Determinant of a Tensor

$$\frac{vol(F\bar{u}_1, F\bar{u}_2, F\bar{u}_3)}{vol(\bar{u}_1, \bar{u}_2, \bar{u}_3)}$$

$$\frac{vol(F(\alpha_{11}\bar{u}_1 + \alpha_{21}\bar{u}_2 + \alpha_{31}\bar{u}_3, F\bar{u}_2, F\bar{u}_3)}{vol(\alpha_{11}\bar{u}_1 + \alpha_{21}\bar{u}_2 + \alpha_{31}\bar{u}_3, \bar{u}_2, \bar{u}_3}$$

$$= \frac{\alpha_{11}\ vol(F\bar{u}_1, F\bar{u}_2, F\bar{u}_3)}{\alpha_{11}\ vol(\bar{u}_1, \bar{u}_2, \bar{u}_3)} = \frac{vol(F\bar{u}_1, F\bar{u}_2, F\bar{u}_3)}{vol(\bar{u}_1, \bar{u}_2, \bar{u}_3)}$$

[...]

$$\frac{vol(F\bar{v}_1, F\bar{v}_2, F\bar{v}_3)}{vol(\bar{v}_1, \bar{v}_2, \bar{v}_3)} = \frac{vol(F\bar{u}_1, F\bar{u}_2, F\bar{u}_3)}{vol(\bar{u}_1, \bar{u}_2, \bar{u}_3)}$$

Definition

$$\det F = \frac{vol(Fe_1, Fe_2, Fe_3)}{vol(e_1, e_2, e_3)}$$

where $\{e_1, e_2, e_3\}$ is any basis

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$$\operatorname{tr} A = \frac{\operatorname{vol}(Ae_1, e_2, e_3) + \operatorname{vol}(e_1, Ae_2, e_3) + \operatorname{vol}(e_1, e_2, Ae_3)}{\operatorname{vol}(e_1, e_2, e_3)}$$

As an application, let us compute in a motion

$$\frac{d}{dt}(\det F)$$

$$\begin{aligned}\frac{d}{dt}\operatorname{vol}(F\bar{u}_1, F\bar{u}_2, F\bar{u}_3) &= \operatorname{vol}(\dot{F}\bar{u}_1, F\bar{u}_2, F\bar{u}_3)\\ &+ \operatorname{vol}(F\bar{u}_1, \dot{F}\bar{u}_2, F\bar{u}_3)\\ &+ \operatorname{vol}(F\bar{u}_1, F\bar{u}_2, \dot{F}\bar{u}_3)\end{aligned}$$

$$\begin{aligned}&= \operatorname{vol}(\dot{F}F^{-1}u_1, u_2, u_3)\\ &+ \operatorname{vol}(u_1, \dot{F}F^{-1}u_2, u_3)\\ &+ \operatorname{vol}(u_1, u_2, \dot{F}F^{-1}u_3)\end{aligned}$$

$$\begin{aligned}&= \operatorname{tr}(\dot{F}F^{-1})\operatorname{vol}(u_1, u_2, u_3)\\ &= \operatorname{tr}(\dot{F}F^{-1})\det F \operatorname{vol}(\bar{u}_1, \bar{u}_2, \bar{u}_3)\end{aligned}$$

$$\Rightarrow \frac{d}{dt}(\det F) = \operatorname{tr}(\dot{F}F^{-1})\det F$$

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$$p_A(t) = p_0(t) + F(t)(\bar{p}_A - \bar{p}_0)$$

velocity

$$v_A(t) = \lim_{\Delta t \to 0} \frac{p_A(t+\Delta t) - p_A(t)}{\Delta t} = \dot{p}_A(t)$$

$$\dot{p}_A(t) = \dot{p}_0(t) + \dot{F}(t)(\bar{p}_A - \bar{p}_0)$$

$$\dot{p}_A(t) = \dot{p}_0(t) + \dot{F}(t)F(t)^{-1}(p_A(t) - p_0(t))$$

$$v(p_A(t)) = v(p_0(t)) + \underbrace{\dot{F}(t)F(t)^{-1}}_{\text{velocity gradient}}(p_A(t) - p_0(t))$$

