$(85-86)_{16}$ Monday[2014-06-09] A1.3
16:00-18:00

$$T = \hat{T}(F) - p\,I$$

$$A = \varphi(F)I - F^T T F^{-T} \det F - a\,I + \eta\,\dot{G}G^{-1}$$

$\det F = 1$

$$[F^T T F^{-T}] = \begin{pmatrix} \lambda & & \\ & \frac{1}{\sqrt{\lambda}} & \\ & & \frac{1}{\sqrt{\lambda}} \end{pmatrix} \begin{pmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{pmatrix} \begin{pmatrix} \frac{1}{\lambda} & & \\ & \sqrt{\lambda} & \\ & & \sqrt{\lambda} \end{pmatrix} = [T]$$

$$[T] = \begin{pmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{pmatrix} \qquad [A] = \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix}$$

$$[\dot{G}G^{-1}] = \begin{pmatrix} 1 & & \\ & -\frac{1}{2} & \\ & & -\frac{1}{2} \end{pmatrix} \frac{\dot{\gamma}}{\gamma} \qquad [\dot{F}F^{-1}] = \begin{pmatrix} 1 & & \\ & -\frac{1}{2} & \\ & & -\frac{1}{2} \end{pmatrix} \frac{\dot{\lambda}}{\lambda}$$

$$[C] = [F^T F] = \begin{pmatrix} \lambda^2 & & \\ & \frac{1}{\lambda} & \\ & & \frac{1}{\lambda} \end{pmatrix}$$

$$\iota_1 = \operatorname{tr} C = \lambda^2 + \frac{2}{\lambda}$$

stress characterization

$\sigma_1 = \hat{\sigma}_1^D - p$

$\sigma_2 = \hat{\sigma}_2^D - p$

$\sigma_3 = \hat{\sigma}_3^D - p$

$T = \hat{T}(F) - pI$

balance

$\sigma_1 = \mu$

$\sigma_2 = 0$

$\sigma_3 = 0$

$T = \frac{M}{V_R}$

$$\hat{\sigma}_1^D - p = \mu$$
$$\hat{\sigma}_2^D - p = 0$$
$$\hat{\sigma}_3^D - p = 0$$

trace $\quad 0 - 3p = \mu \quad \Rightarrow \quad p = -\frac{1}{3}\mu$

$$\hat{\sigma}_1^D = \mu + p = \frac{2}{3}\mu$$
$$\hat{\sigma}_2^D = p = -\frac{1}{3}\mu$$
$$\hat{\sigma}_3^D = p = -\frac{1}{3}\mu$$

response function

$$\hat{T}(F)\cdot\dot{F}F^{-1} = \left(\hat{\sigma}_1 - \frac{1}{2}(\hat{\sigma}_2 + \hat{\sigma}_3)\right)\frac{\dot{\lambda}}{\lambda} \qquad \hat{\sigma}_0 := \hat{\sigma}_1 - \frac{1}{2}(\hat{\sigma}_2 + \hat{\sigma}_3)$$

for a neo-Hookean material

$$\frac{d}{dt}\varphi(F) = 2c_1\left(\lambda^2 - \frac{1}{\lambda}\right)\frac{\dot{\lambda}}{\lambda} \quad \Rightarrow \quad \hat{\sigma}_0 = 2c_1\left(\lambda^2 - \frac{1}{\lambda}\right)$$

Because of incompressibility only the deviatoric part of the response $\hat{T}(F)$ is delivered by the strain energy.

Hence $$\hat{\sigma}_1^D = \hat{\sigma}_1 - \frac{1}{3}\left(\hat{\sigma}_1 + \hat{\sigma}_2 + \hat{\sigma}_3\right) = \frac{2}{3}\left(\hat{\sigma}_1 - \frac{1}{2}\left(\hat{\sigma}_2 + \hat{\sigma}_3\right)\right)$$

$$\Rightarrow \quad \hat{\sigma}_1^D = \frac{2}{3}\hat{\sigma}_0$$

$$\hat{\sigma}_1^D = \frac{2}{3}p \qquad \text{(balance \& stress characterization)}$$

$$\frac{2}{3}\hat{\sigma}_0 = \frac{2}{3}p$$

$$\boxed{2\, c_1\left(\lambda^2 - \frac{1}{\lambda}\right) = p}$$

balance with neo-Hookean response

The deformation gradient depends on both λ and γ.

