(17-18) Thursday [2015-11-12] 14:30-16:30

Polar decomposition of the deformation gradient

Theorem: any tensor $F : \mathcal{V} \to \mathcal{V}$ with $\det F > 0$, can be decomposed into the product of a rotation R and a stretch U

$$F = RU$$

where $R \in Orth^+$ is a proper orthogonal tensor

$$R^T R = I \qquad \det R = 1$$

and $U \in Psym$ is a positive definite symmetric tensor:

$$U^T = U \qquad \begin{cases} Ua \cdot a > 0 & \forall a \neq 0 \\ Ua \cdot a = 0 \Leftrightarrow a = 0 \end{cases}$$

Proof:

$$C := F^T F \qquad \text{Cauchy-Green tensor}$$

$$C^T = C, \quad Cu \cdot u = F^T F u \cdot u = Fu \cdot Fu > 0 \quad \forall u \neq 0$$

$$\det F > 0 \Rightarrow \{ Fu = 0 \Leftrightarrow u = 0 \}$$

$C \in Psym \Rightarrow$ its eigenvalues are positive numbers and its eigenspaces are orthogonal to each other:

$$C = \lambda_1^2 a_1 \otimes a_1 + \lambda_2^2 a_2 \otimes a_2 + \lambda_3^2 a_3 \otimes a_3$$

with $\{a_1, a_2, a_3\}$ unit eigenvectors orthogonal to each other

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$(u \otimes v)\, w = u\,(v \cdot w)$ Tensor product

$(a_1 \otimes a_1)\, a_1 = a_1$

$(a_1 \otimes a_1)\, a_2 = a_1 (a_1 \cdot a_2) = 0$ [...]

$C\, a_1 = \lambda_1^2\, a_1$ $\lambda_1^2 > 0$

$C\, a_2 = \lambda_2^2\, a_2$ $\lambda_2^2 > 0$

$C\, a_3 = \lambda_3^2\, a_3$ $\lambda_3^2 > 0$

Matrix of $(a_1 \otimes a_1)$

$(a_1 \otimes a_1)\, e_1 = a_1 (a_1 \cdot e_1)$

$(a_1 \otimes a_1)\, e_2 = a_1 (a_1 \cdot e_2)$

$(a_1 \otimes a_1)\, e_3 = a_1 (a_1 \cdot e_3)$

$a_1 = a_{11}\, e_1 + a_{21}\, e_2 + a_{31}\, e_3$ $e_i \cdot e_j = \delta_{ij}$

$$[a_1 \otimes a_1] = \begin{pmatrix} a_{11}^2 & a_{11}\, a_{21} & a_{11}\, a_{31} \\ a_{21}\, a_{11} & a_{21}^2 & a_{21}\, a_{31} \\ a_{31}\, a_{11} & a_{31}\, a_{21} & a_{31}^2 \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} \begin{pmatrix} a_{11} & a_{21} & a_{31} \end{pmatrix}$$

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$$C = \lambda_1^2\, a_1 \otimes a_1 + \lambda_2^2\, a_2 \otimes a_2 + \lambda_3^2\, a_3 \otimes a_3$$

$$[C] = \lambda_1^2 \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} \begin{pmatrix} a_{11} & a_{21} & a_{31} \end{pmatrix}$$

$$+ \lambda_2^2 \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} \begin{pmatrix} a_{12} & a_{22} & a_{32} \end{pmatrix}$$

$$+ \lambda_3^2 \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix} \begin{pmatrix} a_{13} & a_{23} & a_{33} \end{pmatrix}$$

$$[C] = [A] \begin{pmatrix} \lambda_1^2 & & \\ & \lambda_2^2 & \\ & & \lambda_3^2 \end{pmatrix} [A]^T$$

$$U := \lambda_1\, a_1 \otimes a_1 + \lambda_2\, a_2 \otimes a_2 + \lambda_3\, a_3 \otimes a_3 \qquad \lambda_i > 0$$

$$\Rightarrow U^2 = U\,U = (\qquad\qquad)(\qquad\qquad) = C$$

$$(a_1 \otimes a_1)(a_1 \otimes a_1)\, u = (a_1 \otimes a_1)\, a_1 (a_1 \cdot u) = a_1 (a_1 \cdot u)$$
$$= (a_1 \otimes a_1)\, u$$

$$(a_1 \otimes a_1)(a_2 \otimes a_2)\, u = (a_1 \otimes a_1)\, a_2 (a_2 \cdot u) = 0$$

[...]