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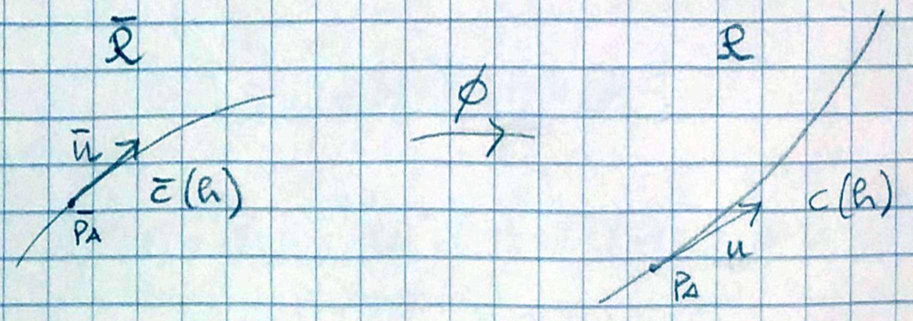
(11-12) Friday [2015-10-30] 9:00-11:00

We assume that in any affine deformation

$$\det F > 0$$

For a generic deformation

$$\bar{c}(h) \overset{\phi}{\longmapsto} c(h)$$



$$\bar{p}_A = \bar{c}(0) \qquad\qquad p_A = c(0)$$

$$\bar{c}'(0) = \lim_{h \to 0} \frac{\bar{c}(h) - c(0)}{h} \qquad\qquad \bar{c}'(0) = \lim_{h \to 0} \frac{c(h) - c(0)}{h}$$

tangent vectors

Let us choose 3 curves on the reference shape such that their tangent vectors at $\bar{p}_A$ are 3 linearly independent vectors $\{\bar{u}_1, \bar{u}_2, \bar{u}_3\}$.

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The tangent vectors to the corresponding curves on $\mathcal{B}$ will be $\{u_1, u_2, u_3\}$

It can be proved that the function

$$\bar{u}_1 \mapsto u_1$$

$$\bar{u}_2 \mapsto u_2$$

$$\bar{u}_3 \mapsto u_3$$

is a linear transformation (i.e. a tensor)

$$F(\bar{p}_A) : \mathcal{V} \to \mathcal{V}$$

which is called the "deformation gradient".

In general

$$F(\bar{p}_0) \neq F(\bar{p}_A)$$

We assume that in any deformation

$$\det F(x) > 0 \qquad \forall x \in \bar{\mathcal{B}}$$

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$$u = F(\bar{p}_A)\,\bar{u}$$

$$u = \lim_{h\to 0} \frac{c(h)-c(0)}{h} \qquad \bar{u} = \lim_{h\to 0} \frac{\bar{c}(h)-\bar{c}(0)}{h}$$

Let us set $o(h) := c(h) - \big(c(0) + F(\bar{p}_A)(\bar{c}(h) - \bar{c}(0))\big)$

where $\bar{c}(0) = \bar{p}_A$ and $c(0) = p_A$

We get

$$\frac{o(h)}{h} = \frac{c(h)-c(0)}{h} - F(\bar{p}_A)\,\frac{\bar{c}(h)-\bar{c}(0)}{h}$$

$$\lim_{h\to 0} \frac{o(h)}{h} = u - F(\bar{p}_A)\,\bar{u} = 0$$

Hence we can write

$$c(h) = \underbrace{c(0) + F(\bar{c}(0))\big(\bar{c}(h)-\bar{c}(0)\big)}_{\text{affine deformation}} + \underbrace{o(h)}_{\text{remainder}}$$

where $\|o(h)\|$ approaches zero faster than h

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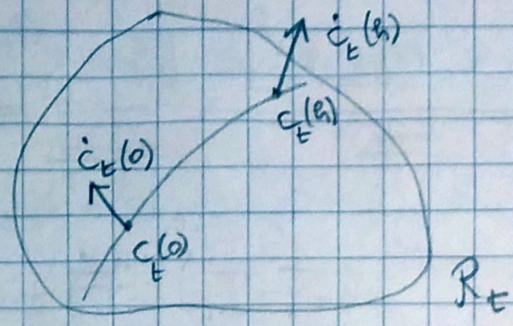
(13-14) Thursday [2015-11-05]
14:30 - 16:30

