(27-28) Friday [2015-11-27]

Cauchy continuum

$$\mathcal{F}^{ext}(v) = \int_{\mathcal{R}} b \cdot v \, dV + \int_{\partial\mathcal{R}} t \cdot v \, dA$$

$$\mathcal{F}^{int}(v) = -\int_{\mathcal{R}} z \cdot v \, dV - \int_{\mathcal{R}} T \cdot \nabla v \, dV$$

material frame indifference (objectivity)

$$z \cdot v + T \cdot \nabla v = 0$$

for any rigid test velocity field

$$v(x) = v_o + W(x - p_o)$$

$$z \cdot v_o + z \otimes (x - p_o) \cdot W + T \cdot W = 0 \quad \forall v_o, \forall W$$

$$\Rightarrow \begin{cases} z = 0 \\ \operatorname{skw} T = 0 \end{cases}$$

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Balance of forces

$$\mathcal{F}^{ext}(v) + \mathcal{F}^{int}(v) = 0 \qquad \forall v$$

$$\int_{\mathcal{R}} b\cdot v\, dV + \int_{\partial\mathcal{R}} t\cdot v\, dA = \int_{\mathcal{R}} T\cdot\nabla v\, dV$$

Defining the divergence of a tensor field T by

$$\mathrm{div}\,T\cdot v = \mathrm{div}(T^T v) - T\cdot\nabla v \qquad \forall v$$

$$\int_{\mathcal{R}} b\cdot v\, dV + \int_{\partial\mathcal{R}} t\cdot v\, dA + \int_{\mathcal{R}} \mathrm{div}\,T\cdot v\, dV - \int_{\mathcal{R}} \mathrm{div}(T^T v)\, dV = 0$$

divergence theorem

with n external unit normal vector field

$$\int_{\mathcal{R}} \mathrm{div}(T^T v)\, dV = \int_{\partial\mathcal{R}} T^T v\cdot n\, dA$$

$$\int_{\mathcal{R}} (b + \mathrm{div}\,T)\cdot v\, dV + \int_{\partial\mathcal{R}} (t - Tn)\cdot v\, dA = 0 \qquad \forall v$$

$$\Rightarrow \quad \begin{cases} \mathrm{div}\,T + b = 0 \\ Tn = t \end{cases}$$

Cauchy balance equations

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(29-30) Thursday [2015-12-03]

$$[\mathrm{T}] = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$ matrix of T in an orthonormal basis e_i

$$\operatorname{div}\mathrm{T}\cdot e_1 = \operatorname{div}(\mathrm{T}^T e_1) = \operatorname{tr}\nabla(\mathrm{T}^T e_1)$$

$$\mathrm{T}^T e_1 = \sigma_{11}\, e_1 + \sigma_{12}\, e_2 + \sigma_{13}\, \sigma_3$$

$$[\nabla(\mathrm{T}^T e_1)] = \begin{pmatrix} \sigma_{11,1} & \sigma_{11,2} & \sigma_{11,3} \\ \sigma_{12,1} & \sigma_{12,2} & \sigma_{12,3} \\ \sigma_{13,1} & \sigma_{13,2} & \sigma_{13,3} \end{pmatrix}$$

$$\operatorname{div}\mathrm{T}\cdot e_1 = \sigma_{11,1} + \sigma_{12,2} + \sigma_{13,3}$$

$$\mathrm{T}^T e_2 = \sigma_{21}\, e_1 + \sigma_{22}\, e_2 + \sigma_{23}\, e_3$$

$$\mathrm{T}^T e_3 = \sigma_{31}\, e_1 + \sigma_{32}\, e_2 + \sigma_{33}\, e_3$$

$$\operatorname{div}\mathrm{T}\cdot e_2 = \sigma_{21,1} + \sigma_{22,2} + \sigma_{23,3}$$

$$\operatorname{div}\mathrm{T}\cdot e_3 = \sigma_{31,1} + \sigma_{32,2} + \sigma_{33,3}$$

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Deviatoric and spherical part of a tensor

For any tensor L, if we define

$$\mathrm{sph}\, L := \frac{1}{3}(\mathrm{tr}\, L) I \qquad \text{SPHERICAL PART}$$

and

$$\mathrm{dev}\, L := L - \frac{1}{3}(\mathrm{tr}\, L) I \qquad \text{DEVIATORIC PART}$$

we get

$$\mathrm{dev}\, L + \mathrm{sph}\, L = L \quad , \quad \mathrm{tr}(\mathrm{dev}\, L) = 0$$

In general

$$\begin{aligned} T \cdot L &= (\mathrm{dev}\, T + \mathrm{sph}\, T) \cdot (\mathrm{dev}\, L + \mathrm{sph}\, L) \\ &= \mathrm{dev}\, T \cdot \mathrm{dev}\, L + \mathrm{sph}\, T \cdot \mathrm{sph}\, L \\ &\quad + \mathrm{dev}\, T \cdot \mathrm{sph}\, L + \mathrm{sph}\, T \cdot \mathrm{dev}\, L \end{aligned}$$

