Algorithms for UNRELIABLE Distributed Systems:

The consensus problem

Failures in Distributed Systems

Let us go back to the **MPS** model; it may undergo the following malfunctioning, among others:

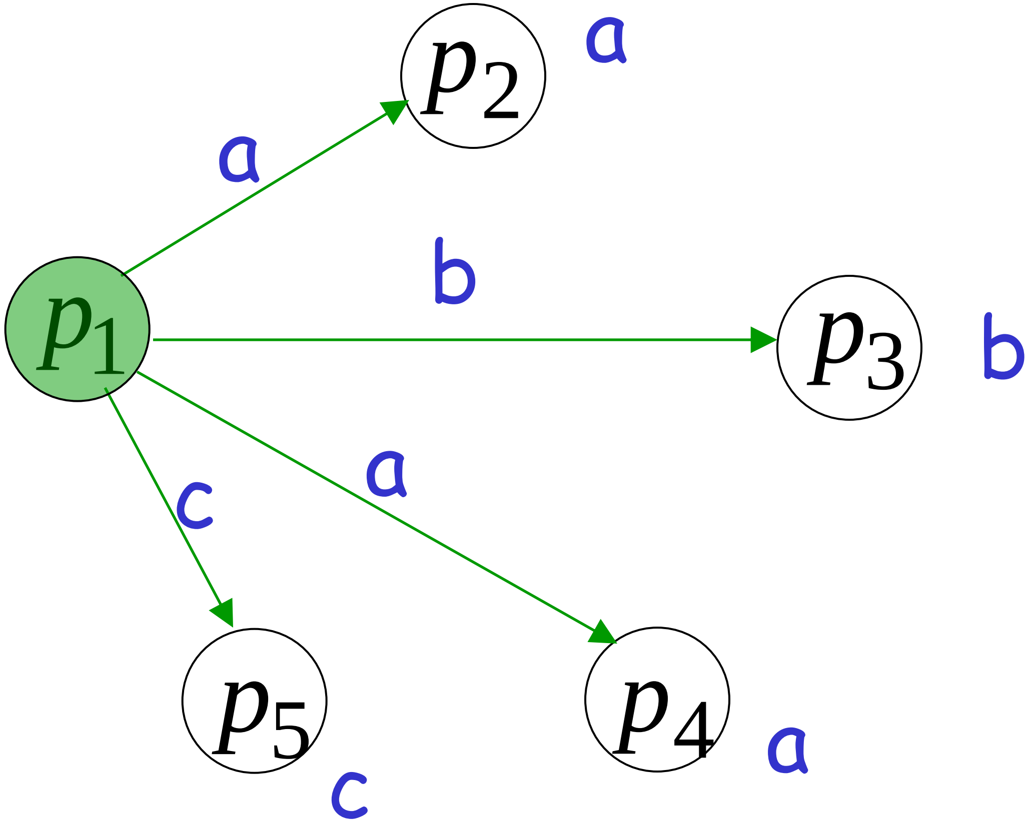
Link failure: A link fails and remains inactive for some time; the network may get disconnected

Processor crash (or benign) failure: At some point, a processor stops forever taking steps; also in this case, the network may get disconnected

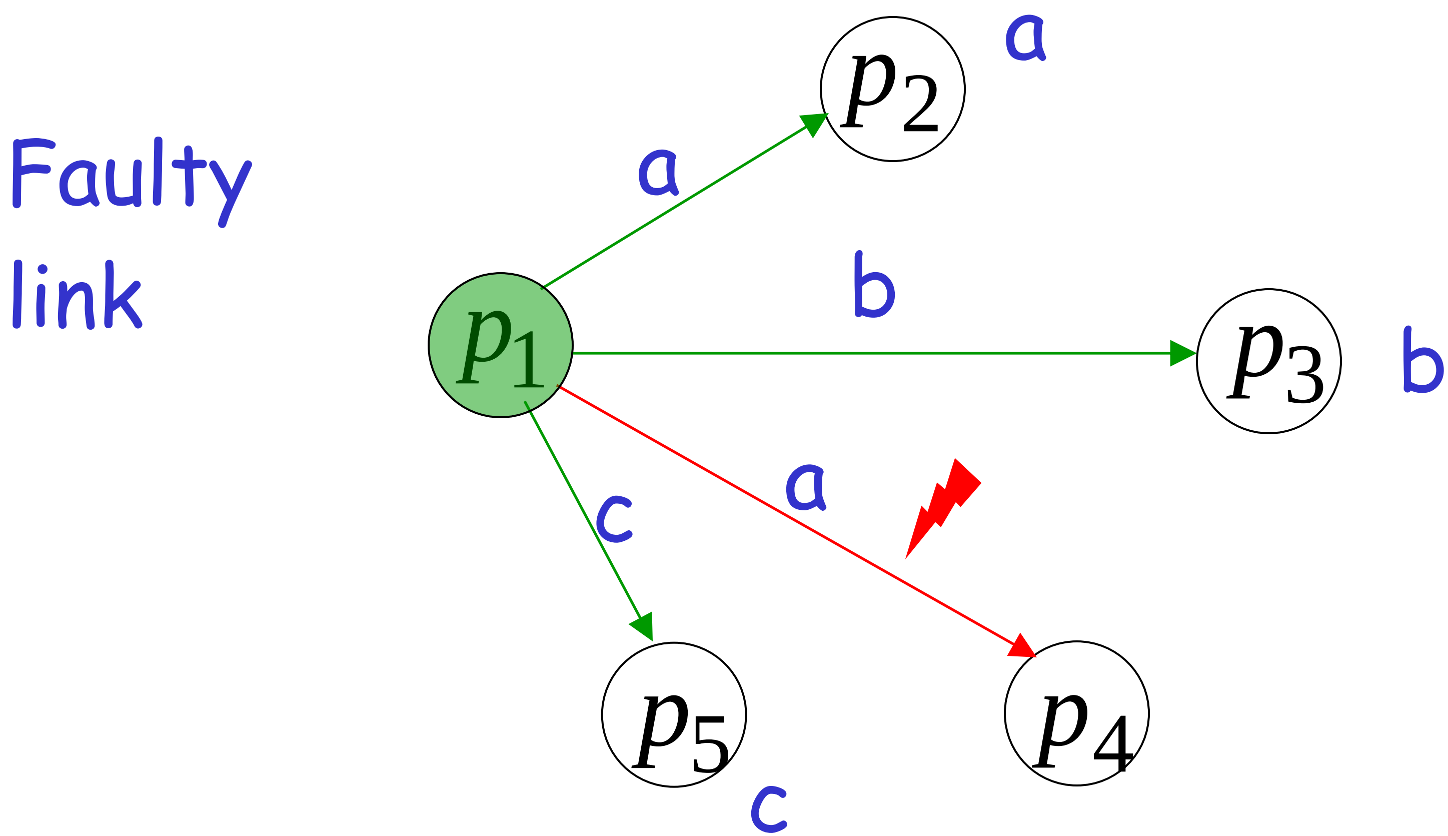
Processor Byzantine (or malicious) failure: during the execution, a processor changes state arbitrarily and sends messages with arbitrary content (name dates back to untrustable Byzantine Generals of Byzantine Empire, IV–XV century A.D.); also in this case, the network may get disconnected