Velocity gradient ∇v

v velocity field on $\mathcal{R}_t$



$$c_t(h) = o + \left(c_{1t}(h)\, e_1 + c_{2t}(h)\, e_2 + c_{3t}(h)\, e_3 \right)$$

$$c'_t(0) = \lim_{h \to 0} \frac{c_t(h) - c_t(0)}{h} = c'_{1t}(0)\, e_1 + c'_{2t}(0)\, e_2 + c'_{3t}(0)\, e_3$$

$$\begin{aligned} v(c_t(h)) = {} & v_1\left(c_{1t}(h), c_{2t}(h), c_{3t}(h)\right) e_1 \\ & + v_2\left(c_{1t}(h), c_{2t}(h), c_{3t}(h)\right) e_2 \\ & + v_3\left(c_{1t}(h), c_{2t}(h), c_{3t}(h)\right) e_3 \end{aligned}$$

velocity field description at time t

with $$\begin{aligned} v(c_t(h)) &= \dot{c}_t(h) \\ &= \dot{c}_{1t}(h)\, e_1 + \dot{c}_{2t}(h)\, e_2 + \dot{c}_{3t}(h)\, e_3 \end{aligned}$$

$$\Rightarrow \quad \begin{aligned} v_1\left(c_{1t}(h), c_{2t}(h), c_{3t}(h)\right) &= \dot{c}_{1t}(h) \\ v_2\left(c_{1t}(h), c_{2t}(h), c_{3t}(h)\right) &= \dot{c}_{2t}(h) \\ v_3\left(c_{1t}(h), c_{2t}(h), c_{3t}(h)\right) &= \dot{c}_{3t}(h) \end{aligned}$$

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$$\lim_{h\to 0} \frac{v(c_t(h)) - v(c_t(0))}{h} =$$

$$= \left(v_{1,1}\, c'_{1t}(0) + v_{1,2}\, c'_{2t}(0) + v_{1,3}\, c'_{3t}(0) \right) e_1$$

$$+ \left(v_{2,1}\, c'_{1t}(0) + v_{2,2}\, c'_{2t}(0) + v_{2,3}\, c'_{3t}(0) \right) e_2$$

$$+ \left(v_{3,1}\, c'_{1t}(0) + v_{2,2}\, c'_{2t}(0) + v_{3,3}\, c'_{3t}(0) \right) e_3$$

with $$v_{i,j} := \frac{\partial}{\partial c_{jt}} v_i \Big|_{h=0}$$

$$\begin{pmatrix} v_{1,1} & v_{1,2} & v_{1,3} \\ v_{2,1} & v_{2,2} & v_{2,3} \\ v_{3,1} & v_{3,2} & v_{3,3} \end{pmatrix} \begin{pmatrix} c'_{1t}(0) \\ c'_{2t}(0) \\ c'_{3t}(0) \end{pmatrix}$$

this matrix product, delivering the three components of the limit vector above, reveals that the derivative of the velocity field along any curve c_t depends linearly on the tangent vector c'_t:

$$\lim_{h\to 0} \frac{v(c_t(h)) - v(c_t(0))}{h} = \nabla v \Big|_{c_t(0)} c'_t(0)$$

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Let us consider again the velocity field v at time t and set

$$o(h) := v(c_t(h)) - \left(v(c_t(0)) + \nabla v\Big|_{c_t(0)} \big(c_t(h) - c_t(0)\big) \right)$$

