdot \bar{a}_i + \alpha_2 \bar{a}_2 \cdot \bar{a}_i + \alpha_3 \bar{a}_3 \cdot \bar{a}_i = 0$$

$$(\alpha_1 \bar{a}_1 + \alpha_2 \bar{a}_2 + \alpha_3 \bar{a}_3) \cdot \bar{a}_i = 0 \qquad i = 1, 2, 3$$

$$\Rightarrow \quad \alpha_1 \bar{a}_1 + \alpha_2 \bar{a}_2 + \alpha_3 \bar{a}_3 = 0$$

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(5-6) [2015-10-22]
Thursday
15:00-17:00

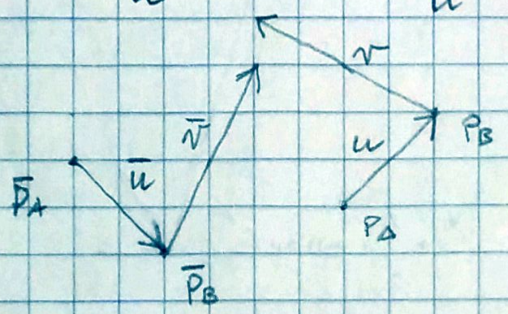
$$\bar{u} \overset{R}{\mapsto} u$$

$$\bar{v} \overset{R}{\mapsto} v$$

$$\alpha \bar{u} + \beta \bar{v} \overset{R}{\mapsto} \alpha u + \beta v$$

$$u = R\bar{u}$$

$$\underbrace{p_B - p_A}_{u} = R(\underbrace{\bar{p}_B - \bar{p}_A}_{\bar{u}})$$



$$p_B = p_A + u$$

$$p_B = p_A + R(\bar{p}_B - \bar{p}_A)$$

$$u \cdot v = R\bar{u} \cdot v = \bar{u} \cdot R^T v \qquad R^T v = \bar{v}$$

$$a_i \cdot a_j = R\bar{a}_i \cdot a_j = \bar{a}_i \cdot R^T a_j \Rightarrow R^T a_j = \bar{a}_j$$

$$R^T = R^{-1}$$

$$u \cdot v = R\bar{u} \cdot R\bar{v} = \bar{u} \cdot R^T R \bar{v}$$

$$a_i \cdot a_j = R\bar{a}_i \cdot R\bar{a}_j = \bar{a}_i \cdot R^T R \bar{a}_j$$

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$$a_i \cdot u = 0 \qquad i = 1, 2, 3 \quad \Rightarrow \quad u = 0$$

$$u = \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3$$

$$\alpha_1 a_1 \cdot u + \alpha_2 a_2 \cdot u + \alpha_3 a_3 \cdot u = 0$$

$$\Rightarrow \quad \underbrace{(\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3)}_{u} \cdot u = 0$$

$$u \cdot u = 0 \quad \Rightarrow \quad u = 0$$

$$\bar{a}_i \cdot \bar{a}_j = a_i \cdot a_j = R\,\bar{a}_i \cdot a_j = \bar{a}_i \cdot R^T a_j$$

$$\bar{a}_i \cdot (\bar{a}_j - R^T a_j) = 0$$

$$\Rightarrow \quad \bar{a}_j - R^T a_j = 0$$

$$R^T = R^{-1}$$

$$R^T R = I \,, \qquad R R^T = I$$

ORTHOGONAL TENSOR

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Uniqueness of the rotation tensor R

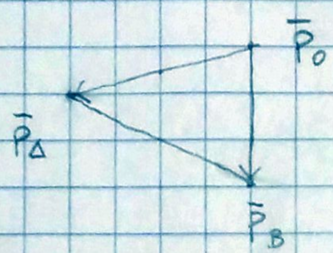
$$\begin{cases} p_B = p_A + R_A(\bar{p}_B - \bar{p}_A)\,; \quad p_O = p_A + R_A(\bar{p}_O - \bar{p}_A) \\ p_B = p_O + R_O(\bar{p}_B - \bar{p}_O) \end{cases}$$

$$0 = (p_A - p_O) + R_A(\bar{p}_B - \bar{p}_A) - R_O(\bar{p}_B - \bar{p}_O)$$

$$0 = \Big(p_A - \big(p_A + R_A(\bar{p}_O - \bar{p}_A)\big)\Big) + R_A(\bar{p}_B - \bar{p}_A) - R_O(\bar{p}_B - \bar{p}_O)$$