Normal operating

Non-faulty links and nodes

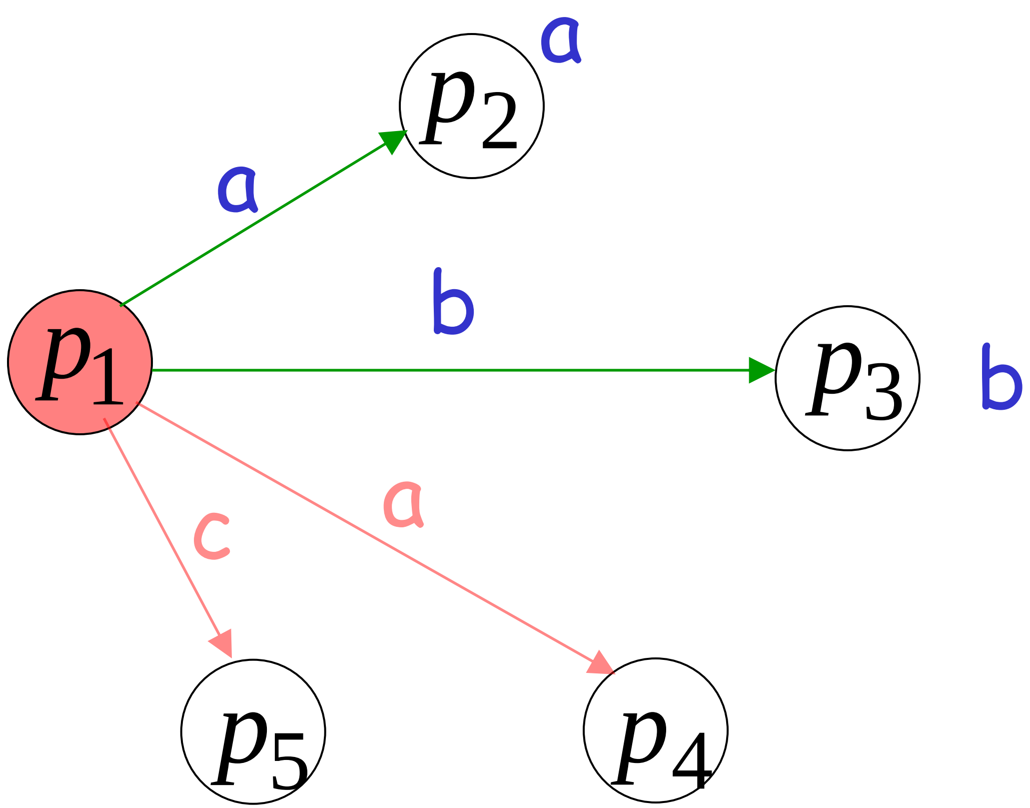


Link (non-permanent) Failures

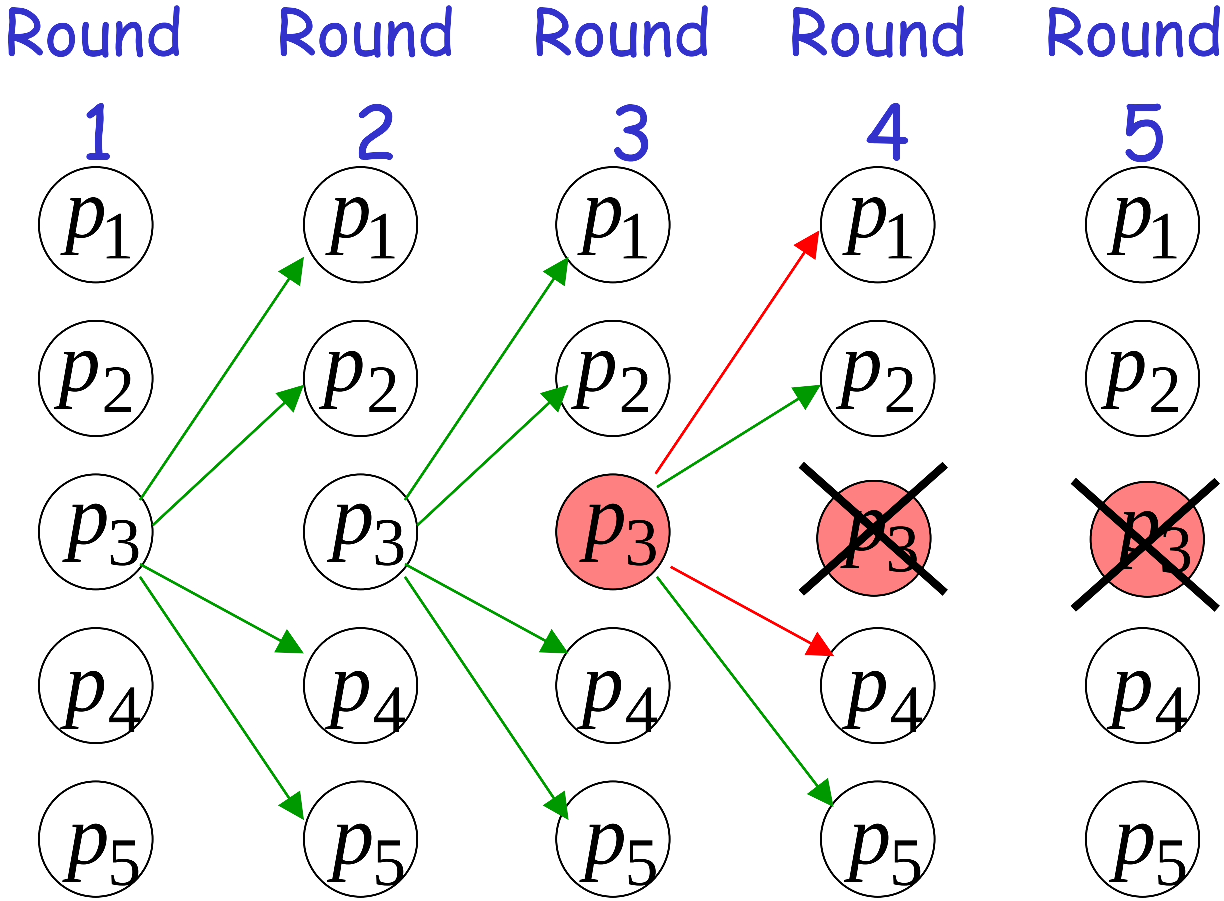


Messages sent on the failed link
are not delivered (for some time), but
they cannot be corrupted

Processor (permanent) crash failure



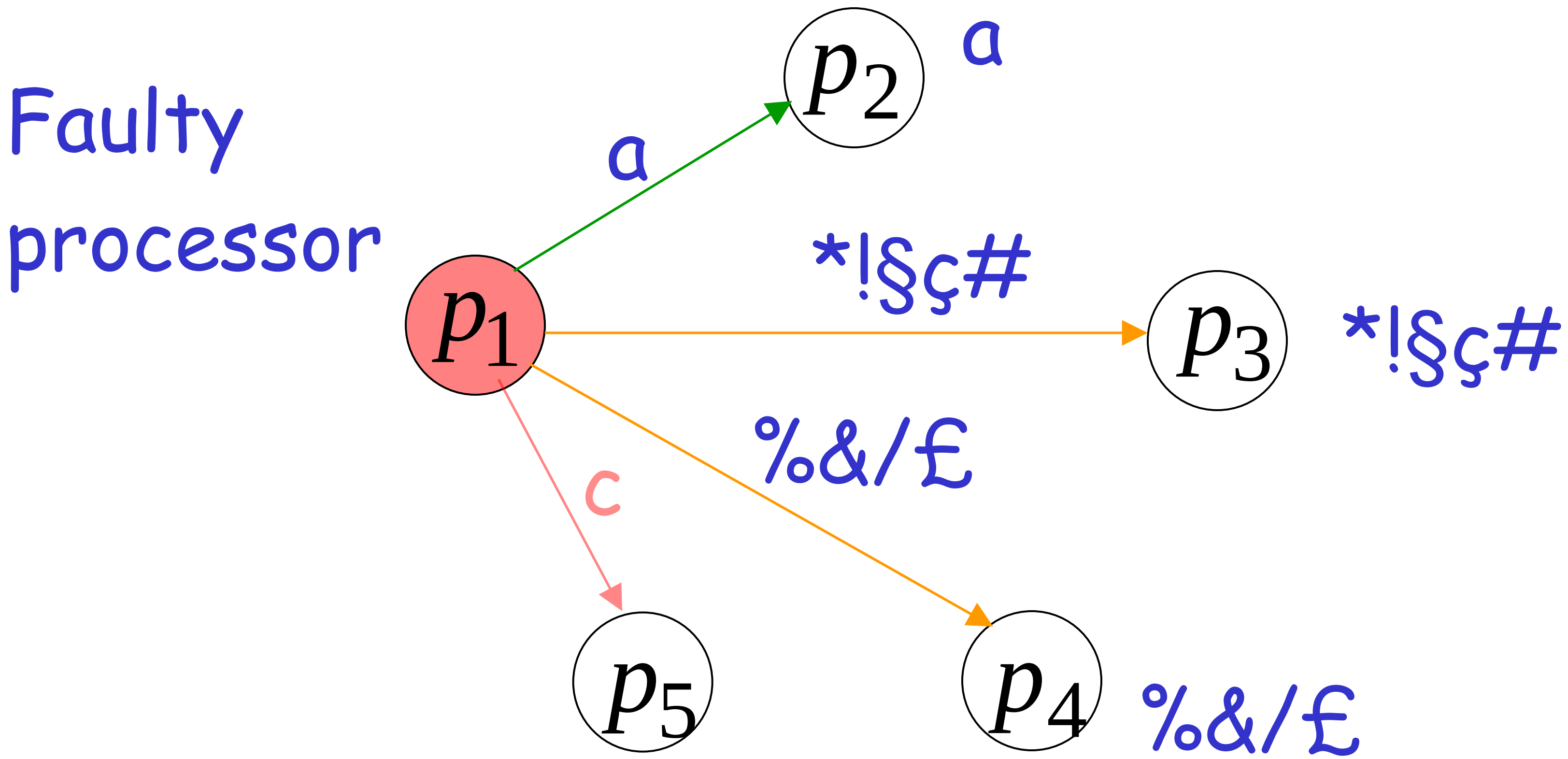
Some of the messages are not sent (forever)



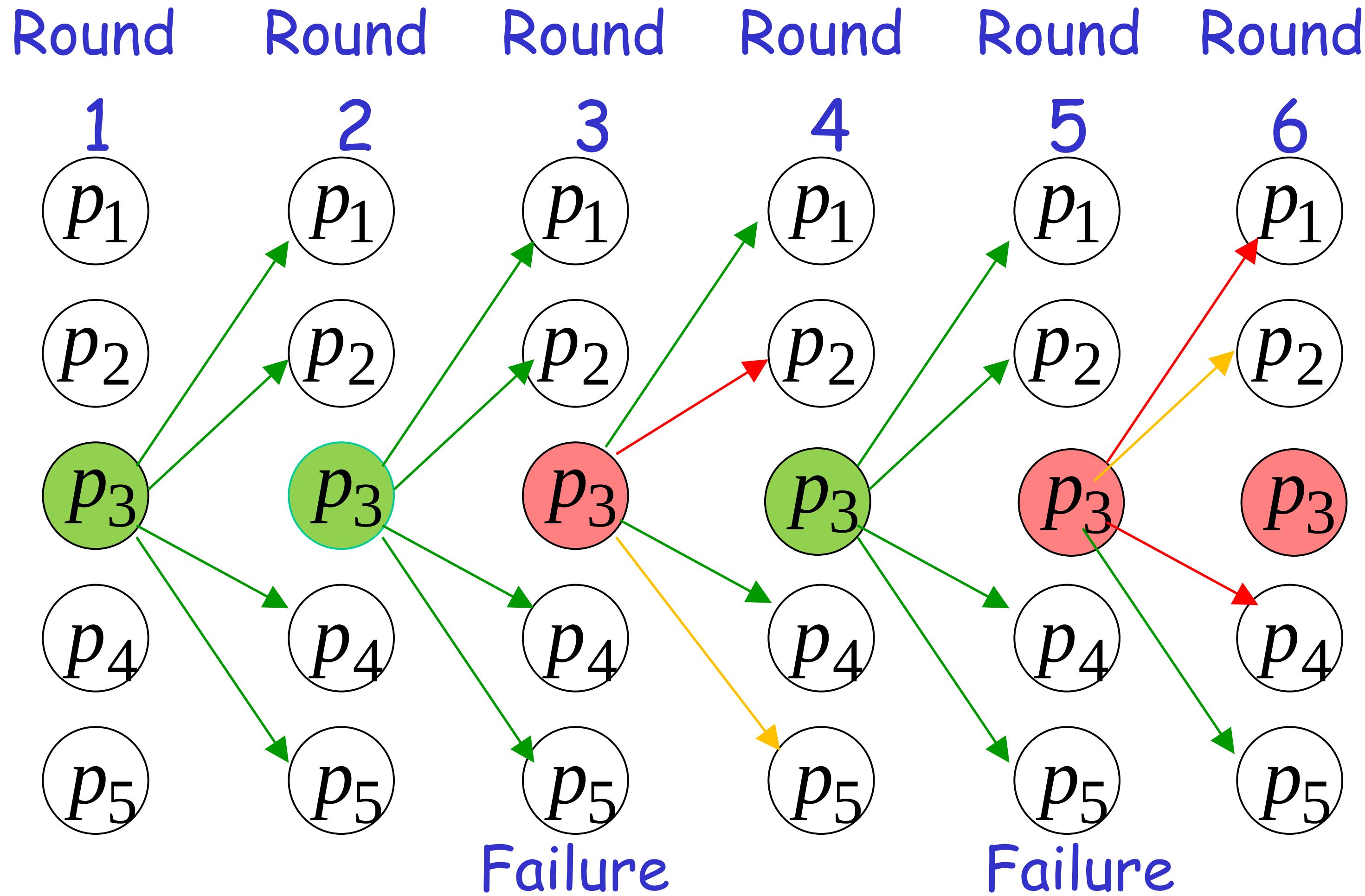
Crash failure in a synchronous MPS

After failure the processor disappears from the network

Processor Byzantine failure



Processor sends arbitrary messages (i.e., they could be either correct or corrupted), plus some messages may be not sent



Byzantine failure in a synchronous MPS

After failure the processor may continue functioning in the network

Consensus Problem

Every processor has an input x∈X (notice that in this way the algorithms running at the processors will depend on their input), and must decide an output y∈Y. Assume that link or node failures can possibly take place in the system. Then, design an algorithm enjoying the following properties:

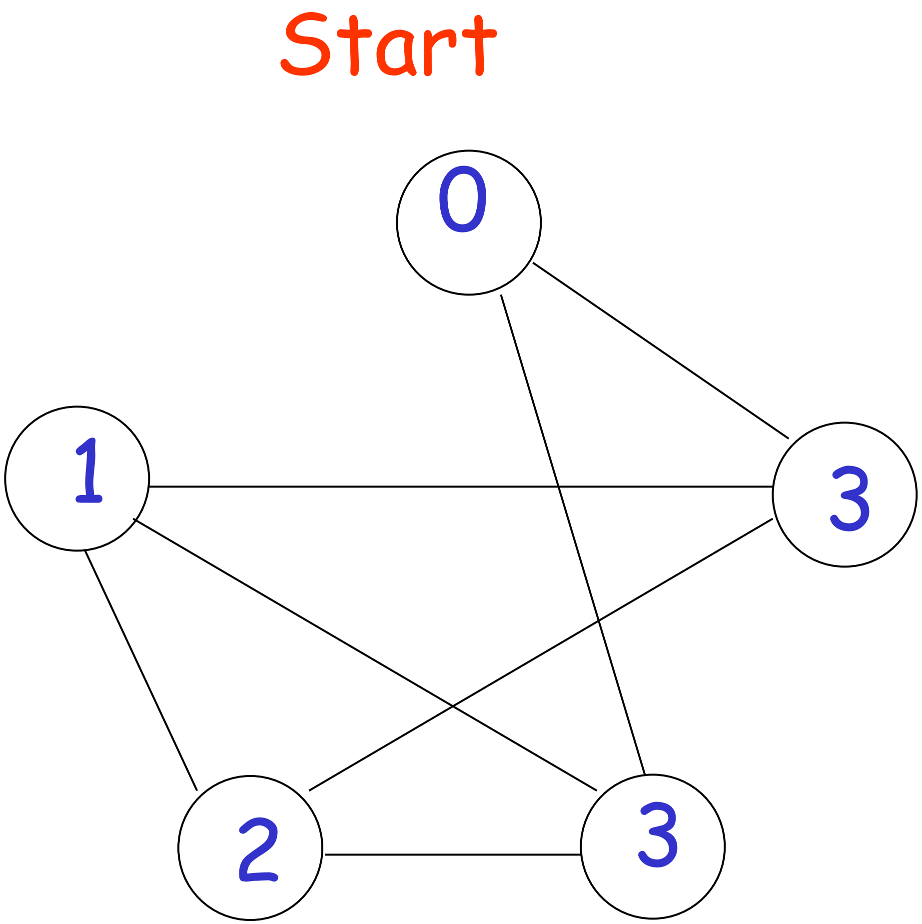
Termination: Eventually, every non-faulty processor decides on a value y∈Y.

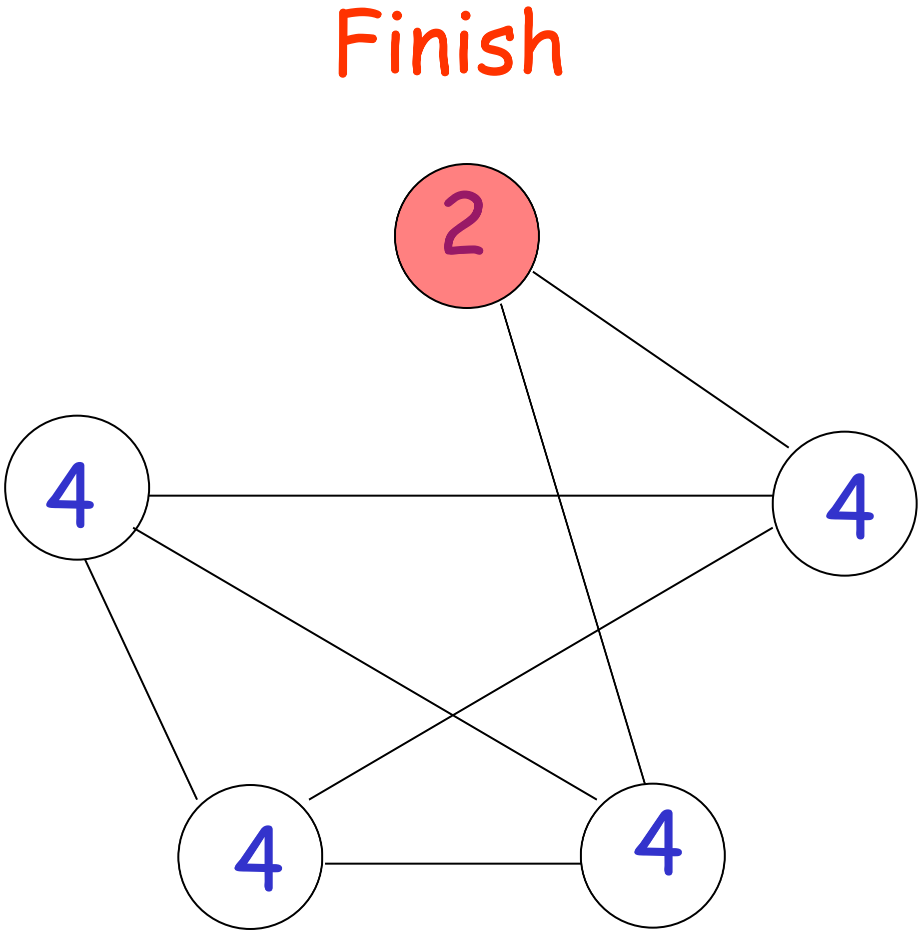
Agreement: All decisions by **non-faulty** processors must be the same.

Validity: If all inputs are the same, then the decision of a non-faulty processor must equal the common input (this avoids trivial solutions).

In the following, we assume that X=Y=**N**

Agreement



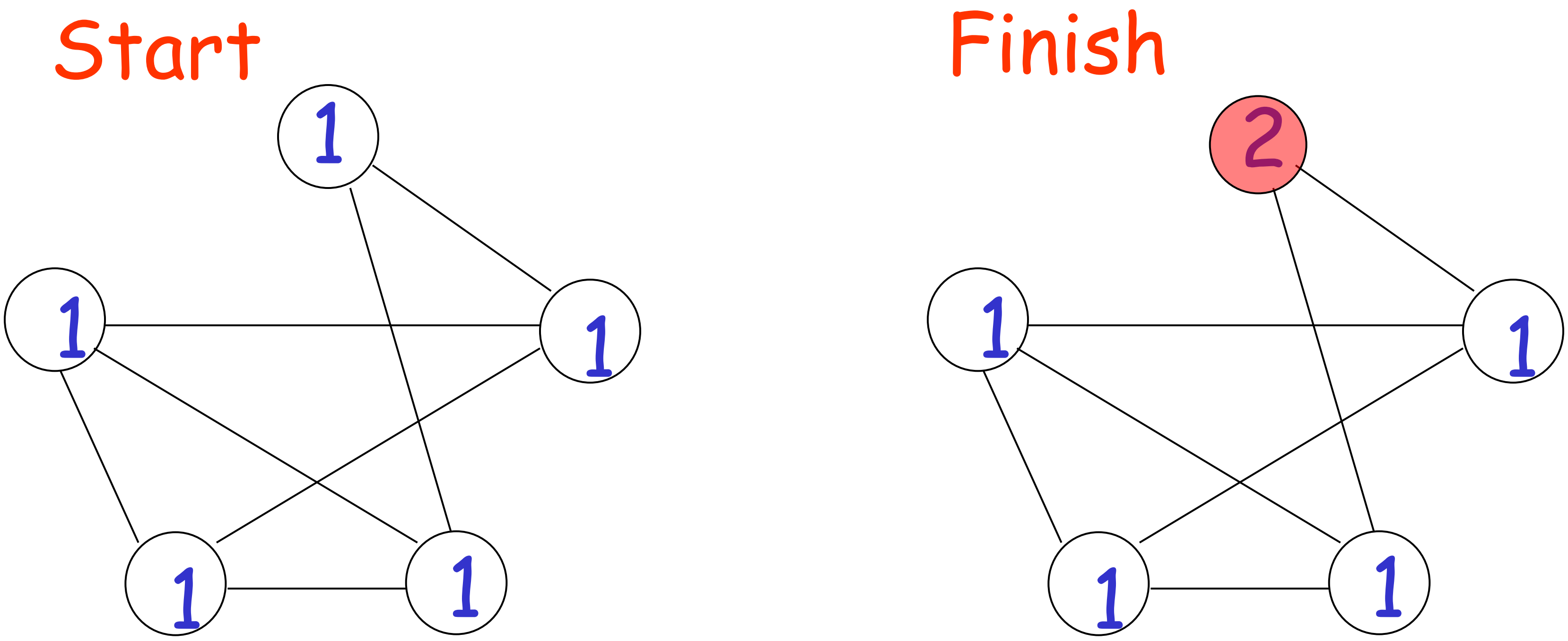


Everybody has an initial value

All non-faulty must decide the same value

Validity

If everybody starts with the same value,
then non-faulty must decide that value

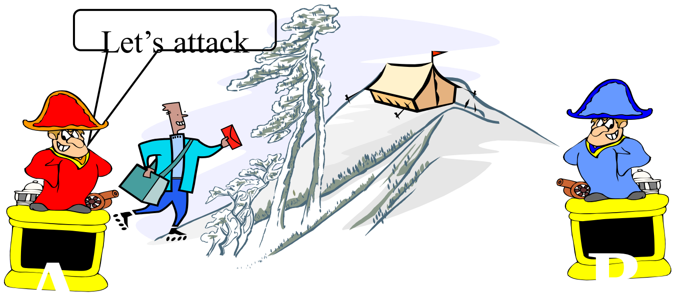


Negative result for link failures

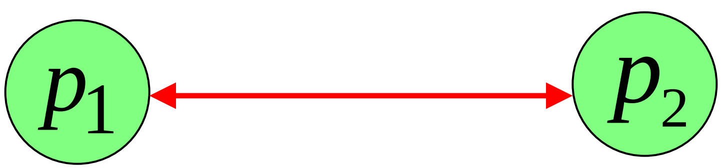
- Although this is the simplest fault a MPS may face, it may already be enough to prevent consensus
- More formally, there exist input instances for which it is impossible to reach consensus in case of single non-permanent link failures, even in the synchronous non-anonymous case
- To illustrate this negative result, we present the very famous problem of the 2 generals

Consensus under non-permanent link failures: the 2 generals problem

There are two generals of the same army who have encamped a short distance apart. Their objective is to decide on whether to capture a hill, which is possible only if they both attack (i.e., if only one general attacks, he will be defeated, and so their common output should be either "not attack" or "attack"). However, they might have different opinion about what to do (i.e., their input). The two generals can only communicate (synchronously) by sending messengers, which could be captured (i.e., link failure), though. Is it possible for them to reach a common decision?



