$(69-70)_{13}$ Monday [2014-05-19] A1.3 16:00-18:00

Inertial forces and free oscillations

$$b^{in}(x) = -\rho(x)\, a(x) \qquad \rightarrow (53-54)_{10}$$

Affine deformations

$$p_A(t) = p_o(t) + F(t)(\bar{p}_A - \bar{p}_o)$$

$$v_A = \dot{p}_A = \dot{p}_o + \dot{F}(\bar{p}_A - \bar{p}_o) = \dot{p}_o + \dot{F}F^{-1}(p_A - p_o)$$

$$a_A = \ddot{p}_A = \ddot{p}_o + \ddot{F}(\bar{p}_A - \bar{p}_o) = \ddot{p}_o + \ddot{F}F^{-1}(p_A - p_o)$$

test velocity field

$$v(x) = v_o + \nabla v\,(x - p_o) \qquad x \in \mathcal{R}$$

$$\bar{v}(x) = v_o + \nabla \bar{v}\,(x - p_o) \qquad x \in \bar{\mathcal{R}}$$

acceleration field

$$a(x) = a_o + \ddot{F}F^{-1}(x - p_o) \qquad x \in \mathcal{R}$$

$$\bar{a}(x) = a_o + \ddot{F}(x - \bar{p}_o) \qquad x \in \bar{\mathcal{R}}$$

$$\mathcal{I}^{in}(v) = \int_{\mathcal{R}} b^{in}(x)\cdot v(x)\, dV = -\int_{\bar{\mathcal{R}}} \rho_o\, \bar{a}(x)\cdot \bar{v}(x)\, dV$$

$$= -\int_{\bar{\mathcal{R}}} \rho_o \big(a_o + \ddot{F}(x - \bar{p}_o)\big)\cdot\big(v_o + \nabla\bar{v}\,(x - \bar{p}_o)\big)\, dV$$

$$\mathcal{F}^{in}(v) = -\int_{\bar{\mathcal{R}}} \rho_o a_o \cdot v_o \, dV - \int_{\bar{\mathcal{R}}} \rho_o a_o \cdot \nabla\bar{v}\,(x - \bar{p}_o)\, dV$$

$$- \int_{\bar{\mathcal{R}}} \rho_o \ddot{F}\,(x - \bar{p}_o) \cdot v_o \, dV - \int_{\bar{\mathcal{R}}} \rho_o \ddot{F}\,(x - \bar{p}_o) \cdot \nabla\bar{v}\,(x - p_o)\, dV$$

The barycenter of the reference shape is the position x_G such that

$$(x_G - p_o) = \frac{1}{V_{\bar{\mathcal{R}}}} \int_{\bar{\mathcal{R}}} \rho_o (x - \bar{p}_o)\, dV$$

We choose $p_o \equiv x_G$ so that $\int_{\bar{\mathcal{R}}} \rho_o (x - \bar{p}_o)\, dV = 0$

and

$$\mathcal{F}^{in}(v) = -\int_{\bar{\mathcal{R}}} \rho_o \, dV \; a_o \cdot v_o$$

$$- \int_{\bar{\mathcal{R}}} \rho_o \ddot{F}\,(x - \bar{p}_o) \otimes (x - \bar{p}_o) \cdot \nabla\bar{v}\, dV$$

$$= -\underset{\substack{\uparrow \\ \text{total mass}}}{m}\, a_o \cdot v_o - \ddot{F} \underbrace{\int_{\bar{\mathcal{R}}} \rho_o (x - p_o) \otimes (x - p_o)\, dV}_{\bar{J}\ \text{Euler tensor}} \cdot \nabla\bar{v}$$

$$\mathcal{J}^{in}(v) = -m\, a_0 \cdot v_0 - \ddot{F}\bar{J} \cdot \nabla\bar{v}$$

$$= -m\, a_0 \cdot v_0 - \ddot{F}\bar{J} \cdot (\nabla v\, F)$$

$$= -m\, a_0 \cdot v_0 - \ddot{F}\bar{J}F^T \cdot \nabla v = f^{in} \cdot v_0 + M^{in} \cdot \nabla v$$