$$0 = -R_A(\bar{p}_O - \bar{p}_A) + R_A(\bar{p}_B - \bar{p}_A) - R_O(\bar{p}_B - \bar{p}_O)$$

$$0 = R_A(\bar{p}_A - \bar{p}_O) + R_A(\bar{p}_B - \bar{p}_A) - R_O(\bar{p}_B - \bar{p}_O)$$



$$0 = R_A(\bar{p}_B - \bar{p}_O) - R_O(\bar{p}_B - \bar{p}_O)$$

$$\forall \bar{p}_B$$

$$R_A \bar{a}_i = R_O \bar{a}_i \qquad i = 1, 2, 3$$

$$\Rightarrow R_A = R_O$$

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Composition of two rigid deformations

$$\phi^{(1)}(\bar{p}_A) = \phi^{(1)}(\bar{p}_o) + R^{(1)}(\bar{p}_A - \bar{p}_o)$$

$$\phi^{(2)}(\phi^{(1)}(\bar{p}_A)) = \phi^{(2)}(\underbrace{\phi^{(1)}(\bar{p}_o)}_{p_o^{(1)}}) + R^{(2)} R^{(1)} \underbrace{(\bar{p}_A - \bar{p}_o)}_{p_A^{(1)} - p_o^{(1)}}$$

$$\phi = \phi^{(2)} \circ \phi^{(1)}$$

$$\phi(\bar{p}_A) = \phi(\bar{p}_o) + R(\bar{p}_A - \bar{p}_o)$$

$$\Rightarrow \quad R = R^{(2)} R^{(1)}$$

$$R^T R = (R^{(2)} R^{(1)})^T (R^{(2)} R^{(1)}) = I$$

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(7-8) [2015-10-23]
Friday 09:11:00

[Review of yesterday's topics]

Affine deformations

$$\phi(\bar{p}_1) = \phi(\bar{p}_0) + F(\bar{p}_1 - \bar{p}_0)$$

(extension of the rigid deformation representation)

The matrix of a tensor, with the usual numbering of entries, and the corresponding numbering of components

$$A e_j = a_{ij} e_i$$

$$\begin{array}{cccc} & A e_1 & A e_2 & A e_3 \\ & \downarrow & \downarrow & \downarrow \\ [A] = & \multicolumn{3}{c}{\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}} \end{array}$$

The matrix of the transposed tensor in a basis made up of unitary vectors orthogonal to each other is the transposed matrix.

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An affine deformation transforms straight lines into straight lines

$$\bar{c}_A(h) = \bar{p}_A + h\bar{u}$$

$$\phi(\bar{c}_A(h)) = \phi(\bar{p}_A) + h u \qquad u = F\bar{u}$$

$$\bar{c}_B(h) = \bar{p}_B + h\bar{u}$$

$$\phi(\bar{c}_B(h)) = \phi(\bar{p}_B) + h u$$

Hence parallel lines are transformed into parallel lines, parallelograms are transformed into parallelograms, parallelepipeds are transformed into parallelepipeds.

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(9-10) [2015-10-29]
Thursday 15:00-17:00

Affine deformation

$$\phi(\bar{p}_\Delta) = \phi(\bar{p}_o) + F(\bar{p}_\Delta - \bar{p}_o)$$

$$p_\Delta = p_o + F(\bar{p}_\Delta - \bar{p}_o)$$

$\bar{u} \to u = F\bar{u}$

$\bar{v} \to v = F\bar{v}$

$$u \cdot v = F\bar{u} \cdot F\bar{v} = \bar{u} \cdot F^T F \bar{v}$$

$$F^T F \neq I \Rightarrow u \cdot v \neq \bar{u} \cdot \bar{v}$$

$$\|u\| \neq \|\bar{u}\|, \ \|v\| \neq \|\bar{v}\|$$

Volume function

$$\mathrm{vol}(u_1, u_2, u_3) \in \mathbb{R}$$

$$\begin{cases} \mathrm{vol}(u_1 + v, u_2, u_3) = \mathrm{vol}(u_1, u_2, u_3) + \mathrm{vol}(v, u_2, u_3) \\ \mathrm{vol}(\alpha u_1, u_2, u_3) = \alpha\, \mathrm{vol}(u_1, u_2, u_3) \\ \mathrm{vol}(u_2, u_1, u_3) = -\mathrm{vol}(u_1, u_2, u_3) \end{cases}$$

$$\Rightarrow \mathrm{vol}(u_1, u_1, u_3) = 0$$

$$\mathrm{vol}(\underbrace{u_1 - u_1}_{0}, u_2, u_3) = \mathrm{vol}(u_1, u_2, u_3) - \mathrm{vol}(u_1, u_2, u_3) = 0$$

$$\mathrm{vol}(\alpha_2 u_2 + \alpha_3 u_3, u_2, u_3) = 0$$

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Let $\{u_1, u_2, u_3\}$ be linearly independent vectors