More formally, we are talking about consensus in the following MPS:



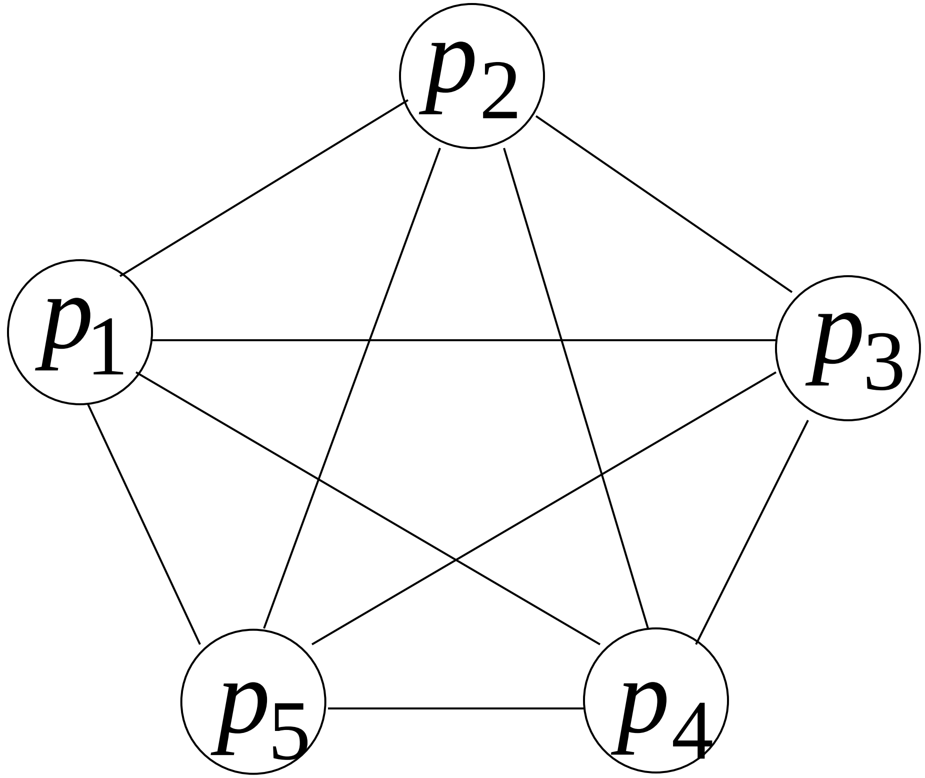
Impossibility of consensus under link failures

- First of all, notice that it is needed to exchange messages to reach consensus (as we said, generals might have different opinions in mind!)
- Assume the problem can be solved, and let Π be the shortest protocol (i.e., a solving algorithm with the minimum number of messages) for a given input configuration.
- Since this protocol is **deterministic**, for such a fixed input configuration, there will be a **sequence** of messages to be exchanged, which however may not be all successfully delivered, due to the possible link failure.
- In particular, suppose now that the last message in Π does not reach the destination (i.e., a link failure takes place). Since Π is correct independent of link failures, consensus must be reached in any case. This means, the last message was useless, and then Π could not be shortest!

Negative result for processor failures in asynchronous systems

- It is not hard to see that a processor failure (both permanent crash and byzantine) is at least as difficult as a non-permanent link failure, and then also in this case not for all the input instances it will be possible to solve the consensus problem
- But even worse, it is not hard to prove that in the asynchronous case, it is impossible to reach consensus for any system topology and already for a single crash failure!
- Notice that for the synchronous case it cannot be given a such general negative result (in the two general problems, the topology was special, in the sense that the link was a bridge, i.e., it was needed for the connectivity of the system) $\Rightarrow$ in search of some positive result, we focus on the synchronous case and we look at the powerful clique topology

Positive results: Assumption on the communication model for crash and byzantine failures



- Complete undirected graph (this implies non-uniformity)
- Synchronous network, synchronous start: w.l.o.g., we assume that messages are sent, delivered and read in the very same round

Overview of Consensus Results

f-resilient consensus algorithms (i.e., algorithms solving consensus for **at most** f faulty processors)

	Crash failures	Byzantine failures
Number of rounds	$f+1$ (tight)	$2(f+1)$ $f+1$ (tight)
Total number of processors	$n \geq f+1$ (tight)	$n \geq 4f+1$ $n \geq 3f+1$ (tight)
Message complexity	$O(n^3)$	$O(n^3)$ $O(n^{O(n)})$ (exponential)

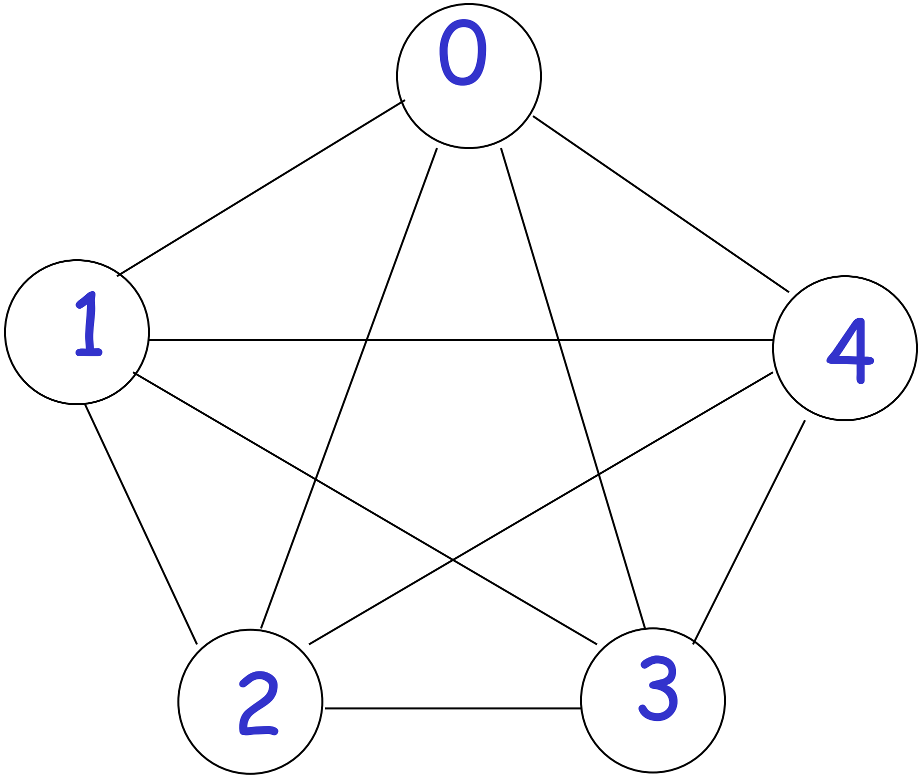
A simple algorithm for fault-free consensus

Each processor:

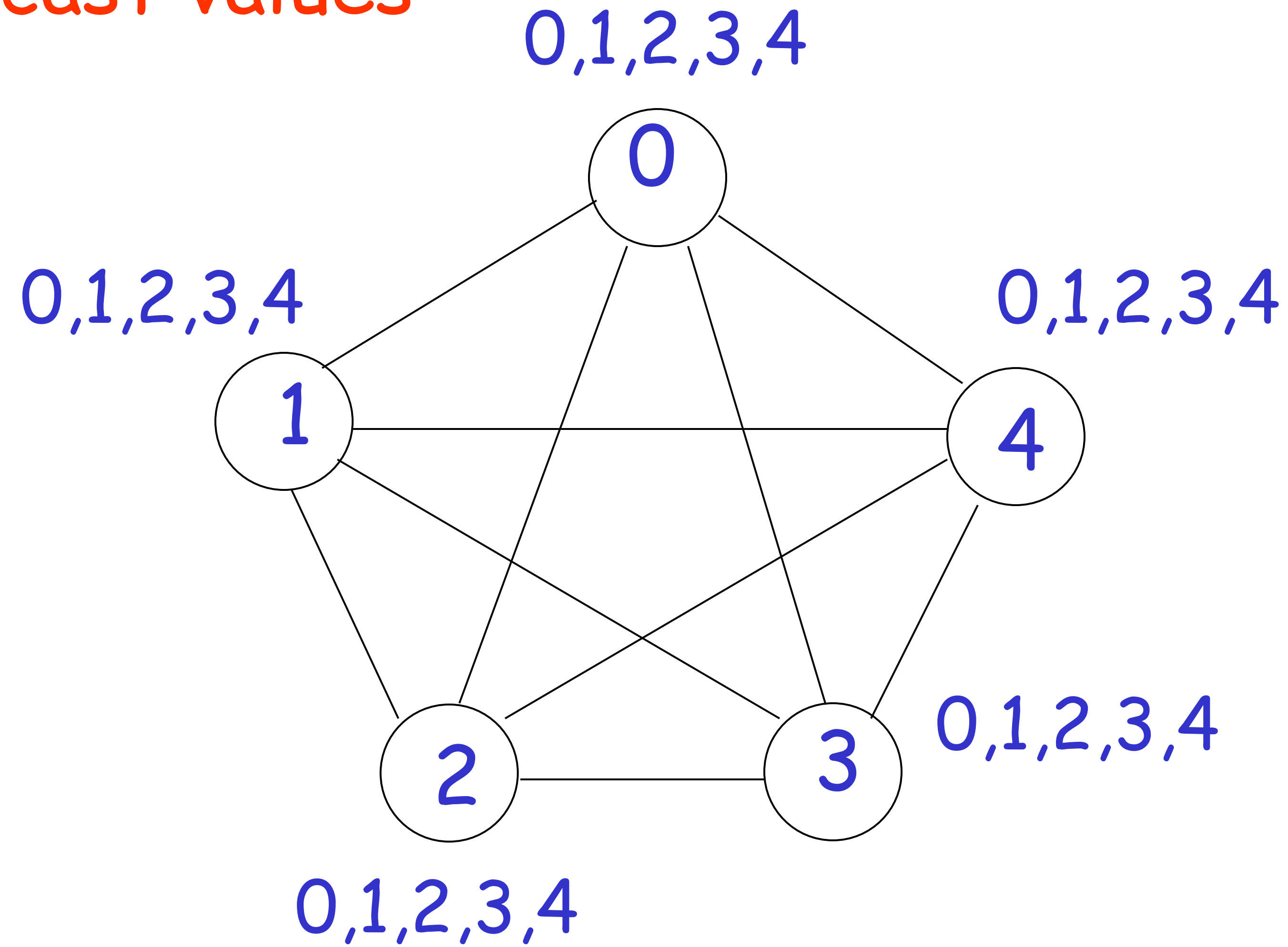
1. Broadcasts its input to all processors
2. Reads all the incoming messages
3. Decides on the minimum received value

(only one round is needed,
since the graph is complete)