Then

$$\lim_{h\to 0} \frac{o(h)}{h} = \lim_{h\to 0} \frac{v(c_t(h)) - v(c_t(0))}{h} - \lim_{h\to 0} \nabla v\Big|_{c_t(0)} \frac{c_t(h) - c_t(0)}{h}$$

$$= \lim_{h\to 0} \frac{v(c_t(h)) - v(c_t(0))}{h} - \nabla v\Big|_{c_t(0)} c_t'(0) = 0$$

Hence it turns out that

$$v(c_t(h)) = v(c_t(0)) + \nabla v\Big|_{c_t(0)} \big(c_t(h) - c_t(0)\big) + o(h)$$

where $\|o(h)\|$ approaches 0 faster than h.

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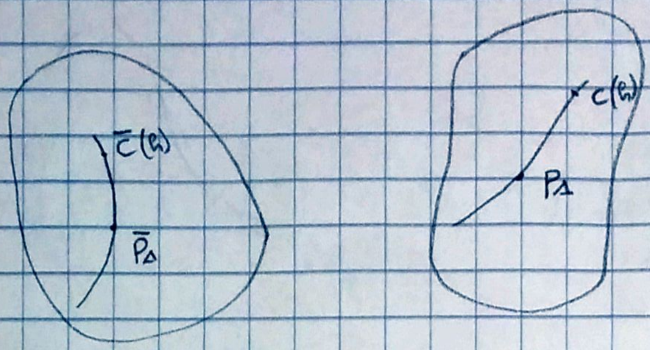
Referential description of the velocity field



$\bar{\mathcal{R}}$ reference shape

$\mathcal{R}$ current shape

Let us define

$$\bar{v}(\bar{p}_\Delta) = v(p_\Delta)$$

Hence

$$\lim_{h\to 0} \frac{\bar{v}(\bar{c}(h)) - \bar{v}(\bar{c}(0))}{h} = \lim_{h\to 0} \frac{v(c(h)) - v(c(0))}{h}$$

$$\nabla\bar{v}\,\bar{c}' = \nabla v\, c'$$

$$c' = F\bar{c}' \Rightarrow \qquad \nabla\bar{v} = \nabla v\, F$$

$$\nabla v = \nabla\bar{v}\, F^{-1}$$

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$$c_t(h) = c_t(0) + F_t\Big|_{\bar{c}(0)} \big(\bar{c}(h) - \bar{c}(0)\big) + o(h)$$

$$\big(c_t(h) - c_t(0)\big) - o_t(h) = F_t\big(\bar{c}(h) - \bar{c}(0)\big)$$

$$c'_t = F_t\, \bar{c}'$$

$$\dot{c}'_t = \dot{F}_t\, \bar{c}' = \dot{F}_t F_t^{-1} c'_t$$

$$\dot{c}'_t(0) = \lim_{h \to 0} \frac{\dot{c}_t(h) - \dot{c}_t(0)}{h} = \lim_{h \to 0} \frac{v(c_t(h)) - v(c_t(0))}{h}$$

$$= \nabla v\Big|_{c_t(0)} c'_t(0)$$

In short

$$\dot{c}'_t = \nabla v\; c'_t$$

Hence

$$\nabla v = \dot{F} F^{-1}$$

As a consequence

$$\nabla \bar{v} = \dot{F}$$

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Rigid motion velocity field

$$p_A(t) = p_0(t) + R(t)(\bar{p}_A - \bar{p}_0)$$

$$\dot{p}_A = \dot{p}_0 + \dot{R}(\bar{p}_A - \bar{p}_0)$$

$$\dot{p}_A = \dot{p}_0 + \dot{R}R^T(p_A - p_0)$$

$$v(c(h)) = v(c(0)) + \dot{R}R^T(c(h) - c(0))$$

$$\lim_{h\to 0} \frac{v(c(h)) - v(c(0))}{h} = \dot{R}R^T \lim_{h\to 0} \frac{c(h) - c(0)}{h}$$