By using coordinates as in

$$x - \bar{p}_0 = s_1 e_1 + s_2 e_2 + s_3 e_3$$

we can compute the matrix of $\bar{J}$ for uniform ρ_0

$$[\bar{J}]_{ij} = \rho_0 \int_{-\frac{\bar{l}_3}{2}}^{\frac{\bar{l}_3}{2}} \int_{-\frac{\bar{l}_2}{2}}^{\frac{\bar{l}_2}{2}} \int_{-\frac{\bar{l}_1}{2}}^{\frac{\bar{l}_1}{2}} s_i s_j \, ds_1\, ds_2\, ds_3$$

$$[\bar{J}] = \frac{\rho_0}{12} \begin{pmatrix} \bar{l}_1^3 \bar{l}_2 \bar{l}_3 & & \\ & \bar{l}_2^3 \bar{l}_3 \bar{l}_1 & \\ & & \bar{l}_3^3 \bar{l}_1 \bar{l}_2 \end{pmatrix}$$

where l_1, l_2, l_3 are the lengths of the edges of the reference shape

If we set $\bar{l}_1 = \bar{l}$, $\bar{l}_2 = \bar{l}_3 = \alpha \bar{l}$ we get

$$V_{\bar{\mathcal{R}}} = \alpha^2 \bar{l}^3$$

$$[\bar{J}] = \frac{\rho_0 \bar{l}^{-2}}{12} \begin{pmatrix} 1 & & \\ & \alpha^2 & \\ & & \alpha^2 \end{pmatrix} V_{\bar{\mathcal{R}}}$$

let us consider again affine deformations such that

$$[F] = \begin{pmatrix} \lambda & & \\ & \frac{1}{\sqrt{\lambda}} & \\ & & \frac{1}{\sqrt{\lambda}} \end{pmatrix}$$

with λ depending on time, and compute time derivatives

$$[\dot{F}] = \begin{pmatrix} 1 & & \\ & -\frac{1}{2}\lambda^{-\frac{3}{2}} & \\ & & -\frac{1}{2}\lambda^{-\frac{3}{2}} \end{pmatrix} \dot{\lambda}$$

$$[\ddot{F}] = \begin{pmatrix} 1 & & \\ & -\frac{1}{2}\lambda^{-\frac{3}{2}} & \\ & & -\frac{1}{2}\lambda^{-\frac{3}{2}} \end{pmatrix} \ddot{\lambda} + \begin{pmatrix} 0 & & \\ & \frac{3}{4}\lambda^{-\frac{5}{2}} & \\ & & \frac{3}{4}\lambda^{-\frac{5}{2}} \end{pmatrix} \dot{\lambda}^2$$

$$[\ddot{F} F^T] = \begin{pmatrix} \lambda & & \\ & -\frac{1}{2}\lambda^{-2} & \\ & & -\frac{1}{2}\lambda^{-2} \end{pmatrix} \ddot{\lambda} + \begin{pmatrix} 0 & & \\ & \frac{3}{4}\lambda^{-3} & \\ & & \frac{3}{4}\lambda^{-3} \end{pmatrix} \dot{\lambda}^2$$

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To make expressions simpler let us set

$$\rho_s := \frac{1}{12}\,\rho_o\,\bar{\ell}^2$$

so we get

$$[\ddot{F}\bar{J}F^T] = \rho_s \begin{pmatrix} \ddot{\lambda}\lambda & 0 & 0 \\ 0 & \alpha^2\left(-\frac{1}{2}\frac{\ddot{\lambda}}{\lambda^2} + \frac{3}{4}\frac{\dot{\lambda}^2}{\lambda^3}\right) & 0 \\ 0 & 0 & \alpha^2\left(-\frac{1}{2}\frac{\ddot{\lambda}}{\lambda^2} + \frac{3}{4}\frac{\dot{\lambda}^2}{\lambda^3}\right) \end{pmatrix} V_{\mathcal{R}}$$