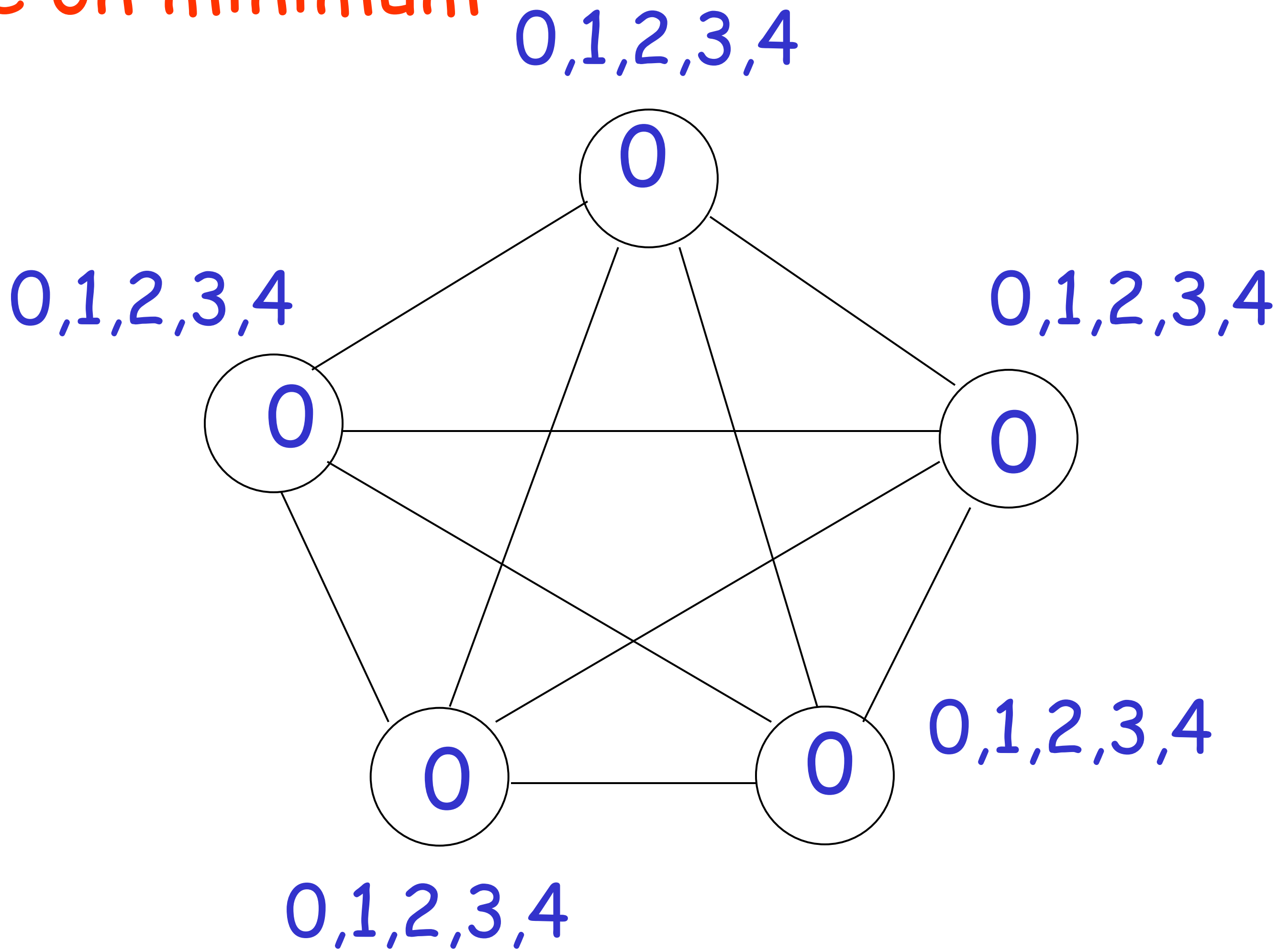
Start



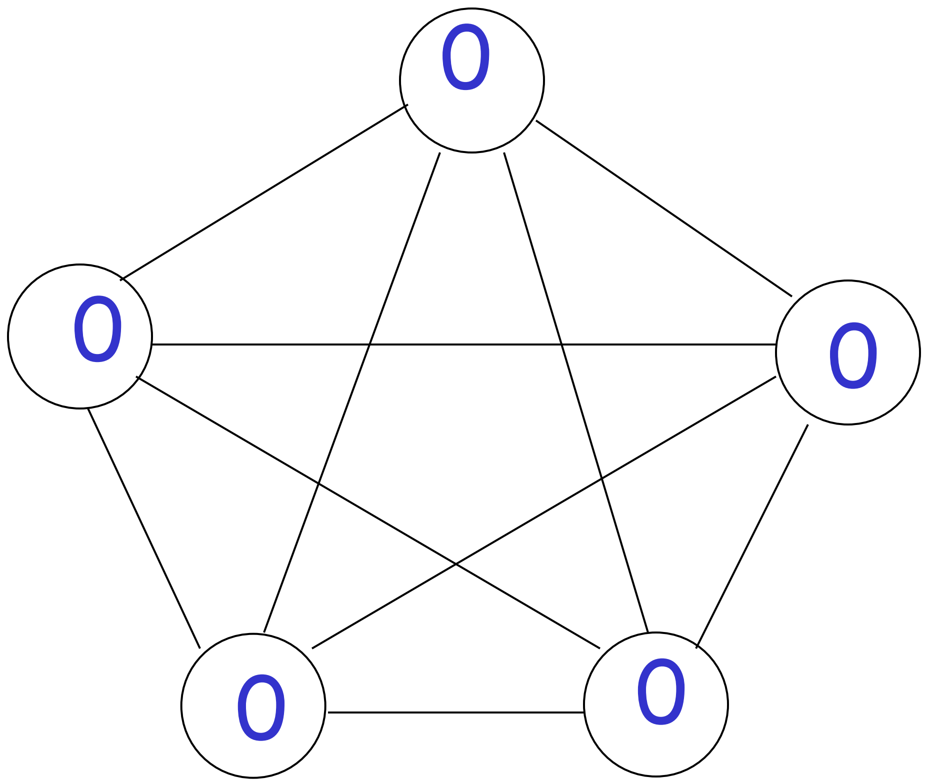
Broadcast values



Decide on minimum

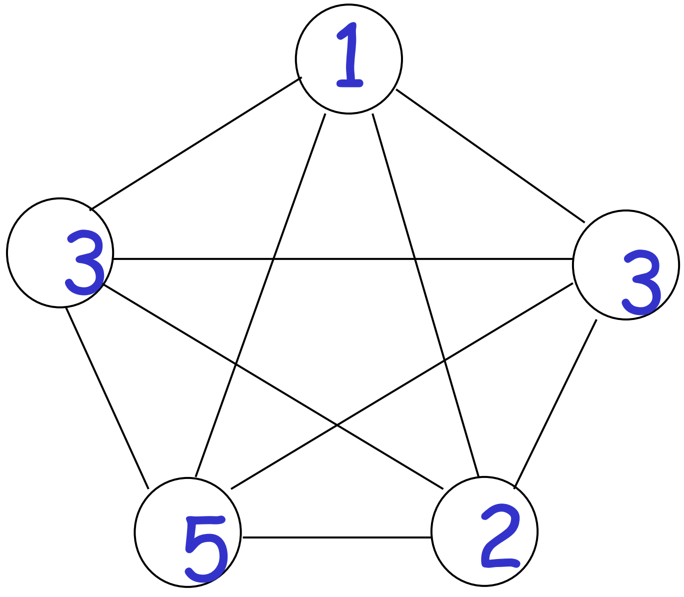


Finish



This algorithm satisfies the agreement

Start



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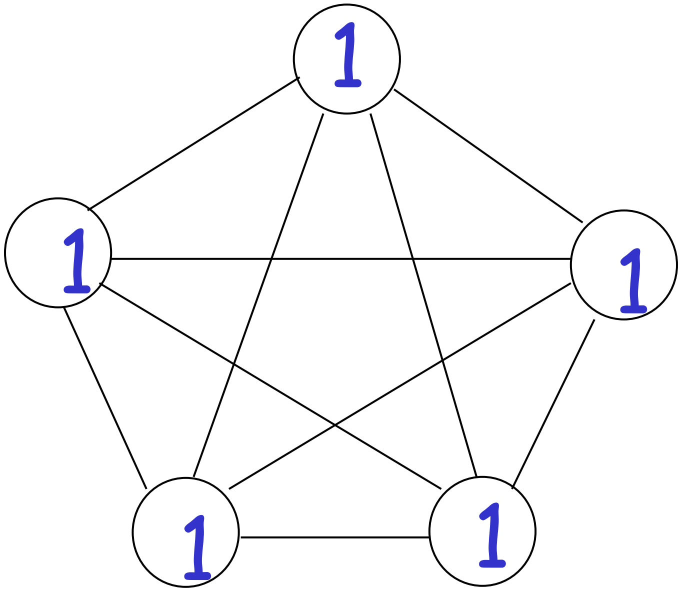
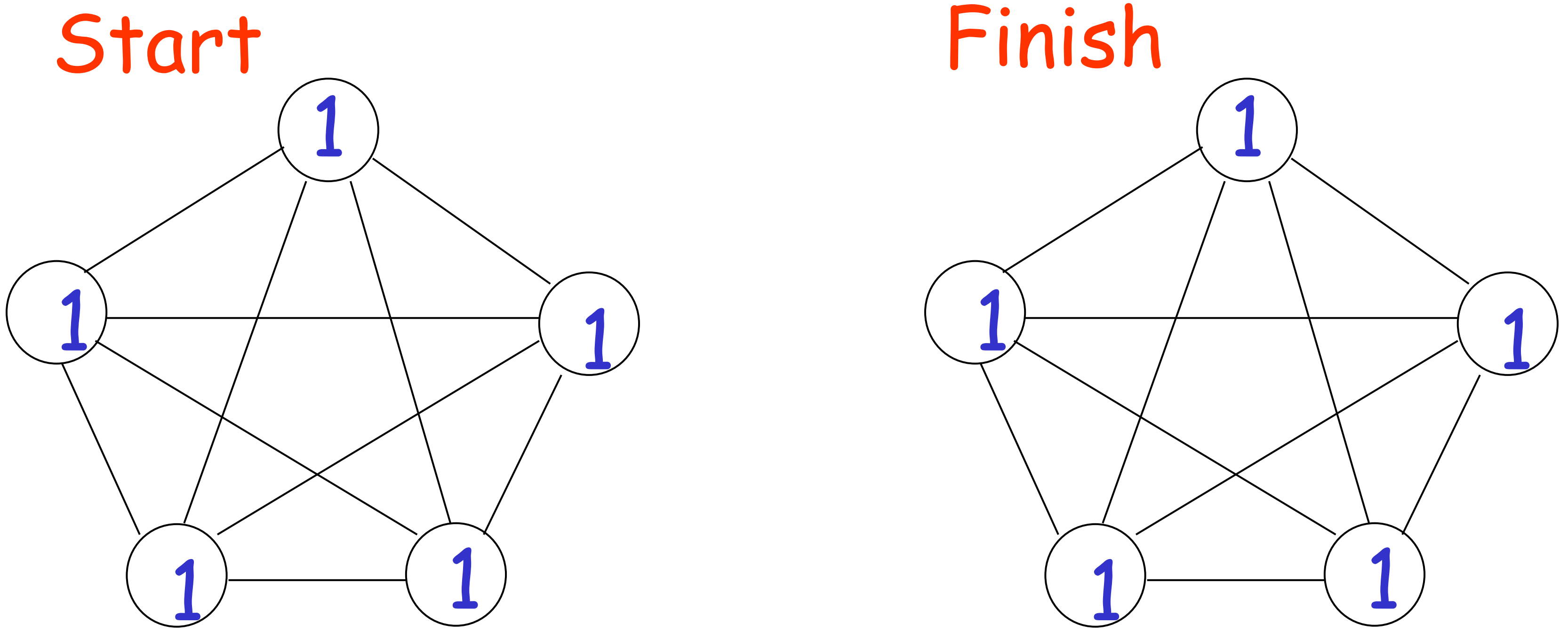


