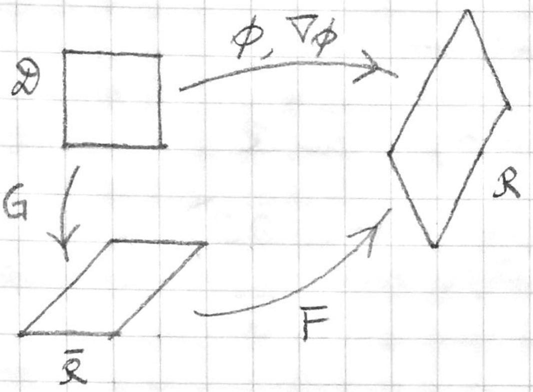
$(81-82)_{15}$ Tuesday [2014-06-03] Δ1.3
9:00 - 11:00

Remodeling



$\nabla\phi = F\,G$ Kröner-Lee decomposition

F elastic distortion

G remodeling distortion with $\det G > 0$

$\mathcal{D}$ reference shape

$\bar{\mathcal{R}}$ relaxed shape

$\mathcal{R}$ current shape

v velocity field on $\mathcal{R}$

$\mathbb{V}$ remodeling velocity tensor field on $\bar{\mathcal{R}}$

φ strain energy density in $\bar{\mathcal{R}}$

Basic assumption: only F affects the strain energy φ.

T Cauchy stress (tensor field on $\mathcal{R}$)

S Piola stress (tensor field on $\bar{\mathcal{R}}$) $S = (\det F)\,T\,F^{-T}$

$\mathbb{S}$ Piola stress (tensor field on $\mathcal{D}$) $\mathbb{S} = (\det\nabla\phi)\,T\,\nabla\phi^{-T}$

$$\mathcal{F}^{ext}(v,\mathbb{V}) = \int_{\mathcal{R}} b\cdot v\, dV + \int_{\partial\mathcal{R}} t\cdot v\, dV + \int_{\bar{\mathcal{R}}} \mathbb{Q}\cdot\mathbb{V}\, dV$$

remodeling force

$$\mathcal{F}^{int}(v,\mathbb{V}) = -\int_{\mathcal{R}} T\cdot\nabla v\, dV - \int_{\bar{\mathcal{R}}} A\cdot\mathbb{V}\, dV$$

remodeling stress

From the force balance principle

$$\mathcal{F}^{ext}(v, \mathbb{V}) + \mathcal{F}^{int}(v, \mathbb{V}) = 0 \qquad \forall v, \forall \mathbb{V}$$

we get the balance equations

$$\operatorname{div} T + b = 0 \qquad \text{in } \mathcal{R}$$

$$T n = t \qquad \text{in } \partial\mathcal{R}$$

$$\mathbb{Q} = A \qquad \text{in } \bar{\mathcal{R}}$$

In any motion

$$\nabla v = \nabla\dot{\phi}\, \nabla\phi^{-1}$$

$$\mathbb{V} = \dot{G}\, G^{-1}$$

and, by the balance principle above,

$$\int_{\mathcal{R}} b \cdot v\, dV + \int_{\partial\mathcal{R}} t \cdot v\, dA + \int_{\bar{\mathcal{R}}} \mathbb{Q} \cdot \dot{G} G^{-1}\, dV$$

$$= \int_{\mathcal{R}} T \cdot \nabla\dot{\phi}\, \nabla\phi^{-1}\, dV + \int_{\bar{\mathcal{R}}} A \cdot \dot{G} G^{-1}\, dV$$

Hence the energy balance principle (dissipation inequality) states that in any motion

$$\int_{\mathcal{R}} T \cdot \nabla\dot{\phi}\, \nabla\phi^{-1}\, dV + \int_{\bar{\mathcal{R}}} A \cdot \dot{G} G^{-1}\, dV - \frac{d}{dt} \int_{\bar{\mathcal{R}}} \varphi(F)\, dV \geq 0$$

After transforming the integrals into integrals over the reference shape

$$\int_{\mathcal{D}} T\nabla\phi^{-T}\cdot\nabla\dot{\phi}\,\det\nabla\phi\,dV+\int_{\mathcal{D}} A\cdot\dot{G}G^{-1}\det G\,dV-\frac{d}{dt}\int_{\mathcal{D}}\varphi(F)\det G\,dV\geq 0$$

we replace the former statement with the localized form

$$S\cdot\nabla\dot{\phi}+A\cdot\dot{G}G^{-1}\det G-\frac{d}{dt}\varphi(F)\det G-\varphi(F)\det G\,\mathrm{tr}(\dot{G}G^{-1})\geq 0$$

where the formula for the time derivative of det G has been used $\rightarrow (17\text{-}18)_3$