remodeling stress	balance
$a_1 = \varphi(F) - \sigma_1 - a + \eta\dfrac{\dot{\gamma}}{\gamma}$	$a_1 = 0$
$a_2 = \varphi(F) - \sigma_2 - a - \dfrac{1}{2}\eta\dfrac{\dot{\gamma}}{\gamma}$	$a_2 = 0$
$a_3 = \varphi(F) - \sigma_3 - a - \dfrac{1}{2}\eta\dfrac{\dot{\gamma}}{\gamma}$	$a_3 = 0$

$$\varphi(F) - \hat{\sigma}_1^D + p - a + \eta\frac{\dot{\gamma}}{\gamma} = 0$$
$$\varphi(F) - \hat{\sigma}_2^D + p - a - \frac{1}{2}\eta\frac{\dot{\gamma}}{\gamma} = 0$$
$$\varphi(F) - \hat{\sigma}_3^D + p - a - \frac{1}{2}\eta\frac{\dot{\gamma}}{\gamma} = 0$$

$$3\varphi(F) + 3p - 3a = 0$$

$$\Rightarrow \quad a = \varphi(F) + \mathrm{p}$$

$$\cancel{\varphi(F)} - \frac{2}{3} p + \cancel{\mathrm{p}} - \cancel{\varphi(F)} - \cancel{\mathrm{p}} + \eta \frac{\dot{\gamma}}{\gamma} = 0$$

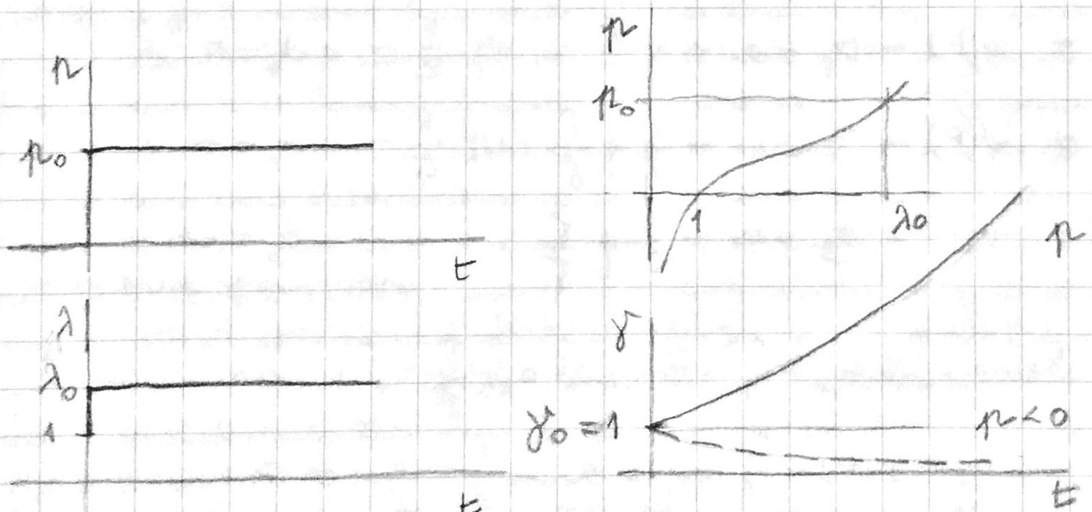
$$\cancel{\varphi(F)} - \cancel{\varphi(F)} - \mathrm{p} - \frac{1}{2} \eta \frac{\dot{\gamma}}{\gamma} = 0$$

$$\cancel{\varphi(F)} - \cancel{\varphi(F)} - \mathrm{p} - \frac{1}{2} \eta \frac{\dot{\gamma}}{\gamma} = 0$$

From the first equation we get

$$(1) \quad \boxed{\eta \frac{\dot{\gamma}}{\gamma} = \frac{2}{3} p}$$

$$\gamma(t) = \gamma_0 \, e^{kt} \qquad k = \frac{1}{\eta} \frac{2}{3} p$$



$$\lambda(t)\gamma(t) = \lambda_0 \gamma(t) \begin{array}{l} \nearrow \infty \quad p > 0 \\ \searrow 0 \quad p < 0 \end{array}$$

with $p(t) = p_0$ and $\lambda(t) = \lambda_0$

We can address a different problem within the same setting.

Let the body be constrained to keep its length unchanged after stretching it at time $t=0$ by $\lambda(0)=\lambda_0$, with $\gamma(0)=1$, so that

$$\lambda(t)\,\gamma(t)=\lambda_0 \qquad t\geq 0$$

Let p_0 be the force needed to get such a stretch.