All the processors decide the minimum exactly over the same set of values

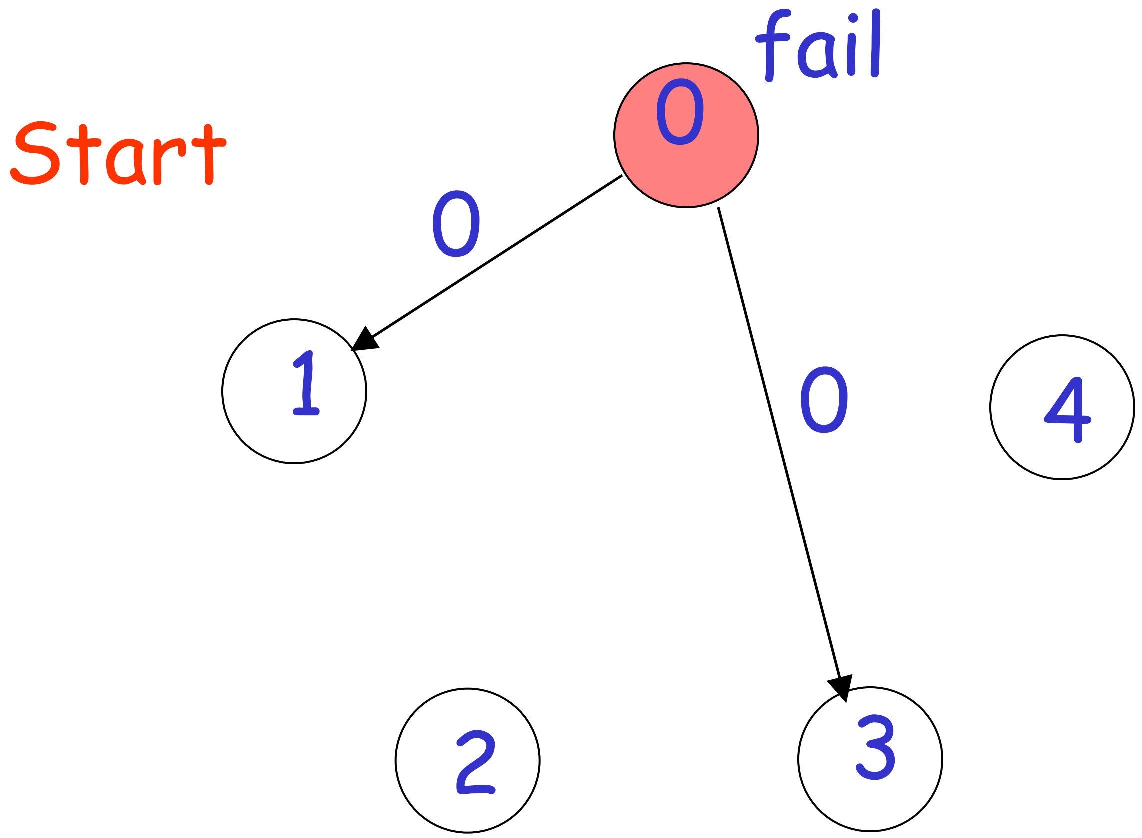
This algorithm satisfies the validity condition



If everybody starts with the same initial value, everybody decides on that value (minimum)