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$$U^{-1} = \frac{1}{\lambda_1} a_1 \otimes a_1 + \frac{1}{\lambda_2} a_2 \otimes a_2 + \frac{1}{\lambda_3} a_3 \otimes a_3$$

$$U U^{-1} = a_1 \otimes a_1 + a_2 \otimes a_2 + a_3 \otimes a_3$$

$$(U U^{-1}) u = (a_1 \cdot u) a_1 + (a_2 \cdot u) a_2 + (a_3 \cdot u) a_3 = u$$

$$U U^{-1} = I$$

$$R = F U^{-1} = \frac{1}{\lambda_1} F (a_1 \otimes a_1) + \dots$$

$$R^T R = \frac{1}{\lambda_1^2} (a_1 \otimes a_1) \underbrace{F^T F}_{C} (a_1 \otimes a_1) + \dots \qquad = I$$

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(19-20) Friday [2015-11-13] 9:00-11:00

POLAR DECOMPOSITION

NUMERICAL PROCEDURE

(assignment)

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POLAR DECOMPOSITION

NUMERICAL PROCEDURE

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(21-22) Thursday [2015-11-19]

FORCE AND POWER

By force distribution $\mathcal{F}$ applied to a body $\mathcal{B}$ in its current shape $\mathcal{R}$ we mean a linear real function over the vector space of the test velocity fields:

we assume that for any two test velocity fields v_1 and v_2, the vector field $(v_1 + v_2)$ such that

$$(v_1 + v_2)(x) = v_1(x) + v_2(x)$$

is again a test velocity field, as it is the vector field αv such that

$$(\alpha v)(x) = \alpha(x)\, v(x)$$

for any scalar field α and any test velocity field v.

The test velocity fields v are assumed to be continuous on $\mathcal{R}$ and differentiable with continuous gradient tensor fields ∇v.

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The force distribution $\mathfrak{F}$ is usually given the following representation

$$\mathfrak{F}(v) = \int_{\mathcal{R}} b(x)\cdot v(x)\, dV + \int_{\partial\mathcal{R}} t(x)\cdot v(x)\, dA \qquad \forall v$$

where b is called the bulk force density per unit current volume and t is called just the traction, which is a force density per unit current area.

We can supplement the expression above with a singular distribution made up of terms like

$$f_A \cdot v(p_A)$$

where f_A is called a force applied to the body point A taking the position $p_A \in \mathcal{R}$.

The real value $\mathfrak{F}(v)$ is called the (total) power, while $b(x)\cdot v(x)$ or $t(x)\cdot v(x)$ are called power density at x per unit current volume or per unit current area respectively.

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Tensor product and scalar product of tensors

$$(u \otimes v)e_j = u(v \cdot e_j) \quad \forall e_j \quad \text{definition for } \otimes$$

$$u \cdot v = \operatorname{tr}(u \otimes v) \quad \text{because of the definition of trace}$$

$$\begin{aligned} \operatorname{vol}((u\otimes v)e_1, e_2, e_3) &= (v \cdot e_1)\operatorname{vol}(u, e_2, e_3) \\ &= (v \cdot e_1)\operatorname{vol}(u_1 e_1, e_2, e_3) \\ &= (v \cdot e_1)u_1 \operatorname{vol}(e_1, e_2, e_3) \end{aligned}$$

$$\operatorname{vol}(e_1, (u\otimes v)e_2, e_3) = (v \cdot e_2)u_2 \operatorname{vol}(e_1, e_2, e_3)$$

$$\operatorname{vol}(e_1, e_2, (u\otimes v)e_3) = (v \cdot e_3)u_3 \operatorname{vol}(e_1, e_2, e_3)$$

$$\operatorname{tr}(u\otimes v) = (v\cdot e_1)u_1 + (v\cdot e_2)u_2 + (v \cdot e_3)u_3 = v \cdot u$$

$$f \cdot Lu = \operatorname{tr}(f \otimes Lu) = \operatorname{tr}((f\otimes u)L^T) \qquad (*)$$

$$f \cdot Lu = L^T f \cdot u = \operatorname{tr}(L^T f \otimes u) = \operatorname{tr}(L^T(f\otimes u))$$

$$f \cdot Lu = u \cdot L^T f = \operatorname{tr}(u \otimes L^T f) = \operatorname{tr}((u\otimes f)L)$$