Let us apply to the body in the shape of a cylinder with a square cross section a normal traction on the opposite end faces while taking into account the inertial forces

The total force is $f = -m\,\ddot{p}_o$

The total moment tensor is

$$M = V_{\mathcal{R}}\,p\,e_1 \otimes e_1 - \ddot{F}\bar{J}F^T$$

The balance equations are

$$f = 0$$

$$\operatorname{skw} M = 0$$

$$\operatorname{sym} M = T\, V_{\mathcal{R}}$$

From the first one we get $\ddot{p}_o = 0$

So the barycenter will move at a constant speed.

The moment tensor turns out to be symmetric, as we can see from its matrix. So the second of the balance equations above is fullfilled.

Since the motion is isochoric

$$V_{\mathcal{R}} = V_{\bar{\mathcal{R}}}$$

Hence, from the last balance equation,

$$T = p\, e_1 \otimes e_1 - \ddot{F} \bar{J} F^T \frac{1}{V_{\mathcal{R}}}$$

whose scalar form is as follows

$$\sigma_1 = p - \rho_s \lambda \ddot{\lambda}$$

$$\sigma_2 = 0 - \rho_s \alpha^2 \left(-\frac{1}{2} \frac{\ddot{\lambda}}{\lambda^2} + \frac{3}{4} \frac{\dot{\lambda}^2}{\lambda^3} \right)$$

$$\sigma_3 = 0 - \rho_s \alpha^2 \left(-\frac{1}{2} \frac{\ddot{\lambda}}{\lambda^2} + \frac{3}{4} \frac{\dot{\lambda}^2}{\lambda^3} \right)$$

Let the material be incompressible and viscoelastic

$$\sigma_1 = \hat{\sigma}_1^D - p + \sigma_1^+$$

$$\sigma_2 = \hat{\sigma}_2^D - p + \sigma_2^+ \qquad \to (65-66)_{12}$$

$$\sigma_3 = \hat{\sigma}_3^D - p + \sigma_3^+$$

Substituting these expressions into the balance equations we get

$$\sigma_1^D - p + \sigma_1^+ = p - \rho_s \lambda \ddot{\lambda}$$

$$\sigma_2^D - p + \sigma_2^+ = - \rho_s \alpha^2 \left(-\frac{1}{2} \frac{\ddot{\lambda}}{\lambda^2} + \frac{3}{4} \frac{\dot{\lambda}^2}{\lambda^3} \right)$$

$$\sigma_3^D - p + \sigma_3^+ = - \rho_s \alpha^2 \left(-\frac{1}{2} \frac{\ddot{\lambda}}{\lambda^2} + \frac{3}{4} \frac{\dot{\lambda}^2}{\lambda^3} \right)$$

By adding terms on the left side and terms on the right side (like taking the trace of the corresponding tensors) we get

$$-3p = \mathfrak{p} - \varrho_s\left(\lambda\ddot{\lambda} + \alpha^2\left(-\frac{\ddot{\lambda}}{\lambda^2} + \frac{3}{2}\frac{\dot{\lambda}^2}{\lambda^3}\right)\right)$$

where

$$\hat{\sigma}_1^D + \hat{\sigma}_2^D + \hat{\sigma}_3^D = 0$$

$$\sigma_1^+ + \sigma_2^+ + \sigma_3^+ = 0$$

because both $\hat{T}(F)$ and T^+ are deviatoric tensors.