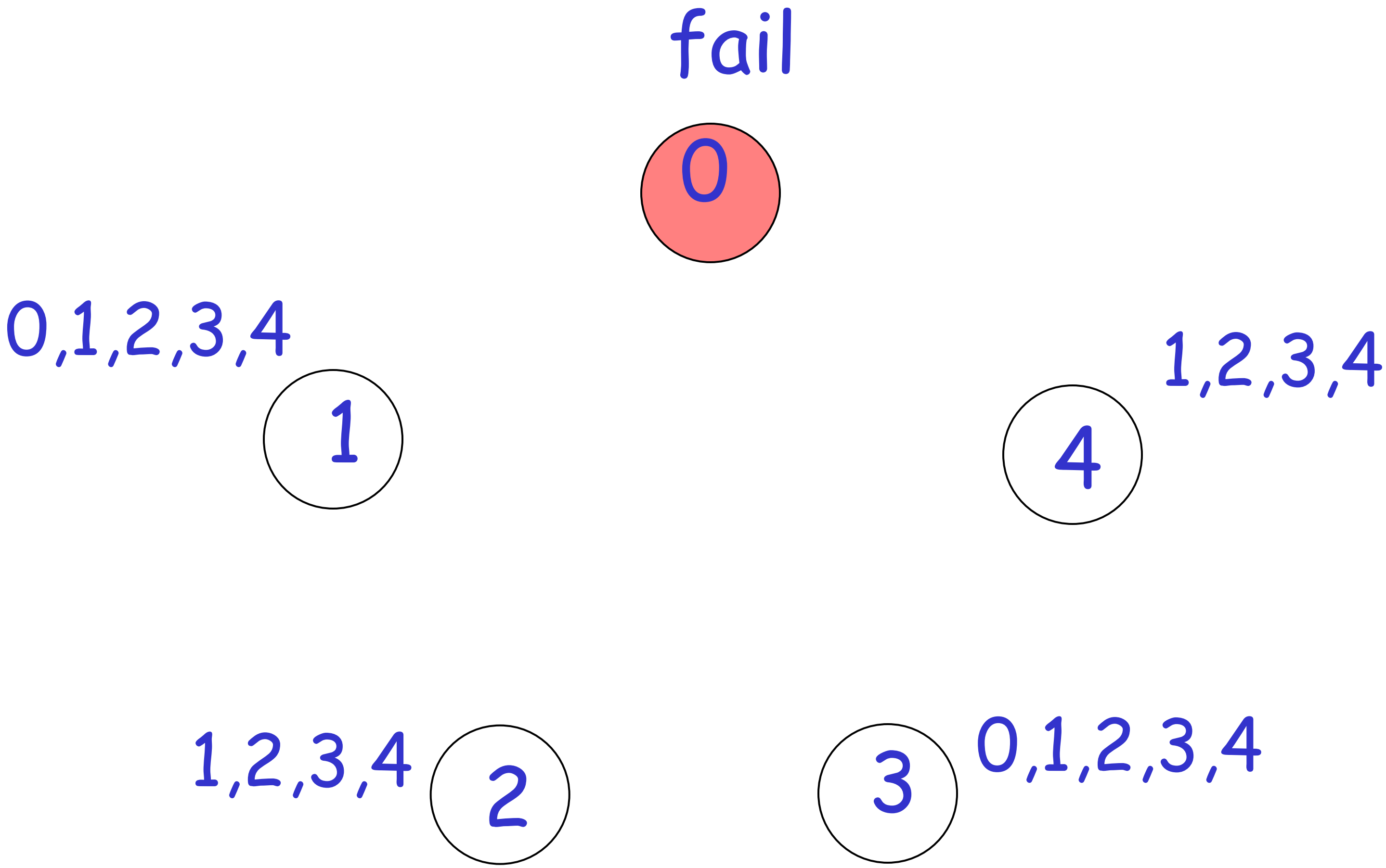
Consensus with Crash Failures

The simple algorithm doesn't work

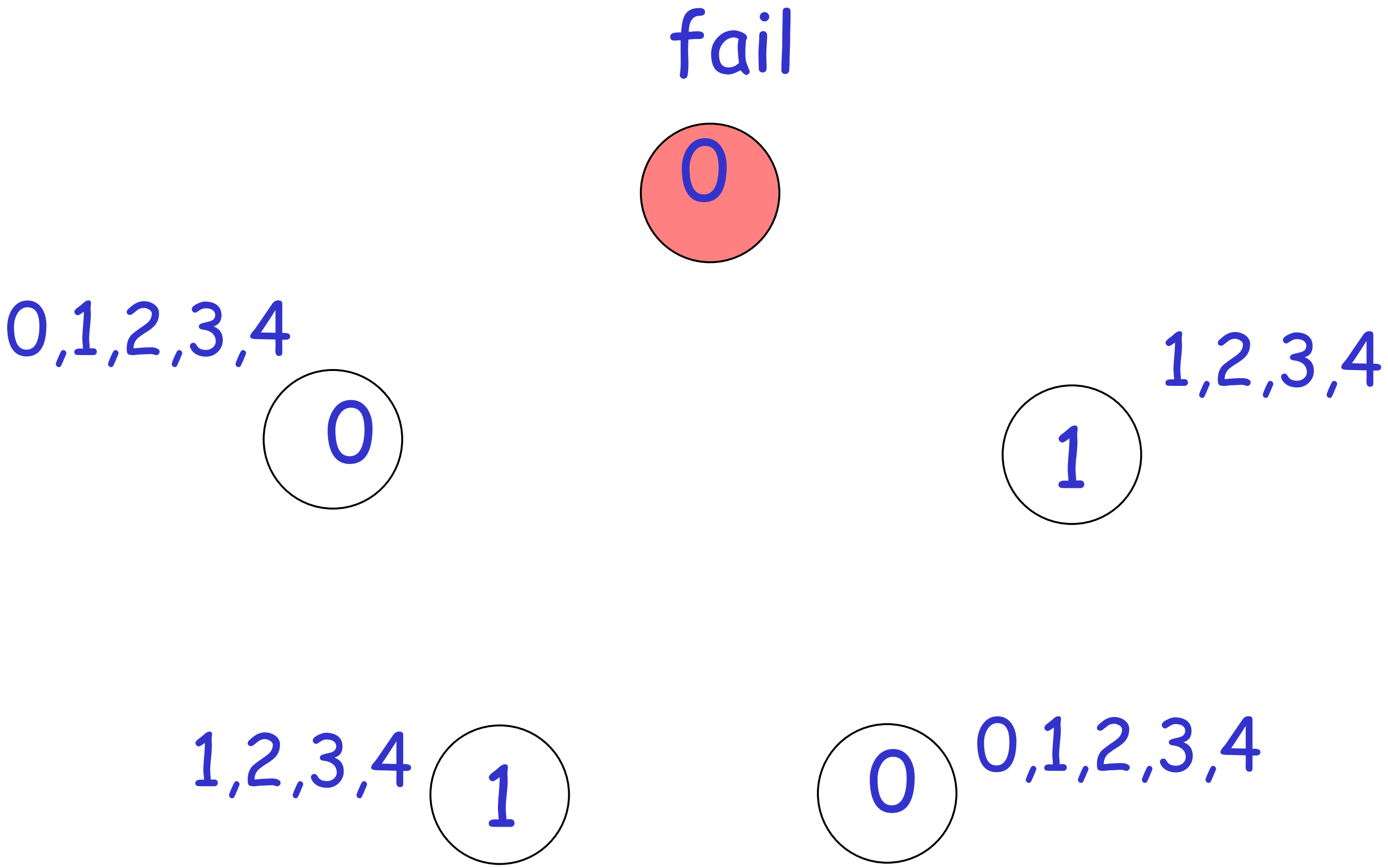


The failed processor doesn't broadcast its value to all processors

Broadcasted values



Decide on minimum



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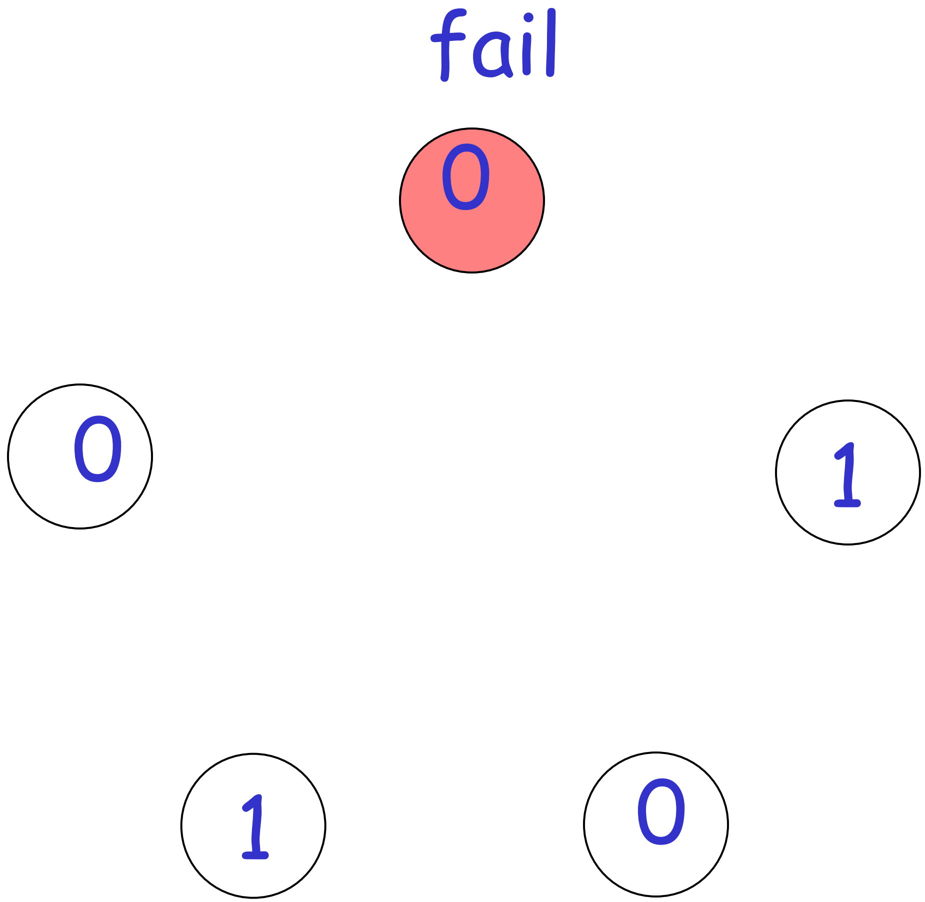


No agreement!!!

An f-resilient to crash failures algorithm

Each processor:

Round 1:

Broadcast to all (including myself) my value;

Read all the incoming values;

Round 2 to round f+1:

Broadcast to all (including myself) any new received values (one message for each value):

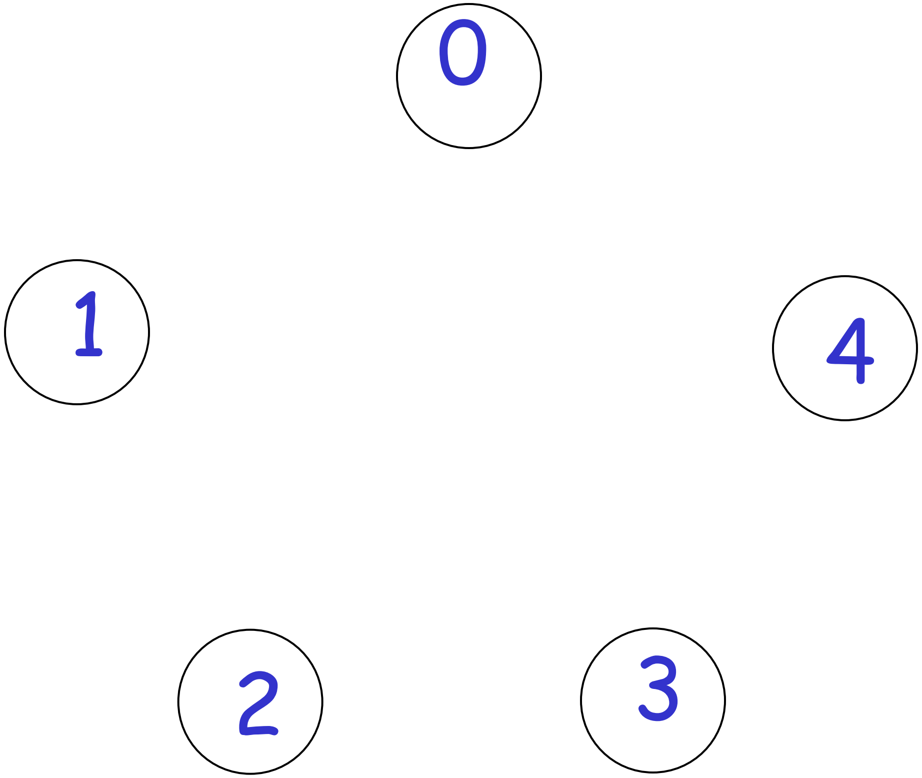
Read all the incoming values;

End of round f+1:

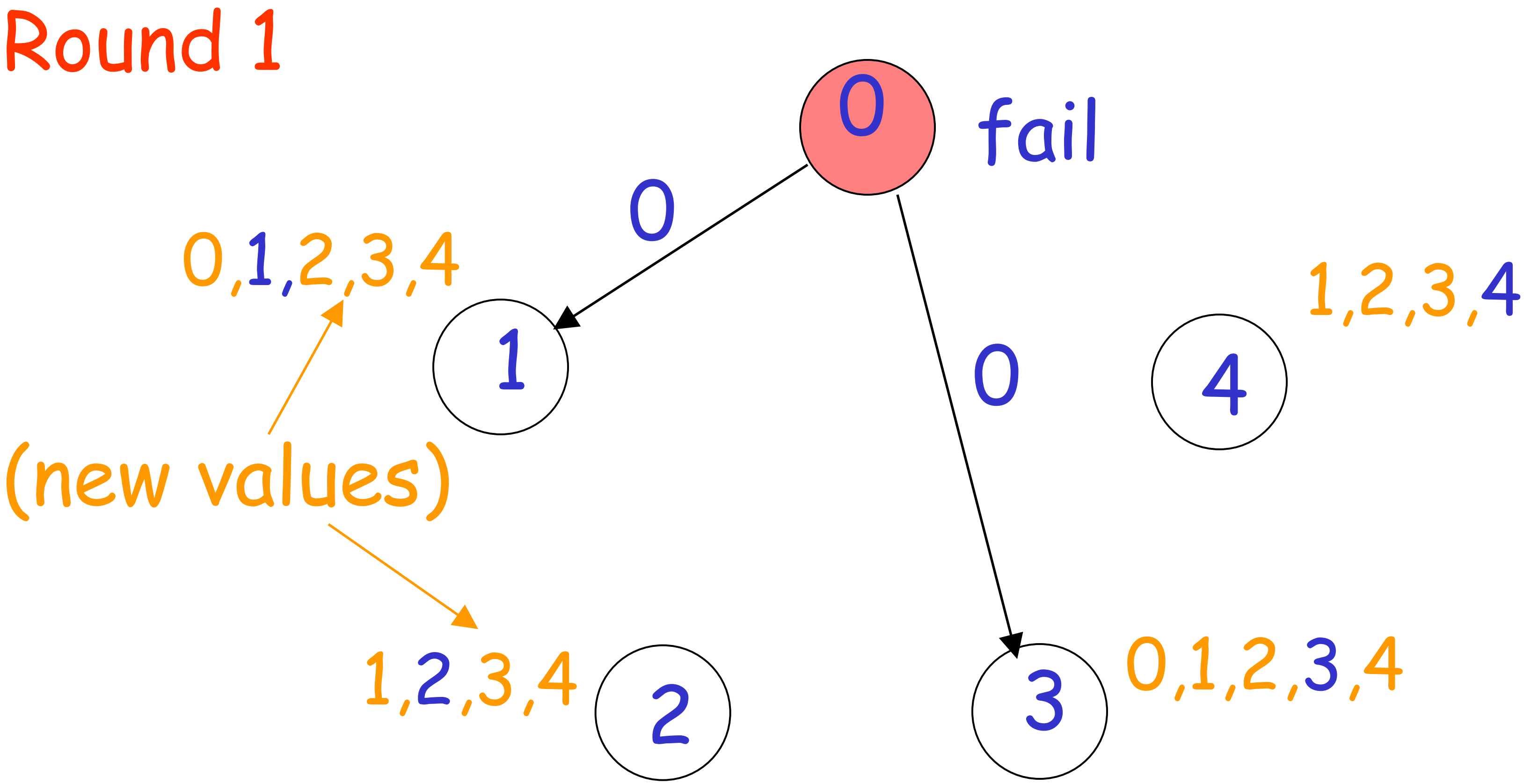
Decide on the minimum value ever received.

Example 1: f=1 failures, f+1 = 2 rounds needed

Start



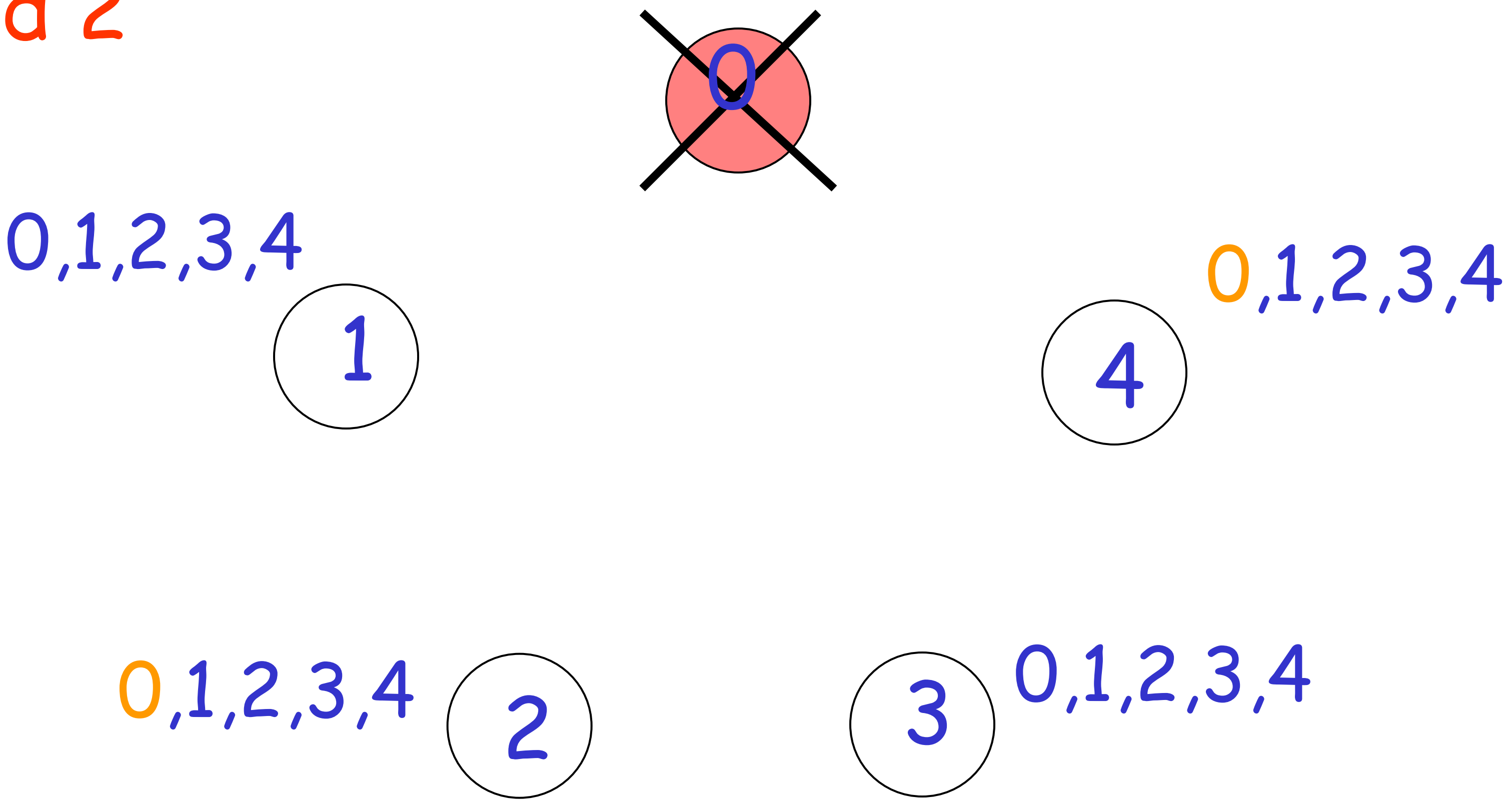
Example 1: f=1 failures, f+1 = 2 rounds needed



