$(51\text{-}52)_{10}$ Monday [2014-04-28] A1.3
16:00-18:00

Newtonian fluids

incompressibility $\operatorname{div} v(x,t) = 0$

$\hat{T}(F) = 0$

$T^+ = 2\mu \operatorname{sym} \nabla v$

$\Rightarrow \qquad T = -pI + 2\mu \operatorname{sym} \nabla v$

balance equation $\operatorname{div} T + b = 0 \qquad b = b_0 + b^{in}$

$\operatorname{div} T = -\operatorname{div}(pI) + \mu(\operatorname{div} \nabla v + \operatorname{div} \nabla v^T)$

$\operatorname{div}(pI) = \nabla p$ (as a vector)

$\operatorname{div} \nabla v = \Delta v$ (Laplacian)

$\operatorname{div} \nabla v^T = \nabla(\underbrace{\operatorname{div} v}_{0}) = 0$

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$$\operatorname{div} pI \cdot a = \operatorname{div}(p\,a) = \operatorname{tr}(\nabla(p\,a))$$
$$= \operatorname{tr}(a \otimes \nabla p) = \nabla p \cdot a$$

$$\nabla(p\,a)e = \lim_{h \to 0} \frac{1}{h}\big(p(x+h\,e) - p(x)\big)a$$
$$= (\nabla p \cdot e)\,a = (a \otimes \nabla p)\,e$$

$$\operatorname{tr}(u \otimes v) = \frac{\operatorname{vol}((u\otimes v)e_1, e_2, e_3)}{\operatorname{vol}(e_1, e_2, e_3)} + \ldots + \ldots$$
$$= \frac{\operatorname{vol}(u(v \cdot e_1), e_2, e_3)}{\operatorname{vol}(e_1, e_2, e_3)} + \ldots + \ldots$$
$$= u_1(v \cdot e_1) + u_2(v \cdot e_2) + u_3(v \cdot e_3)$$
$$= v \cdot (u_1 e_1 + u_2 e_2 + u_3 e_3) = v \cdot u$$

$$\operatorname{div} \nabla v^T \cdot a = \operatorname{div}\big((\nabla v)a\big) = \operatorname{tr} \nabla w \qquad w := (\nabla v)a$$

$$(\nabla w)e = \lim_{h \to 0} \frac{1}{h}\big(w(x+h\,e) - w(x)\big)$$
$$= \lim_{h \to 0} \frac{1}{h}\big(\nabla v(x+h\,e)a - \nabla v(x)a\big)$$

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$$\nabla v(x+he)\,a = \lim_{d\to 0}\frac{1}{d}\Big(v(x+he+da)-v(x+he)\Big)$$

$$\nabla v(x)\,a = \lim_{d\to 0}\frac{1}{d}\Big(v(x+da)-v(x)\Big)$$

$$(\nabla w)e = \lim_{d\to 0}\frac{1}{d}\Big(\lim_{h\to 0}\frac{1}{h}\big(v(x+he+da)-v(x+da)\big)$$
$$-\lim_{h\to 0}\frac{1}{h}\big(v(x+he)-v(x)\big)\Big)$$

$$= \lim_{d\to 0}\frac{1}{d}\Big(\nabla v(x+da)e-\nabla v(x)e\Big)$$

$$\Rightarrow \quad \nabla w = \lim_{d\to 0}\frac{1}{d}\Big(\nabla v(x+da)-\nabla v(x)\Big)$$

$$\operatorname{tr}\nabla w = \lim_{d\to 0}\frac{1}{d}\Big(\underset{\operatorname{div} v(x+da)}{\operatorname{tr}(\nabla v(x+da))}-\underset{\operatorname{div} v(x)}{\operatorname{tr}(\nabla v(x))}\Big)$$

$$\operatorname{tr}\nabla w = \big(\nabla(\operatorname{div} v)\big)\cdot a$$

$$\Rightarrow \quad \operatorname{div}\nabla v^T = \nabla(\operatorname{div} v)$$

$\operatorname{div} v = 0$ (incompressibility) $\Rightarrow \operatorname{div}\nabla v^T = 0$

Finally let us define $\Delta v := \operatorname{div}\nabla v$

it is a vector field $\uparrow$ Laplacian

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$$\operatorname{div} \nabla v \cdot e_i = \operatorname{div}(\nabla v^T e_i) = \operatorname{tr} \nabla(\nabla v^T e_i)$$

$$\nabla v^T e_1 = v_{1,1}\, e_1 + v_{1,2}\, e_2 + v_{1,3}\, e_3$$

$$\nabla v^T e_i = v_{i,1}\, e_1 + v_{i,2}\, e_2 + v_{i,3}\, e_3$$

$$[\nabla(\nabla v^T e_i)] = \begin{pmatrix} v_{i,11} & v_{i,12} & v_{i,13} \\ v_{i,21} & v_{i,22} & v_{i,23} \\ v_{i,31} & v_{i,32} & v_{i,33} \end{pmatrix}$$

$$\operatorname{tr} \nabla(\nabla v^T e_i) = v_{i,11} + v_{i,22} + v_{i,33}$$

$\operatorname{div} \nabla v = \Delta v$ Laplacian

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$$\operatorname{div} \nabla v^T \cdot e_i = \operatorname{div}(\nabla v\, e_i) = \operatorname{tr} \nabla(\nabla v\, e_i)$$

$$\nabla v\, e_1 = v_{1,1}\, e_1 + v_{2,1}\, e_2 + v_{3,1}\, e_3$$

$$\nabla v\, e_i = v_{1,i}\, e_1 + v_{2,i}\, e_2 + v_{3,i}\, e_3$$

$$[\nabla(\nabla v\, e_i)] = \begin{pmatrix} v_{1,i1} & v_{1,i2} & v_{1,i3} \\ v_{2,i1} & v_{2,i2} & v_{2,i3} \\ v_{3,i1} & v_{3,i2} & v_{3,i3} \end{pmatrix}$$

$$\begin{aligned} \operatorname{tr} \nabla(\nabla v\, e_i) &= v_{1,i1} + v_{2,i2} + v_{3,i3} \\ &= v_{1,1i} + v_{2,2i} + v_{3,3i} \\ &= \underbrace{(v_{1,1} + v_{2,2} + v_{3,3})}_{\operatorname{tr} \nabla v}{}_{,i} \end{aligned}$$

$$\operatorname{tr} \nabla v = 0 \;\Rightarrow\; \operatorname{div} \nabla v^T = 0$$