$$\Rightarrow \qquad \nabla v = \dot{R}R^T$$

$W := \dot{R}R^T$ SPIN TENSOR

$$R^{-1} = R^T \quad \Rightarrow \quad RR^T = I$$

$$\dot{R}R^T + R\dot{R}^T = 0$$

$$W + W^T = 0$$

$$W^T = -W$$

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Axial vector of a skewsymmetric tensor

$$\frac{1}{2}\left(\nabla v - \nabla v^T\right) u = \omega \times u$$

$$\begin{pmatrix} 0 & \frac{1}{2}(v_{1,2}-v_{2,1}) & \frac{1}{2}(v_{1,3}-v_{3,1}) \\ \frac{1}{2}(v_{2,1}-v_{1,2}) & 0 & \frac{1}{2}(v_{2,3}-v_{3,2}) \\ \frac{1}{2}(v_{3,1}-v_{1,3}) & \frac{1}{2}(v_{3,2}-v_{2,3}) & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

$$\omega_3 := -\frac{1}{2}\left(v_{1,2} - v_{2,1}\right)$$

$$\omega_2 := \frac{1}{2}\left(v_{1,3} - v_{3,1}\right)$$

$$\omega_1 := -\frac{1}{2}\left(v_{2,3} - v_{3,2}\right)$$

$$\begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} -\omega_3 u_2 + \omega_2 u_3 \\ \omega_3 u_1 - \omega_1 u_3 \\ -\omega_2 u_1 + \omega_1 u_2 \end{pmatrix}$$

$$\left(\omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3\right) \times \left(u_1 e_1 + u_2 e_2 + u_3 e_3\right)$$

$$= \omega_1 u_2 e_3 - \omega_1 u_3 e_2 - \omega_2 u_1 e_3 + \omega_2 u_3 e_1 + \omega_3 u_1 e_2 - \omega_3 u_2 e_1$$

$$= \left(-\omega_3 u_2 + \omega_2 u_3\right) e_1 + \left(\omega_3 u_1 - \omega_1 u_3\right) e_2 + \left(-\omega_2 u_1 + \omega_1 u_2\right) e_3$$

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curl v

$L := \nabla v$

$L = D + W$

$D = \operatorname{sym} L = \frac{1}{2}(\nabla v + \nabla v^T) \qquad W = \operatorname{skw} L = \frac{1}{2}(\nabla v - \nabla v^T)$

($W = 0$ IRROTATIONAL FLOW)

$\frac{1}{2} \operatorname{curl} v = \omega \qquad Wu = \omega \times u \quad \forall u$

for any scalar field α, the matrix of $\nabla(\nabla\alpha)$ is symmetric

$$[\nabla(\nabla\alpha)] = \nabla \begin{pmatrix} \alpha_{,1} \\ \alpha_{,2} \\ \alpha_{,3} \end{pmatrix} = \begin{pmatrix} \alpha_{,11} & \alpha_{,12} & \alpha_{,13} \\ \alpha_{,21} & \alpha_{,22} & \alpha_{,23} \\ \alpha_{,31} & \alpha_{,32} & \alpha_{,33} \end{pmatrix}$$

$\Rightarrow \operatorname{curl} \nabla\alpha = 0$

A scalar field φ is called a potential for the velocity field v if $\qquad v = -\nabla\varphi$

Because $\operatorname{curl} \nabla\varphi = 0$, for a potential to exist the velocity field has to be irrotational