If λ changes in time the force will change according to

$$2\,c_1\left(\lambda(t)^2-\frac{1}{\lambda(t)}\right)=p(t) \qquad \text{(neo-Hookean material)}$$

Therefore

$$\eta\,\frac{\dot\gamma(t)}{\gamma(t)}=\frac{2}{3}\,p(t)$$

becomes

$$\eta\,\frac{\dot\gamma(t)}{\gamma(t)}=\frac{4}{3}\,c_1\left(\lambda(t)^2-\frac{1}{\lambda(t)}\right)$$

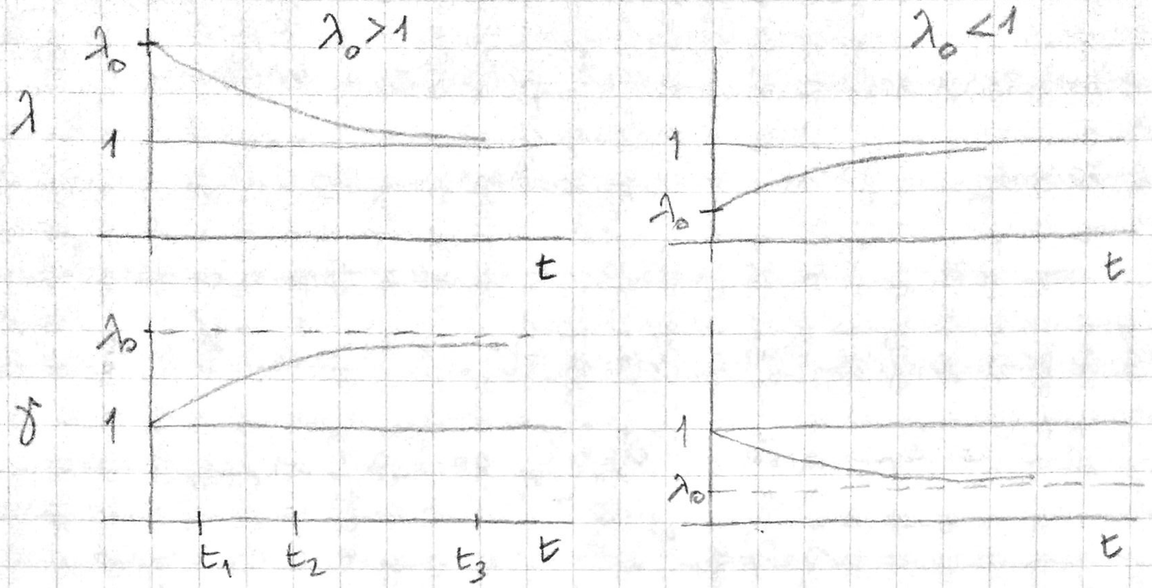
From the constraint $\lambda(t)\gamma(t)=\lambda_0$ we get