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$(53\text{-}54)_{10}$ Tuesday [2014-04-29] A1.3
9:00 - 11:00

Kinetic energy (density per unit current volume)

$$\frac{1}{2}\rho\|v\|^2 = \frac{1}{2}\rho\, v\cdot v$$

$$K := \int_{\mathcal{R}} \frac{1}{2}\rho\|v\|^2 dV = \int_{\bar{\mathcal{R}}} \frac{1}{2}\underbrace{(\det F)\rho}_{\rho_0 \text{ mass density per unit reference volume}}\|\bar{v}\|^2 dV$$ total kinetic energy

$$= \int_{\bar{\mathcal{R}}} \underbrace{\frac{1}{2}\rho_0\, \bar{v}\cdot\bar{v}}_{\text{kinetic energy density per unit reference volume}} dV$$

Inertial force (density per unit current volume)

$$\int_{\mathcal{R}} b^{in}\cdot v\, dV = -\frac{d}{dt}\int_{\mathcal{R}} \frac{1}{2}\rho\|v\|^2 dV$$

$$= -\frac{d}{dt}\int_{\bar{\mathcal{R}}} \frac{1}{2}\rho_0\|\bar{v}\|^2 dV = -\int_{\bar{\mathcal{R}}} \frac{d}{dt}\frac{1}{2}\rho_0\, \bar{v}\cdot\bar{v}\, dV$$

independent of time — costant in time

$$= -\int_{\bar{\mathcal{R}}} \rho_0\, \bar{a}\cdot\bar{v}\, dV = -\int_{\mathcal{R}} \underbrace{\rho_0 \frac{1}{\det F}}_{\rho \text{ mass density per unit current volume}} a\cdot v\, dV$$

$$\Rightarrow\ b^{in}(x) = -\rho(x)\, a(x)$$

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$$\operatorname{div} T + b^{in} + b_o = 0$$

$$-\nabla p + \mu \Delta v - \rho a + b_o = 0$$

$$-\nabla p + \mu \Delta v - \rho \left((\nabla v) v + v' \right) + b_o = 0$$

(velocity and acceleration spatial description)

↳ $(21-22)_4$

Navier-Stokes equation

for Newtonian fluids

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$(55-56)_{10}$ Wednesday [2014-04-30] A1.3
9:00-11:00

In any motion, because of the balance principle,

$$\int_{\mathcal{R}} b \cdot v \, dV + \int_{\partial\mathcal{R}} t \cdot v \, dA = \int_{\mathcal{R}} T \cdot \nabla v \, dV$$

If $b = b^{in}$ and $t \cdot v = 0$ on $\partial\mathcal{R}$ then

$$\int_{\mathcal{R}} b^{in} \cdot v \, dV = \int_{\mathcal{R}} T \cdot \nabla v \, dV$$

with $\displaystyle\int_{\mathcal{R}} b^{in} \cdot v \, dV = -\frac{d}{dt} K$

and $\displaystyle\int_{\mathcal{R}} T \cdot \nabla v \, dV = \int_{\mathcal{R}} (\hat{T}(F) - pI + T^+) \cdot \nabla v \, dV$

Further if the material is incompressible then

$$pI \cdot \nabla v = p \, \mathrm{tr} \nabla v = p \, \mathrm{div} \, v = 0$$

otherwise we set $p = 0$ because there is no reason to split $\hat{T}(F)$ into a deviatoric and a spherical parts.

So whether the material is incompressible or not

$$\int_{\mathcal{R}} T \cdot \nabla v \, dV = \int_{\mathcal{R}} \hat{T}(F) \cdot \nabla v \, dV + \int_{\mathcal{R}} T^+ \cdot \nabla v \, dV$$

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Recalling that for an hyperelastic material

$$\int_{\mathcal{R}} \hat{T}(F)\cdot \dot{F}F^{-1}\,dV = \int_{\bar{\mathcal{R}}} \hat{S}(F)\cdot \dot{F}\,dV$$

$$= \int_{\bar{\mathcal{R}}} \frac{d}{dt}\varphi(F)\,dV = \frac{d}{dt}\int_{\bar{\mathcal{R}}} \varphi(F)\,dV$$

denoting by

$$\Phi = \int_{\bar{\mathcal{R}}} \varphi(F)\,dV$$

the total strain energy (or elastic energy)

we get

$$\int_{\mathcal{R}} \hat{T}(F)\cdot \nabla v\,dV = \frac{d}{dt}\Phi$$

Hence

$$\int_{\mathcal{R}} T\cdot \nabla v\,dV = \frac{d}{dt}\Phi + \int_{\mathcal{R}} T^{+}\cdot \nabla v\,dV$$

By replacing these expressions in the starting power balance equation we get

$$-\frac{d}{dt}K - \frac{d}{dt}\Phi = \int_{\mathcal{R}} T^{+}\cdot \nabla v\,dV \geq 0$$

$(47\text{-}48)_8$

$$\Rightarrow \quad \frac{d}{dt}K + \frac{d}{dt}\Phi \leq 0$$

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