[2016-04-26]

Incompressible hyperelastic material

$$\hat{S}(F)\cdot\dot{F} = \frac{d}{dt}\varphi(F) \qquad \forall \text{ isochoric motion}$$

isochoric

$$\underset{\downarrow\, 1}{(\det F)}\,\hat{T}(F)\cdot\overbrace{\dot{F}F^{-1}}^{} = \frac{d}{dt}\varphi(F)$$

$$\hat{T}(F) = \operatorname{dev}\hat{T}(F) + \operatorname{sph}\hat{T}(F)$$

$$\operatorname{dev}\hat{T}(F)\cdot\dot{F}F^{-1} = \frac{d}{dt}\varphi(F)$$

$$T = \hat{T}_e(F) - pI \qquad \operatorname{tr}\hat{T}_e(F) = 0$$

neo-Hookean material

$$\varphi_H(F) = c_1(\iota_1 - 3)$$

$$\hat{S}(F) = 2c_1F \;;\; \hat{T}(F) = 2c_1FF^T$$

$$\operatorname{dev}\hat{T}(F) = 2c_1\left(FF^T - \frac{1}{3}\operatorname{tr}(FF^T)I\right) \quad \leftarrow \hat{T}_e(F)$$

$$\operatorname{dev}\hat{S}(F) = 2c_1\left(F - \frac{1}{3}\operatorname{tr}(FF^T)F^{-T}\right) \quad \leftarrow \hat{S}_e(F)$$

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Uniaxial deformation (for incompressible materials)

$$-p\,e_1 \leftarrow \square \rightarrow p\,e_1$$

$$[F] = \begin{pmatrix} \lambda & & \\ & 1/\sqrt{\lambda} & \\ & & 1/\sqrt{\lambda} \end{pmatrix}$$

$$M = A_{\mathcal{F}_1}\, p\, e_1 \otimes (\lambda\, \ell_1\, e_1)$$
$$= V_{\mathcal{R}}\, p \quad e_1 \otimes e_1$$

$$f^{ext}/V_{\mathcal{R}} = b + \frac{t_1^+ - t_1^-}{h_1} + \dots = 0$$

$$b \downarrow 0 \qquad t_1^+ = p\, e_1 \quad -t_1^- = -p\, e_1$$

$$\mathrm{skw}\, M = 0$$

$$M/V_{\mathcal{R}} = p\; e_1 \otimes e_1$$

$$p\, e_1 \otimes e_1 = \hat{T}_e(F) - pI \qquad \text{balance}$$

$$\varphi(F) = c_1\,(\iota_1 - 3) \qquad \text{neo-Hookean}$$

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$$[C] = [F^T F] = \begin{pmatrix} \lambda^2 & & \\ & 1/\lambda & \\ & & 1/\lambda \end{pmatrix}$$

$$\iota_1 = \operatorname{tr} C = \lambda^2 + \frac{2}{\lambda} \qquad \iota_3 = 1$$

$$\frac{d}{dt}\iota_1 = \left(2\lambda - \frac{2}{\lambda^2}\right)\dot{\lambda} = 2\left(\lambda^2 - \frac{1}{\lambda}\right)\frac{\dot{\lambda}}{\lambda}$$

$$[\dot{F} F^{-1}] = \begin{pmatrix} 1 & & \\ & -\frac{1}{2}\lambda^{-\frac{3}{2}} & \\ & & -\frac{1}{2}\lambda^{-\frac{3}{2}} \end{pmatrix} \begin{pmatrix} \lambda^{-1} & & \\ & \lambda^{\frac{1}{2}} & \\ & & \lambda^{\frac{1}{2}} \end{pmatrix} \dot{\lambda}$$

$$= \begin{pmatrix} \lambda^{-1} & & \\ & -\frac{1}{2}\lambda^{-1} & \\ & & -\frac{1}{2}\lambda^{-1} \end{pmatrix} \dot{\lambda} = \begin{pmatrix} 1 & & \\ & -\frac{1}{2} & \\ & & -\frac{1}{2} \end{pmatrix} \frac{\dot{\lambda}}{\lambda}$$