$$\operatorname{curl} v = 0$$

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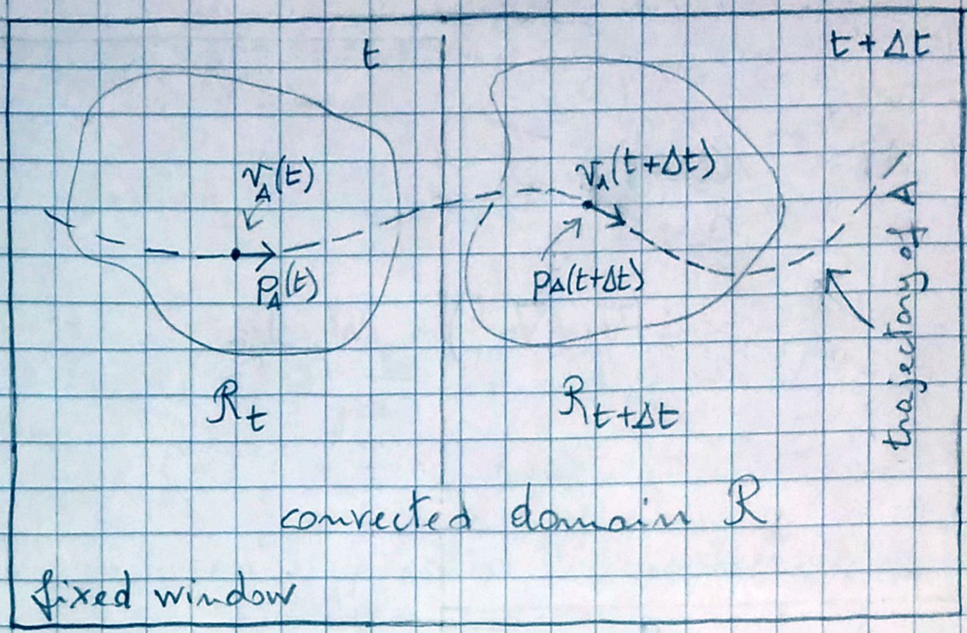
(15–16) Friday [2015-11-06]



$p_A(t)$, $v_A(t) = \dot{p}_A(t)$, $a_A(t) = \ddot{p}_A(t)$ acceleration

Spatial description (over the fixed window)

$$v_t(x) = \dot{p}_A(t) \quad \text{with } x = p_A(t)$$

$$a_t(x) = \ddot{p}_A(t) \quad \text{with } x = p_A(t)$$

$$\ddot{p}_A(t) = \lim_{\Delta t \to 0} \frac{\dot{p}_A(t+\Delta t) - \dot{p}_A(t)}{\Delta t}$$

$$\dot{p}_A(t+\Delta t) = v_{t+\Delta t}(y) , \qquad y = p_A(t+\Delta t)$$

$$\lim_{\Delta t \to 0} \frac{v_{t+\Delta t}(p_A(t+\Delta t)) - v_t(p_A(t))}{\Delta t} = a_t(p_A(t))$$

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$$a(p_A(t)) = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\Big(v_{t+\Delta t}(p_A(t+\Delta t)) - v_t(p_A(t))$$

$$- v_{t+\Delta t}(p_A(t)) + v_{t+\Delta t}(p_A(t))\Big)$$

$$a(p_A(t)) = (\nabla v_t(x))\,\dot{p}_A(t) + \frac{\partial}{\partial t} v_t(x), \qquad x = p_A(t)$$

tangent vector to the trajectory

As a time dependent vector field over a fixed window it can be given the more descriptive form

$$a(x,t) = \nabla v\Big|_{(x,t)} v(x,t) + \frac{\partial}{\partial t} v(x,t)$$

In short

$$a = (\nabla v)\, v + v'$$

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Mass conservation

$$\rho_0 V_{\bar{\mathcal{R}}} = \rho V_{\mathcal{R}}$$

$$\rho V_{\mathcal{R}} = \rho V_{\bar{\mathcal{R}}} \det F$$

$$\frac{d}{dt} \rho V_{\mathcal{R}} = 0$$

$$V_{\bar{\mathcal{R}}} \left(\dot{\rho} \det F + \rho \frac{d}{dt} \det F \right) = 0$$

$$\dot{\rho} \det F + \rho \det F \operatorname{tr} \dot{F} F^{-1} = 0$$

$$\dot{\rho} + \rho \operatorname{div} v = 0 \qquad (\operatorname{div} v := \operatorname{tr} \nabla v)$$

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