Then $vol(u_1, u_2, u_3) = 0 \Rightarrow vol = 0$

just because for any three vectors $\{v_1, v_2, v_3\}$

$$vol(v_1, v_2, v_3) = (\dots) \, vol(u_1, u_2, u_3) = 0$$

by using $\{u_1, u_2, u_3\}$ as a basis.

Let n_1 a unit vector orthogonal to u_2 and u_3

$$n_1 \cdot u_2 = 0 \, , \quad n_1 \cdot u_3 = 0$$

then the vector

$$w_1 := u_1 - (u_1 \cdot n_1) \, n_1$$

is orthogonal to n_1:

$$w_1 \cdot n_1 = u_1 \cdot n_1 - (u_1 \cdot n_1) \underbrace{(n_1 \cdot n_1)}_{1} = 0$$

Hence $w_1 \in \mathrm{span}\{u_2, u_3\}$

Since $u_1 = w_1 + (u_1 \cdot n_1) \, n_1$ then

$$vol(u_1, u_2, u_3) = \cancel{vol(w_1, u_2, u_3)}$$
$$+ \underbrace{(u_1 \cdot n_1)}_{\substack{h_1 \\ \text{height}}} \underbrace{vol(n_1, u_2, u_3)}_{\substack{A_{\mathcal{F}_1} \\ \text{area}}}$$

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We can go further and set

$$w_2 := u_2 - (u_2 \cdot n_1)\, n_1 - (u_2 \cdot n_2)\, n_2$$

where n_2 is a unit vector orthogonal to both n_1 and u_3

$$n_2 \cdot n_1 = 0 \qquad n_2 \cdot u_3 = 0$$

It turns out

$$w_2 \cdot n_1 = u_2 \cdot n_1 - u_2 \cdot n_1 - (u_2 \cdot n_2)(n_2 \cdot n_1) = 0$$

$$w_2 \cdot n_2 = u_2 \cdot n_2 - (u_2 \cdot n_1)(n_1 \cdot n_2) - u_2 \cdot n_2 = 0$$

Hence $\quad w_2 \in \operatorname{span}\{u_3\}$

Since $\quad u_2 = w_2 + (u_2 \cdot n_1)\, n_1 + (u_2 \cdot n_2)\, n_2 \quad$ then

$$\begin{aligned} \operatorname{vol}(u_1, u_2, u_3) &= (u_1 \cdot n_1) \operatorname{vol}(n_1, u_2, u_3) \\ &= (u_1 \cdot n_1) \Big(\operatorname{vol}(n_1, w_2, u_3) \\ &\qquad + (u_2 \cdot n_1) \operatorname{vol}(n_1, n_1, u_3) \\ &\qquad + (u_2 \cdot n_2) \operatorname{vol}(n_1, n_2, u_3) \Big) \\ &= \underbrace{(u_1 \cdot n_1)}_{\substack{h_1 \\ \text{height}}} \underbrace{(u_2 \cdot n_2)}_{\substack{h_2 \\ \text{height}}} \underbrace{\operatorname{vol}(n_1, n_2, u_3)}_{\substack{\ell_3 \\ \text{length}}} \end{aligned}$$

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We could choose n_1 and n_2 in such a way that

$$h_1 = u_1 \cdot n_1 > 0$$

$$h_2 = u_2 \cdot n_2 > 0$$

Then we are left with the choice of the volume function such that

$$\mathrm{vol}\,(n_1, n_2, u_3) = \ell_3 = \|u_3\|$$

or

$$\mathrm{vol}\,(n_1, n_2, u_3) = -\,\ell_3 = -\|u_3\|$$

This is a procedure for relating a volume function to distances and lengths.

Because a rigid deformation leaves any distance or length unchanged it should leave the volume unchanged as well.
But that is not a consequence of leaving the distances unchanged and it has to be stated as an additional assumption.

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