Since

$$\begin{aligned} \mathrm{dev}\, T \cdot \mathrm{sph}\, L &= \left(T - \frac{1}{3}(\mathrm{tr}\, T) I\right) \cdot \left(\frac{1}{3}(\mathrm{tr}\, L) I\right) \\ &= \frac{1}{3}(\mathrm{tr}\, L)\, T \cdot I - \frac{1}{9}(\mathrm{tr}\, T)(\mathrm{tr}\, L)\, I \cdot I \\ &= \frac{1}{3}(\mathrm{tr}\, L)(\mathrm{tr}\, T) - \frac{1}{9}(\mathrm{tr}\, T)(\mathrm{tr}\, L)\, \mathrm{tr}(I) = 0 \end{aligned}$$

then

$$T \cdot L = \mathrm{dev}\, T \cdot \mathrm{dev}\, L + \mathrm{sph}\, T \cdot \mathrm{sph}\, L$$

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Cauchy stress tensor and Piola stress tensor

$T \cdot \nabla v$ stress power density per unit current volume

$\nabla v\, F = \nabla \bar{v}$ velocity gradient in the reference shape

$$\underset{\substack{\uparrow \\ \text{CAUCHY} \\ \text{STRESS}}}{(T} \cdot \nabla v) V_{\mathcal{R}} = (T \cdot \nabla v)(\det F)\, V_{\bar{\mathcal{R}}}$$

$$= (\det F)\,(T \cdot \nabla \bar{v} F^{-1})\, V_{\bar{\mathcal{R}}}$$

$$= \underbrace{(\det F)(T \cdot F^{-T})}_{S} \cdot \nabla \bar{v}\, V_{\bar{\mathcal{R}}} = \underset{\substack{\uparrow \\ \text{PIOLA} \\ \text{STRESS}}}{S} \cdot \nabla \bar{v}\, V_{\bar{\mathcal{R}}}$$

$F^{c} := (\det F) F^{-T}$ cofactor of F

$$\mathcal{F}^{int}(v) = -\int_{\mathcal{R}} T \cdot \nabla v \, dV = -\int_{\bar{\mathcal{R}}} (T \cdot \nabla v) \det F \, dV$$

$$= -\int_{\bar{\mathcal{R}}} S \cdot \nabla \bar{v} \, dV$$

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Force balance (power balance)

$$\mathcal{F}^{ext}(v) + \mathcal{F}^{int}(v) = 0 \quad \forall v$$

Energy imbalance principle

$$\mathcal{F}^{ext}(v) - \frac{d}{dt}\int_{\bar{\mathcal{R}}} \psi \, dV \geq 0$$

in any motion

ψ free energy density per unit reference volume

We assume

$$\psi = \varphi(F) \quad \text{strain energy density}$$

Since

$$\mathcal{F}^{ext}(v) = -\mathcal{F}^{int}(v) = \int_{\mathcal{R}} T \cdot \nabla v \, dV$$

$$\int_{\mathcal{R}} T \cdot \nabla v \, dV - \frac{d}{dt}\int_{\bar{\mathcal{R}}} \varphi(F) \, dV \geq 0$$

$$\int_{\bar{\mathcal{R}}} S \cdot \nabla \bar{v} \, dV - \int_{\bar{\mathcal{R}}} \frac{d}{dt}\varphi(F) \, dV \geq 0$$

We extend this inequality to any subset of $\bar{\mathcal{R}}$

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Energy imbalance (localized)

$$S \cdot \nabla \bar{v} - \frac{d}{dt} \varphi(F) \geq 0 \qquad \nabla \bar{v} = \dot{F}$$

$$\hat{S}(F) \cdot \dot{F} = \frac{d}{dt} \varphi(F)$$

$$\underbrace{\left(S - \hat{S}(F)\right)}_{\text{dissipative stress } S^{+}} \cdot \dot{F} \geq 0$$

Replacing the definition of Piola stress

$$\underbrace{(\det F)}_{>0} \left(T - \hat{T}(F)\right) F^{-T} \cdot \dot{F} \geq 0$$

$$\underbrace{\left(T - \hat{T}(F)\right)}_{\text{dissipative stress } T^{+}} \cdot \dot{F} F^{-1} \geq 0$$

We get also

$$(\det F) \hat{T}(F) F^{-T} \cdot \dot{F} = \frac{d}{dt} \varphi(F)$$

$$\hat{T}(F) \cdot \dot{F} F^{-1} = (\det F)^{-1} \frac{d}{dt} \varphi(F)$$

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Viscous dissipation

We can fullfil the imbalance principle in either form

$$S^+ \cdot \nabla \bar{v} \geq 0$$

or

$$T^+ \cdot \nabla v \geq 0$$

by choosing

$$T^+ = 2\mu \,\mathrm{sym}\, \nabla v$$

Since the inequality

$$\mu \,\mathrm{sym}\, \nabla v \cdot \nabla v \geq 0$$

can be written as

$$\mu \underbrace{(\mathrm{sym}\, \nabla v) \cdot (\mathrm{sym}\, \nabla v)}_{\geq 0} \geq 0$$

then

$$\mu > 0$$

is the only condition to get the dissipation principle (energy imbalance) fulfilled