Now let us substitute the relation

$$\hat{S}(F)\cdot\dot{F}=\frac{d}{dt}\varphi(F)$$

together with the expressions for the Piola stress tensors and the Kröner-Lee decomposition, and get in turn

$$S\cdot(\dot{F}G+F\dot{G})\frac{1}{\det G}+A\cdot\dot{G}G^{-1}-\hat{S}(F)\cdot\dot{F}-\varphi(F)I\cdot\dot{G}G^{-1}\geq 0$$

$$\frac{\det\nabla\phi}{\det G}TF^{-T}G^{-T}\cdot(\dot{F}G+F\dot{G})-\hat{S}(F)\cdot\dot{F}+(A-\varphi(F)I)\cdot\dot{G}G^{-1}\geq 0$$

$$SG^{-T}\cdot(\dot{F}G+F\dot{G})-\hat{S}(F)\cdot\dot{F}+(A-\varphi(F)I)\cdot\dot{G}G^{-1}\geq 0$$

$$S\cdot\dot{F}GG^{-1}+F^{T}S\cdot\dot{G}G^{-1}-\hat{S}(F)\cdot\dot{F}+(A-\varphi(F)I)\cdot\dot{G}G^{-1}\geq 0$$

$(83-84)_{15}$ Wednesday [2014-06-04] A1.3
9:00–11:00

The final expression for the dissipation inequality is

$$\left(S - \hat{S}(F)\right)\cdot \dot{F} + \left(A - \varphi(F)I + F^T S\right)\cdot \dot{G}G^{-1} \geq 0$$

or, equivalently,

$$\det F\left(T - \hat{T}(F)\right)\cdot \dot{F}F^{-1} + \left(A - \underbrace{\left(\varphi(F)I - F^T S\right)}_{\text{Eshelby tensor}}\right)\cdot \dot{G}G^{-1} \geq 0$$

If we set

$$T^+ := T - \hat{T}(F), \quad A^+ := A - \left(\varphi(F)I - F^T S\right)$$

the inequality can be written as

$$(\det F)\, T^+ \cdot \dot{F}F^{-1} + A^+ \cdot \dot{G}G^{-1} \geq 0$$

We can now characterize the dissipation through suitable expressions for T^+ and A^+ consistently with the above inequality in any motion.

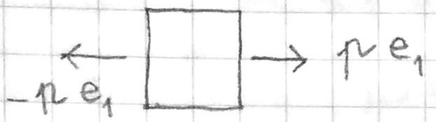
A consistent choice is for example

$$T^+ = 2\mu \,\mathrm{sym}(\dot{F}F^{-1})$$

$$A^+ = \eta\, \dot{G}G^{-1} \qquad (\eta \geq 0)$$

Note that while T is a symmetric tensor because of the objectivity condition, no such a property has been derived for A.

Let us consider again the body undergoing cylindrical deformations



with

$$[F] = \begin{pmatrix} \lambda & & \\ & \frac{1}{\sqrt{\lambda}} & \\ & & \frac{1}{\sqrt{\lambda}} \end{pmatrix} \qquad [G] = \begin{pmatrix} \gamma & & \\ & \frac{1}{\sqrt{\gamma}} & \\ & & \frac{1}{\sqrt{\gamma}} \end{pmatrix}$$

Hence

$$[\nabla\phi] = \begin{pmatrix} \lambda\gamma & & \\ & \frac{1}{\sqrt{\lambda\gamma}} & \\ & & \frac{1}{\sqrt{\lambda\gamma}} \end{pmatrix}$$

Balance equations

$$f = 0$$

$$\text{skw}\, M = 0$$

$$\frac{M}{V_{\mathcal{R}}} = T$$

$$\mathcal{Q} = A$$

where

$$M = p\, e_1 \otimes e_1\, V_{\mathcal{R}}$$

Let us set the remodeling force

$$\mathcal{Q} = 0$$

From the balance equations we get

$$T = p\, e_1 \otimes e_1$$
$$A = 0$$

Material characterization

$$T = \hat{T}(F) - p I + T^+$$
$$A = \varphi(F) I - F^T S - a I + A^+$$

where the inner pressure p is the reactive stress defined by the property that its power is zero, because $\dot{F}F^{-1}$ is deviatoric ($\mathrm{tr}\, \dot{F}F^{-1} = 0$) here.

Since $\dot{G}G^{-1}$ is deviatoric as well, because of the assumption of isochoric remodeling, we have to add the remodeling pressure a, whose power is $aI \cdot \dot{G}G^{-1} = 0$, because $\mathrm{tr}\, \dot{G}G^{-1} = 0$ here.

Let us set now $\quad T^+ = 0 \quad (\Leftrightarrow \mu = 0)$

$$A^+ = \eta\, \dot{G}G^{-1} \quad (\eta > 0)$$

and choose a neo-Hookean strain energy

$$\varphi(F) = c_1(\iota_1 - 3)$$