We solve the last equation for the inner pressure p and replace its expression in the first of the previous equations

$$\hat{\sigma}_1^D + \sigma_1^+ = \mathfrak{p} - \frac{1}{3}\mathfrak{p} - \varrho_s\lambda\ddot{\lambda} + \frac{1}{3}\varrho_s\left(\lambda\ddot{\lambda} + \alpha^2\left(-\frac{\ddot{\lambda}}{\lambda^2} + \frac{3}{2}\frac{\dot{\lambda}^2}{\lambda^3}\right)\right)$$

$$\hat{\sigma}_1^D + 2\mu\frac{\dot{\lambda}}{\lambda} = \frac{2}{3}\mathfrak{p} + \frac{1}{3}\varrho_s\left(-2\lambda\ddot{\lambda} + \alpha^2\left(-\frac{\ddot{\lambda}}{\lambda^2} + \frac{3}{2}\frac{\dot{\lambda}^2}{\lambda^3}\right)\right)$$

$$\hat{\sigma}_1^D = \frac{2}{3}\mathfrak{p} - 2\mu\frac{\dot{\lambda}}{\lambda} - \frac{2}{3}\varrho_s\left(\lambda\ddot{\lambda} + \frac{\alpha^2}{2}\left(\frac{\ddot{\lambda}}{\lambda^2} - \frac{3}{2}\frac{\dot{\lambda}^2}{\lambda^3}\right)\right)$$

$$\hat{\sigma}_1^D = \frac{2}{3}\hat{\sigma}_0\,; \quad \hat{\sigma}_0 = 2c_1\left(\lambda^2 - \frac{1}{\lambda}\right) \quad \text{neo-Hookean material} \rightarrow (63\text{-}64)_{12}$$

$$\hat{\sigma}_0 = \mathfrak{p} - 3\mu\frac{\dot{\lambda}}{\lambda} - \varrho_s\left(\lambda\ddot{\lambda} + \frac{\alpha^2}{2}\left(\frac{\ddot{\lambda}}{\lambda^2} - \frac{3}{2}\frac{\dot{\lambda}^2}{\lambda^3}\right)\right)$$

The final expression for the equation of motion is

$$2c_1\left(\lambda^2-\frac{1}{\lambda}\right)=p-3\mu\frac{\dot\lambda}{\lambda}-\rho_s\left(\lambda\ddot\lambda+\frac{\alpha^2}{2}\left(\frac{\ddot\lambda}{\lambda^2}-\frac{3}{2}\frac{\dot\lambda^2}{\lambda^3}\right)\right)$$

We can linearize this equation according to the following procedure

Let us set $\quad \lambda(t)=\lambda_0+\beta\,\tilde\varepsilon(t)$

$$p=p_0$$

and replace the original expression with a series expansion with respect to β starting at $\beta=0$, up to order 1. Then collect separately terms of order 0 and terms of order 1 with respect to β. We get

$$2c_1\left(\lambda_0^2-\frac{1}{\lambda_0}\right)=p_0$$

$$4c_1(1+\lambda_0^3)\,\tilde\varepsilon+6\lambda_0\mu\,\dot{\tilde\varepsilon}+(\alpha^2+2\lambda_0^3)\,\rho_s\,\ddot{\tilde\varepsilon}=0$$

$(73-74)_{13}$ Wednesday [2014-05-21] A1.3
9:00-11:00

Summary about the equation of motion and its linearization, describing small oscillations around the solution (λ_0, p_0).

Small oscillations

Let us consider solutions in the general form

$$\tilde{\varepsilon}(t) = \tilde{\varepsilon}_0 \, e^{kt}$$

The characteristic equation turns out to be

$$4 c_1 (1 + 2\lambda_0^3) + (6 \lambda_0 \mu) k + \rho_s (\alpha^2 + 2 \lambda_0^3) k^2 = 0$$

$$\Rightarrow \quad k = \frac{-3\lambda_0 \mu \pm \sqrt{9\lambda_0^2 \mu^2 - 4 c_1 \rho_s (1 + 2\lambda_0^3)(\alpha^2 + 2\lambda_0^3)}}{(\alpha^2 + 2\lambda_0^3)\rho_s}$$

Critical viscosity value μ_0 (depending on λ_0)

$$\mu_0^2 = \frac{2}{3} c_1 \rho_s \frac{1}{\lambda_0^2} (1 + 2\lambda_0^3)(\alpha^2 + 2\lambda_0^3)$$

$\mu < \mu_0$ damped oscillations

$\mu > \mu_0$ overdamping, no oscillations