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(31-32) Friday [2015-12-04] 9:00-11:00

Inner pressure

An incompressible material is characterized by

$$\operatorname{div} v = 0 \quad \text{(isochoric motion)}$$

Since $\operatorname{div} v = \operatorname{tr} \nabla v = 0$ then

$$\operatorname{sph}(\nabla v) = \frac{1}{3} \operatorname{tr} \nabla v = 0 \quad \Rightarrow \nabla v = \operatorname{dev} \nabla v$$

Hence

$$\begin{aligned} T \cdot \nabla v &= (\operatorname{dev} T + \operatorname{sph} T) \cdot (\operatorname{dev} \nabla v + \operatorname{sph} \nabla v) \\ &= \operatorname{dev} T \cdot \operatorname{dev} \nabla v = \operatorname{dev} T \cdot \nabla v \end{aligned}$$

Response function from the strain energy

$$\hat{S}(F) \cdot \nabla \bar{v} = \frac{d}{dt} \varphi(F)$$

$$\underbrace{(\det F)}_{1} \hat{T}(F) F^{-T} \cdot \nabla \bar{v} = \frac{d}{dt} \varphi(F)$$

$$\hat{T}(F) \cdot \nabla v = \frac{d}{dt} \varphi(F)$$

Because of incompressibility any spherical part of the stress is filtered out from the stress power above and the response function turns out to be a deviatoric tensor.

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This is why the spherical part of the stress, denoted by $-p\mathbf{I}$, enters explicitly the general characterization of the stress

$$\mathbf{T} = \hat{\mathbf{T}}(\mathbf{F}) - p\mathbf{I} + \mathbf{T}^{+}$$

The dissipative stress $\mathbf{T}^{+}$ is subject to the dissipation inequality

$$\mathbf{T}^{+} \cdot \nabla v \geq 0$$

which can possibly be fulfilled by choosing

$$\mathbf{T}^{+} = 2\mu \,\mathrm{sym}\, \nabla v$$

Should the material be not incompressible even the spherical part of the stress will be delivered by the rate of change of the strain energy as a response function.

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Newtonian fluids

$$\operatorname{div} v = 0 \qquad \text{(incompressibility)}$$

$$\hat{T}(F) = 0 \qquad \text{(no strain energy)}$$

$$T^{+} = 2\mu \operatorname{sym} \nabla v \qquad \text{(viscous dissipation)}$$

stress

$$T = -pI + 2\mu \operatorname{sym} \nabla v$$

force balance

$$\operatorname{div} T + b = 0$$

$$\operatorname{div} T = -\operatorname{div}(pI) + \mu\left(\operatorname{div} \nabla v + \operatorname{div} \nabla v^{T}\right)$$

$$\operatorname{div}(pI)e = \operatorname{div}(pe) = \operatorname{tr}(\nabla(pe))$$

$$= \operatorname{tr}(e \otimes \nabla p) = \nabla p \cdot e \quad \Rightarrow \operatorname{div}(pI) = \nabla p$$

$$\operatorname{div} \nabla v = \Delta v \qquad \text{(Laplacian)}$$

$$\operatorname{div} \nabla v^{T} = \nabla(\underbrace{\operatorname{div} v}_{0}) = 0$$

$$\operatorname{div} T = -\nabla p + \mu \Delta v$$

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$$b = b^{in} + b_o \quad \text{(bulk force density)}$$

inertial force density

mass density

$$b^{in} = -\rho\, a$$

acceleration field

with

$$a = (\nabla v)v + v' \qquad [\to (15\text{-}16)]$$

Replacing the expressions for div T and b into the balance equation we get the Navier-Stokes equation

$$-\nabla p + \mu \Delta v + \rho (\nabla v) v + \rho v' + b_o = 0$$

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Viscoelastic incompressible materials

$\det F = 1$ (incompressibility)

$$T = \hat{T}(F) - pI + 2\mu \,\mathrm{sym}\, \nabla v$$

neo-Hookean strain energy

$$\varphi(F) = c_1(\iota_1 - 3)$$

$\iota_1 := \mathrm{tr}\, C$ $\qquad C = F^T F$

Mooney-Rivlin strain energy

$$\varphi(F) = c_1(\iota_1 - 3) + c_2(\iota_2 - 3)$$

$$\iota_2 := \frac{1}{2}\left((\mathrm{tr}\, C)^2 - \mathrm{tr}\, C^2\right)$$

Almost incompressible neo-Hookean

$$\varphi(F) = c_1(\bar{\iota}_1 - 3) + c_3(\det F - 1)^2$$

$$\bar{\iota}_1 := \iota_1 (\det F)^{-\frac{2}{3}}$$

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