$$\dot\lambda(t)\,\gamma(t)+\lambda(t)\,\dot\gamma(t)=0$$

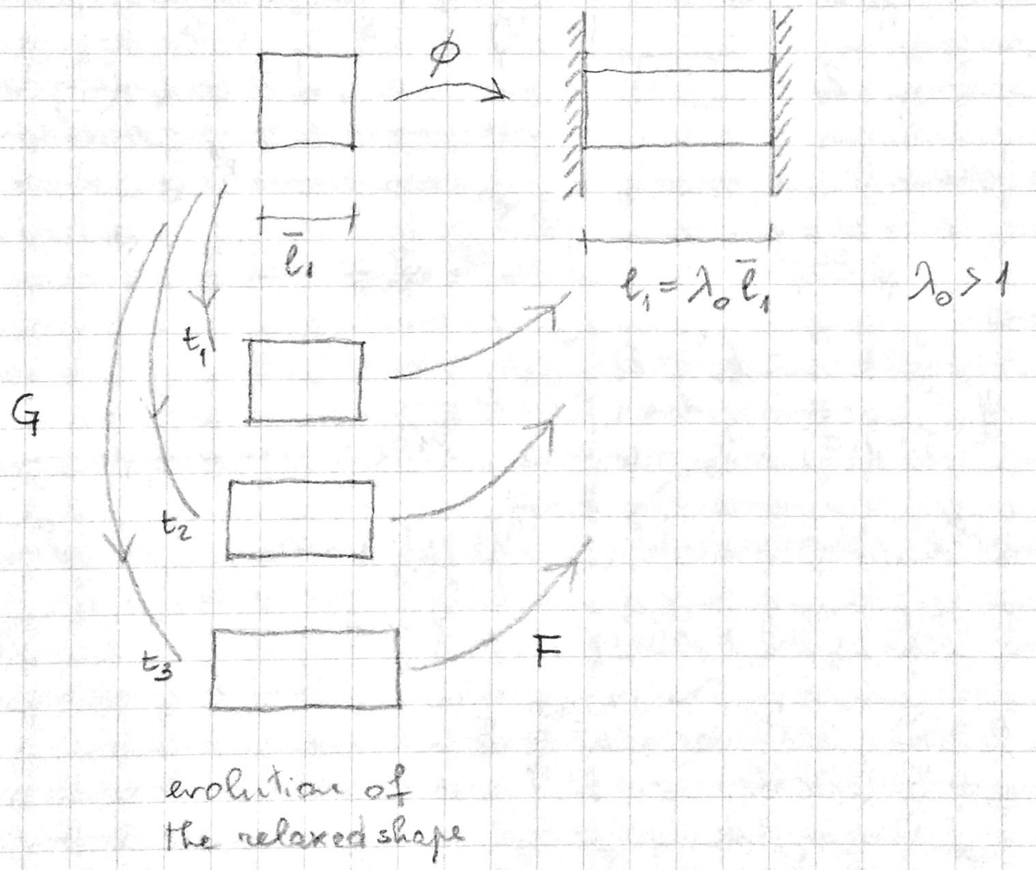
$$\frac{\dot\lambda(t)}{\lambda(t)}=-\frac{\dot\gamma(t)}{\gamma(t)}$$

Hence the evolution of the elastic stretch is described by

$$(2)\qquad \boxed{\eta\,\frac{\dot\lambda}{\lambda}=-\frac{4}{3}\,c_1\left(\lambda^2-\frac{1}{\lambda}\right)} \quad \text{with } \lambda(0)=\lambda_0$$



Note that the only stationary solution is $\lambda = 1$



$(87-88)_{16}$ Wednesday [2014-06-11] A1.3
9:00 - 11:00

Let us now consider a case where the remodeling force is different from zero

$$\mathcal{Q} = q\, e_1 \otimes e_1$$

From the balance equation

$$A = \mathcal{Q}$$

we get

$$a_1 = q$$

$$a_2 = 0$$

$$a_3 = 0$$

and, from the remodeling stress characterization,

$$\varphi(F) - \sigma_1^D + p - a + \eta \frac{\dot\gamma}{\gamma} = q$$

$$\varphi(F) - \sigma_2^D + p - a - \frac{1}{2}\eta \frac{\dot\gamma}{\gamma} = 0$$

$$\varphi(F) - \sigma_3^D + p - a - \frac{1}{2}\eta \frac{\dot\gamma}{\gamma} = 0$$

$$3\varphi(F) + 3p - 3a = q$$

$$\Rightarrow \quad a = \varphi(F) + p - \frac{1}{3} q$$

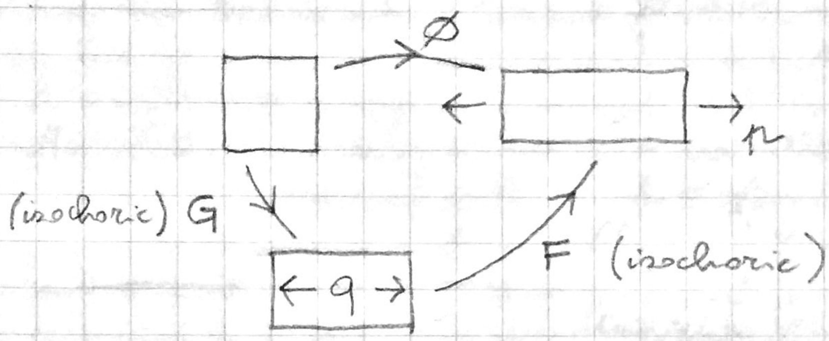
Substituting this expression into the equations above we get

$$\cancel{\varphi(F)} - \frac{2}{3} p + \cancel{p} - \cancel{\varphi(F)} - \cancel{p} + \frac{1}{3} q + \eta \frac{\dot\gamma}{\gamma} = q$$

which simplifies to

$$\boxed{\eta\,\frac{\dot\gamma}{\gamma} = \frac{2}{3}\left(p + q\right)} \qquad (3)$$