$$[T] = \begin{pmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{pmatrix} \qquad \hat{T}_e(F) = \begin{pmatrix} \hat{\sigma}_1(\lambda) & & \\ & \hat{\sigma}_2(\lambda) & \\ & & \hat{\sigma}_3(\lambda) \end{pmatrix}$$

$$\operatorname{tr} \hat{T}_e(F) = 0$$

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$$\hat{T}(F) \cdot \dot{F}F^{-1} = \left(\hat{\sigma}_1(\lambda) - \frac{1}{2}\left(\hat{\sigma}_2(\lambda) + \hat{\sigma}_3(\lambda)\right)\right)\frac{\dot{\lambda}}{\lambda}$$

$$\mathrm{tr}\, \hat{T}_e(F) = 0 \quad \Rightarrow \quad \hat{\sigma}_2(\lambda) + \hat{\sigma}_3(\lambda) = -\hat{\sigma}_1(\lambda)$$

$$\hat{T}_e(F) \cdot \dot{F}F^{-1} = \frac{3}{2}\hat{\sigma}_1(\lambda)\frac{\dot{\lambda}}{\lambda}$$

$$\varphi(F) = c\,(\iota_1 - 3) \Rightarrow \frac{d}{dt}\varphi(F) = 2c\left(\lambda^2 - \frac{1}{\lambda}\right)\frac{\dot{\lambda}}{\lambda}$$

$$\Rightarrow \quad \frac{3}{2}\hat{\sigma}_1(\lambda) = 2c\left(\lambda^2 - \frac{1}{\lambda}\right)$$

$$T = \hat{T}_e(\lambda) - pI \qquad\qquad T = M/V_\Omega$$

$$\sigma_1 = \hat{\sigma}_1(\lambda) - p \qquad\qquad \sigma_1 = \pi$$

$$\sigma_2 = \hat{\sigma}_2(\lambda) - p \qquad\qquad \sigma_2 = 0$$

$$\sigma_3 = \hat{\sigma}_3(\lambda) - p \qquad\qquad \sigma_3 = 0$$

$$\hat{\sigma}_1(\lambda) - p = \pi$$

$$\hat{\sigma}_2(\lambda) - p = 0$$

$$\hat{\sigma}_3(\lambda) - p = 0$$

taking the trace

$$0 \; - 3p = \pi \qquad \Rightarrow \qquad p = -\frac{1}{3}\pi$$

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$$\hat{\sigma}_1(\lambda) = \frac{4}{3} c \left(\lambda^2 - \frac{1}{\lambda}\right)$$

$$\frac{4}{3} c \left(\lambda^2 - \frac{1}{\lambda}\right) + \frac{1}{3} p = p$$

$$\hat{\sigma}_2(\lambda) - \hat{\sigma}_3(\lambda) = 0$$

$$\hat{\sigma}_2(\lambda) + \hat{\sigma}_3(\lambda) - 2p = 0$$

$$\hat{\sigma}_2(\lambda) + \hat{\sigma}_3(\lambda) = -\hat{\sigma}_1(\lambda) = -\frac{4}{3} c \left(\lambda^2 - \frac{1}{\lambda}\right)$$

$$\Rightarrow \quad 2\,\hat{\sigma}_2(\lambda) = -\frac{4}{3} c \left(\lambda^2 - \frac{1}{\lambda}\right)$$

$$\hat{\sigma}_2(\lambda) = -\frac{2}{3} c \left(\lambda^2 - \frac{1}{\lambda}\right)$$

$$\hat{\sigma}_3(\lambda) = -\frac{2}{3} c \left(\lambda^2 - \frac{1}{\lambda}\right)$$

$$\frac{4}{3} c \left(\lambda^2 - \frac{1}{\lambda}\right) + \frac{1}{3} p = p$$

$$\boxed{2c\left(\lambda^2 - \frac{1}{\lambda}\right) = p}$$

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Equivalently, we can use the general response function

$$\hat{T}_e(F) = 2c_1\left(FF^T - \frac{1}{3}tr(FF^T)I\right) \qquad \text{neo-Hookean}$$