Hence $f \cdot Lu = (f\otimes u)\cdot L$

by defining

$$M \cdot L = \operatorname{tr}(ML^T) = \operatorname{tr}(L^T M) = \operatorname{tr}(M^T L)$$

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$$(f \otimes u)\, e_i \cdot e_j = (u \cdot e_i)(f \cdot e_j) = (u \otimes f)\, e_j \cdot e_i$$

$$A e_i \cdot e_j = e_i \cdot A^T e_j = A^T e_j \cdot e_i$$

$$\Rightarrow \qquad u \otimes f = (f \otimes u)^T$$

(*)

$$(f \otimes Lu)\, e_i = f\,(Lu \cdot e_i) = f\,(u \cdot L^T e_i)$$

$$= (f \otimes u)\, L^T e_i$$

$$(L^T f \otimes u)\, e_i = L^T f\,(u \cdot e_i) = L^T (f \otimes u)\, e_i$$

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(23-24) Friday [2015-11-20]

Rigid test velocity field

$$v(x) = v_o + W(x - p_o)$$

$$t(x) \cdot v(x) = t(x) \cdot v_o + t(x) \cdot W(x - x_o)$$

$$t \cdot Wu = t \otimes u \cdot W = M \cdot W$$

$$t \cdot Wu = u \cdot W^T t = (u \otimes t) \cdot W^T = M^T \cdot W^T$$

$$M \cdot W = M^T \cdot W^T = -M^T \cdot W$$

$$(M + M^T) \cdot W = 0$$

$$\operatorname{sym} M \cdot W = 0$$

How to compute the moment tensor for a few exemplary force distributions.

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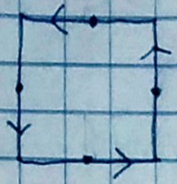
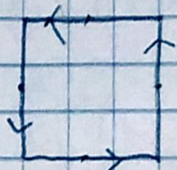




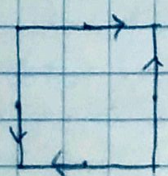
rigid test velocity field

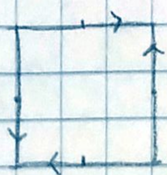
force distribution

W

M

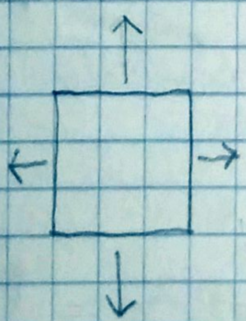
Affine test velocity fields

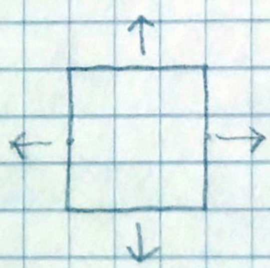




test velocity field

force distribution





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(25-26) Thursday [2015-11-26]

About the assignment
(and its time extension)

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$$U = \lambda_1\, a_1 \otimes a_1 + \lambda_2\, a_2 \otimes a_2 + \lambda_3\, a_3 \otimes a_3$$

$$\begin{aligned} R = {} & \cos\theta\, e_1 \otimes e_1 - \sin\theta\, e_1 \otimes e_2 \\ & + \sin\theta\, e_2 \otimes e_1 + \cos\theta\, e_2 \otimes e_2 \\ & + e_3 \otimes e_3 \end{aligned}$$

$$\theta = \omega t = 2\pi\nu\, t = \frac{2\pi}{T}\, t$$

$$\lambda_1 = 1 + \varepsilon_1 \sin(2\pi\nu t) \qquad 0 \le \varepsilon_1 < 1$$

$$\lambda_2 = 1 + \varepsilon_2 \sin(2\pi\nu t) \qquad 0 \le \varepsilon_2 < 1$$

$$\begin{aligned} \dot{R} = \dot{\theta}\,\big(& -\sin\theta\, e_1 \otimes e_1 - \cos\theta\, e_1 \otimes e_2 \\ & + \cos\theta\, e_2 \otimes e_1 - \sin\theta\, e_2 \otimes e_2 \big) \end{aligned}$$

$$\dot{R} R^T = \dot{\theta}\,\big(-e_1 \otimes e_2 + e_2 \otimes e_1 \big)$$

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