Note that the remodeling force q can either increase or decrease the effect of p



Let us now consider the body constrained to keep its length unchanged while under the action of q so that $\lambda(t)\,\gamma(t) = \lambda_0$, with $\gamma(0) = 1$.
Since $\dfrac{\dot\gamma}{\gamma} = -\dfrac{\dot\lambda}{\lambda}$, by using the expression for p derived from a neo-Hookean strain energy we get

$$\boxed{\eta\,\frac{\dot\lambda}{\lambda} = -\frac{2}{3}\left(2c_1\left(\lambda^2 - \frac{1}{\lambda}\right) + q\right)} \qquad (4)$$

There is a unique stationary solution for each value of q, as shown by the graph $(\lambda, q) \equiv (\lambda, -p)$

In order to describe a volumetric growth let us consider a case where G is a spherical tensor

$$[G] = \begin{pmatrix} \gamma & & \\ & \gamma & \\ & & \gamma \end{pmatrix}$$

while F is still isochoric

$$[F] = \begin{pmatrix} \lambda & & \\ & \frac{1}{\sqrt{\lambda}} & \\ & & \frac{1}{\sqrt{\lambda}} \end{pmatrix} \qquad [\nabla\phi] = \begin{pmatrix} \lambda\gamma & & \\ & \frac{\gamma}{\sqrt{\lambda}} & \\ & & \frac{\gamma}{\sqrt{\lambda}} \end{pmatrix}$$

The remodeling velocity will be a spherical tensor

$$\dot{G}G^{-1} = \frac{\dot{\gamma}}{\gamma} I$$

That is why we assume the remodeling force be described by a spherical tensor as well

$$Q = q I$$

The scalar form of the remodeling stress characterization is

$$a_1 = \varphi(F) - \sigma_1 + a_0 + \eta \frac{\dot{\gamma}}{\gamma}$$

$$a_2 = \varphi(F) - \sigma_2 - \frac{1}{2} a_0 + \eta \frac{\dot{\gamma}}{\gamma}$$

$$a_3 = \varphi(F) - \sigma_3 - \frac{1}{2} a_0 + \eta \frac{\dot{\gamma}}{\gamma}$$

where a_0 has been used to describe the deviatoric part of the remodeling stress whose power is zero since the remodeling velocity is now a spherical tensor.

Substituting the balance equations

$$\sigma_1 = p \qquad a_1 = q$$
$$\sigma_2 = 0 \qquad a_2 = q$$
$$\sigma_3 = 0 \qquad a_3 = q$$

we get

$$\varphi(F) - p + a_0 + \eta\frac{\dot{\gamma}}{\gamma} = q$$
$$\varphi(F) - \frac{1}{2}a_0 + \eta\frac{\dot{\gamma}}{\gamma} = q$$
$$\varphi(F) - \frac{1}{2}a_0 + \eta\frac{\dot{\gamma}}{\gamma} = q$$

$$3\varphi(F) - p + 3\eta\frac{\dot{\gamma}}{\gamma} = 3q$$

$$\Rightarrow \quad (5) \quad \boxed{\eta\frac{\dot{\gamma}}{\gamma} = -\left(\varphi(F) - \frac{1}{3}p\right) + q}$$

and, from any of the equations above,

$$a_0 = \frac{2}{3}p$$

For a neo-Hookean material

$$\varphi(F) = c_1(\iota_1 - 3) = c_1\left(\lambda^2 + \frac{2}{\lambda} - 3\right)$$

$$p = 2c_1\left(\lambda^2 - \frac{1}{\lambda}\right)$$

$$\varphi(F) - \frac{1}{3}p = \frac{1}{3}c_1\left(\lambda^2 + \frac{8}{\lambda} - 9\right)$$

Let us consider again the case where the body is constrained between two walls in such a way that

$$\lambda(t)\gamma(t) = \lambda_0 \qquad t \geq 0$$

with $\lambda(0) = \lambda_0$ and $\gamma(0) = 1$.

From the condition above we get again

$$\frac{\dot\lambda}{\lambda} = -\frac{\dot\gamma}{\gamma}$$

leading to the evolution equation for the elastic stretch

$$\boxed{\eta\frac{\dot\lambda}{\lambda} = \frac{1}{3}c_1\left(\lambda^2 + \frac{8}{\lambda} - 9\right) - q} \qquad (6)$$