Since

$$[FF^T] = \begin{pmatrix} \lambda^2 & & \\ & 1/\lambda & \\ & & 1/\lambda \end{pmatrix}$$

we get

$$tr(FF^T) = \lambda^2 + \frac{2}{\lambda} \qquad \text{sph } FF^T$$

$$\lambda^2 - \frac{1}{3}\left(\lambda^2 + \frac{2}{\lambda}\right) = \frac{2}{3}\left(\lambda^2 - \frac{1}{\lambda}\right)$$

$$\frac{1}{\lambda} - \frac{1}{3}\left(\lambda^2 + \frac{2}{\lambda}\right) = -\frac{1}{3}\left(\lambda^2 - \frac{1}{\lambda}\right) \qquad \text{dev } FF^T$$

$$\hat{T}_e(F) = 2c_1\begin{pmatrix} \frac{2}{3} & & \\ & -\frac{1}{3} & \\ & & -\frac{1}{3} \end{pmatrix}\left(\lambda^2 - \frac{1}{\lambda}\right) \qquad \text{deviatoric response}$$

$$\hat{T}_e(F) - pI = M/V_R \qquad \text{balance}$$

$$\begin{cases} dev\left(\hat{T}_e(F) - pI\right) = dev\,(M/V_R) \\ sph\left(\hat{T}_e(F) - pI\right) = sph\,(M/V_R) \end{cases}$$

$$\frac{M}{V_R} = \mu\, e_1 \otimes e_1 = \mu\left(e_1 \otimes e_1 - \frac{1}{3}I + \frac{1}{3}I\right)$$

$$\left.\begin{array}{ll} dev & \hat{T}_e(F) = \mu\left(e_1 \otimes e_1 - \frac{1}{3}I\right) \\ sph & -p = \mu\,\frac{1}{3} \end{array}\right\} \text{balance}$$

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$$\begin{cases} \frac{4}{3} c\left(\lambda^2 - \frac{1}{\lambda}\right) = p - \frac{1}{3} p \\ -\frac{2}{3} c\left(\lambda^2 - \frac{1}{\lambda}\right) = -\frac{1}{3} p \\ -\frac{2}{3} c\left(\lambda^2 - \frac{1}{\lambda}\right) = -\frac{1}{3} p \end{cases}$$

dev balance

$$\boxed{2c\left(\lambda^2 - \frac{1}{\lambda}\right) = p}$$

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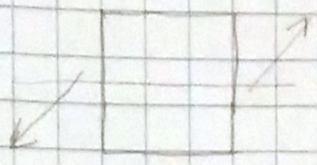
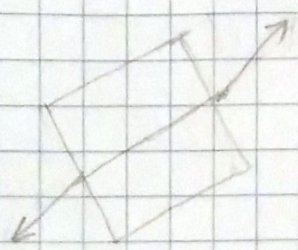
[2016-05-04]

Uniaxial stretch and rotation



$$[F] = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda & & \\ & \frac{1}{\sqrt{\lambda}} & \\ & & \frac{1}{\sqrt{\lambda}} \end{pmatrix}$$



$$t_1 = p\,\frac{\sqrt{2}}{2}(e_1 + e_2)$$

$$u_1 = F\bar{u}_1 = F(l_1 e_1)$$

$$M = A_{\mathcal{F}_1}\, p\,\frac{\sqrt{2}}{2}(e_1+e_2)\otimes F(l_1 e_1)$$

$$= A_{\mathcal{F}_1}\, p\,\frac{\sqrt{2}}{2}\, l_1\,(e_1+e_2)\otimes R(\lambda e_1)$$

$$= A_{\mathcal{F}_1}\, p\,\frac{\sqrt{2}}{2}\,\lambda l_1\,(e_1+e_2)\otimes(\cos\theta\, e_1 + \sin\theta\, e_2)$$

$$= \frac{\sqrt{2}}{2} V_{\mathcal{B}}\, p\,\Big(\cos\theta\,(e_1\otimes e_1 + e_2\otimes e_1) + \sin\theta\,(e_1\otimes e_2 + e_2\otimes e_2)\Big)$$

$$\operatorname{skw} M = \frac{\sqrt{2}}{2} V_{\mathcal{B}}\, p\,\frac{1}{2}\Big(\cos\theta\,(e_2\otimes e_1 - e_1\otimes e_2) + \sin\theta\,(e_1\otimes e_2 - e_2\otimes e_1)\Big)$$

$$= \frac{\sqrt{2}}{2} V_{\mathcal{B}}\, p\,\frac{1}{2}(\cos\theta - \sin\theta)(e_2\otimes e_1 - e_1\otimes e_2)$$

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$$\mathrm{skw}\, M = 0 \quad \Rightarrow \quad \cos\vartheta = \sin\vartheta$$

$$\pm\frac{\sqrt{2}}{2} = \pm\frac{\sqrt{2}}{2}$$

$$\cos\vartheta = \frac{\sqrt{2}}{2}, \quad \sin\vartheta = \frac{\sqrt{2}}{2}$$

$$[R] = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} \qquad [F] = \frac{\sqrt{2}}{2}\begin{pmatrix} \lambda & -1/\sqrt{\lambda} & 0 \\ \lambda & 1/\sqrt{\lambda} & 0 \\ 0 & 0 & \sqrt{2}/\sqrt{\lambda} \end{pmatrix}$$

$$\cos\vartheta = -\frac{\sqrt{2}}{2}, \quad \sin\vartheta = -\frac{\sqrt{2}}{2}$$

$$[R] = -\frac{\sqrt{2}}{2}\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -\sqrt{2} \end{pmatrix} \qquad [F] = -\frac{\sqrt{2}}{2}\begin{pmatrix} \lambda & -1/\sqrt{\lambda} & 0 \\ \lambda & 1/\sqrt{\lambda} & 0 \\ 0 & 0 & -\sqrt{2}/\sqrt{\lambda} \end{pmatrix}$$

$$\varphi(F) = \varphi(U)$$

$$[U] = \begin{pmatrix} \lambda & & \\ & 1/\sqrt{\lambda} & \\ & & 1/\sqrt{\lambda} \end{pmatrix}$$

$$\hat{T}(F) \cdot \dot{F}F^T = \dot{\varphi}(U)$$

It is convenient to resort to the general expression for the response function

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$\theta = \frac{\pi}{4}$ (uniaxial rotated)

$$[B] = \begin{pmatrix} \frac{1+\lambda^3}{2\lambda} & \frac{-1+\lambda^3}{2\lambda} & 0 \\ \frac{-1+\lambda^3}{2\lambda} & \frac{1+\lambda^3}{2\lambda} & 0 \\ 0 & 0 & \frac{1}{\lambda} \end{pmatrix}$$

$$\hat{T}_e(F) = 2c\left(B - \frac{1}{3}(\mathrm{tr}\, B)I\right)$$ neo-Hookean

$$T = \hat{T}_e(F) - pI$$

$$[\hat{T}_e(F)] = 2c\,\frac{\lambda^3 - 1}{2\lambda} \begin{pmatrix} \frac{1}{3} & 1 & 0 \\ 1 & \frac{1}{3} & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix}$$

$$[M] = \frac{1}{2}\mu \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} V_R$$

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$$\operatorname{tr}\left(T - M/V_R\right) = -3p - \pi$$

$$\operatorname{dev}\left(T - M/V_R\right) = \left(2c\left(\lambda^2 - \frac{1}{\lambda}\right) - \pi\right) \begin{pmatrix} \frac{1}{6} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{6} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}$$

$$\boxed{\begin{aligned} \pi + 3p &= 0 \\ 2c\left(\lambda^2 - \frac{1}{\lambda}\right) &= \pi \end{aligned}}$$

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