Broadcast all values to everybody

Example 1: f=1 failures, f+1 = 2 rounds needed

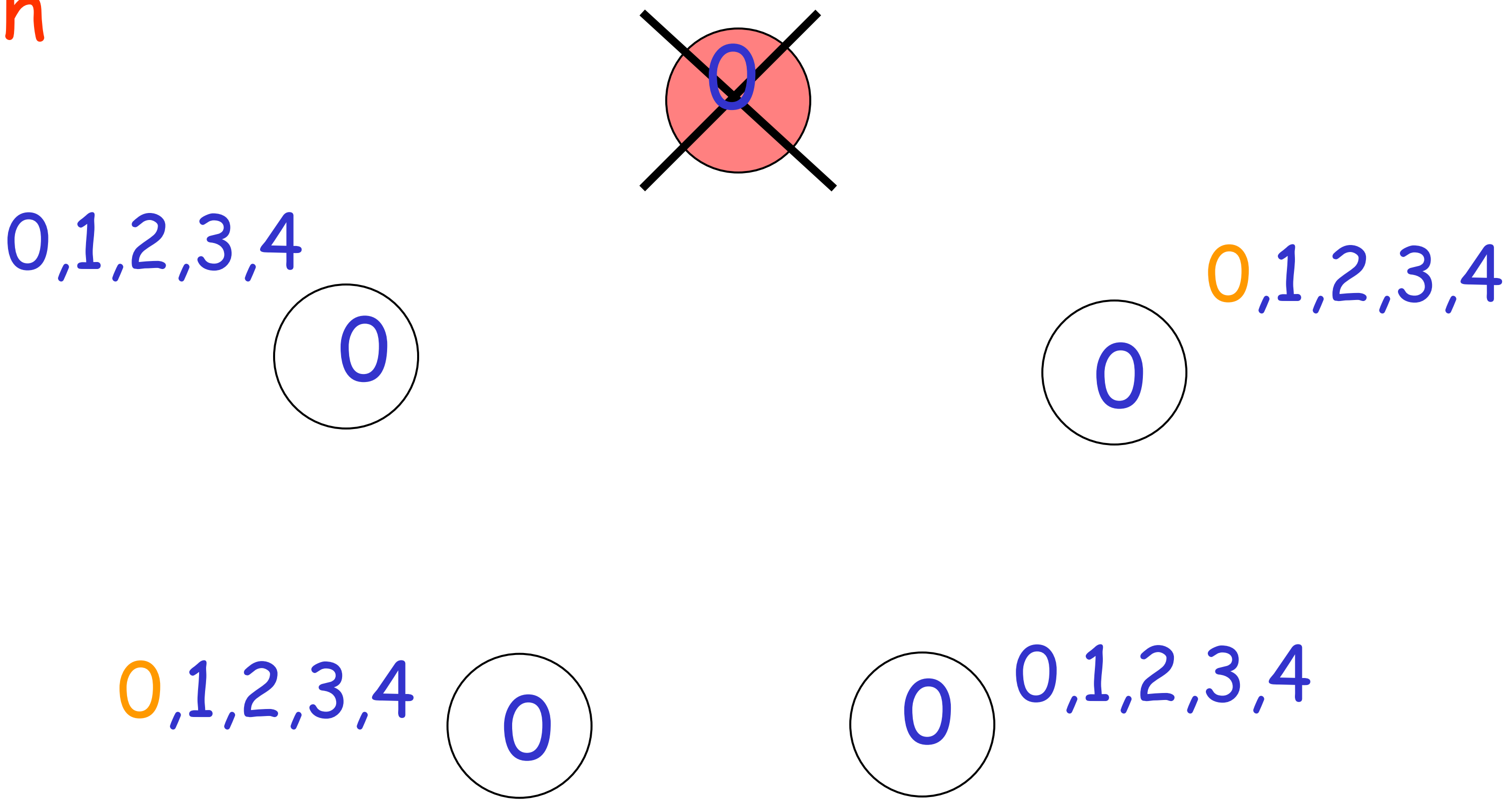
Round 2



Broadcast all new values to everybody

Example 1: f=1 failures, f+1 = 2 rounds needed

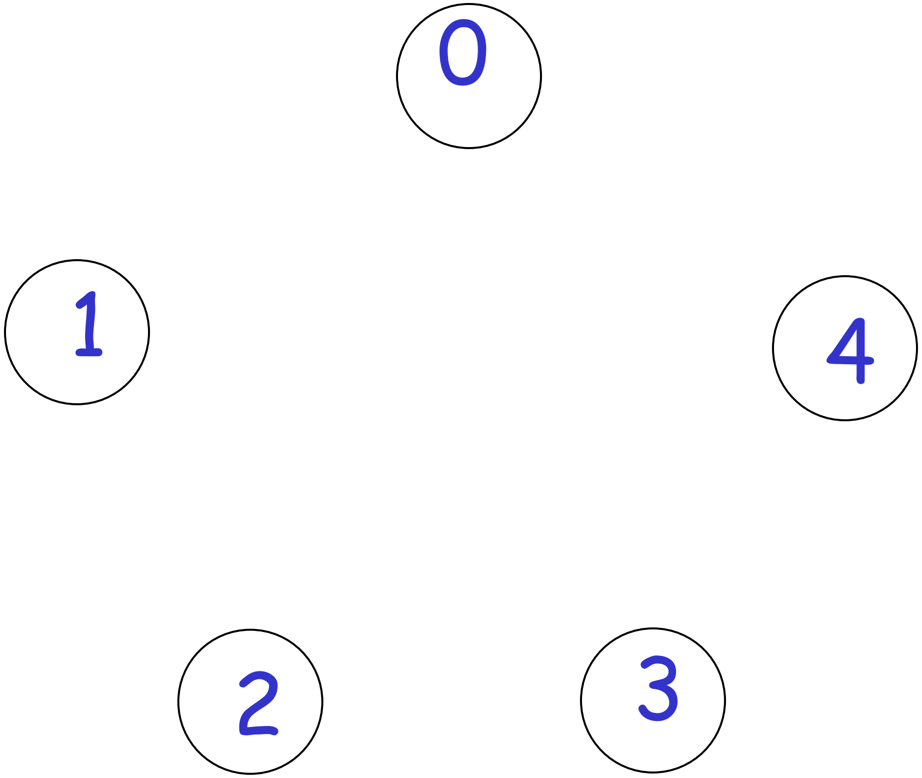
Finish



Decide on minimum value

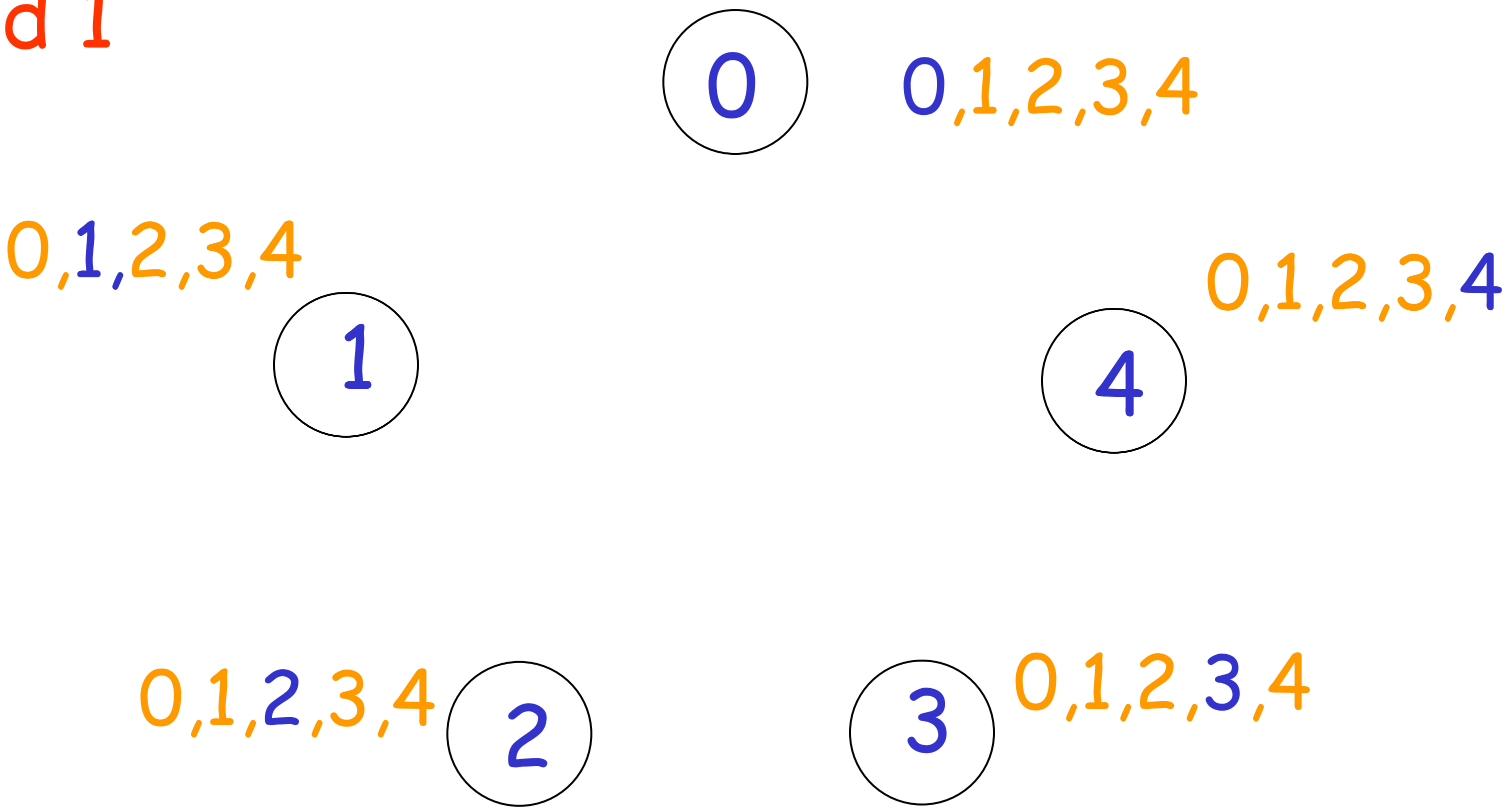
Example 2: f=1 failures, f+1 = 2 rounds needed

Start



Example 2: f=1 failures, f+1 = 2 rounds needed

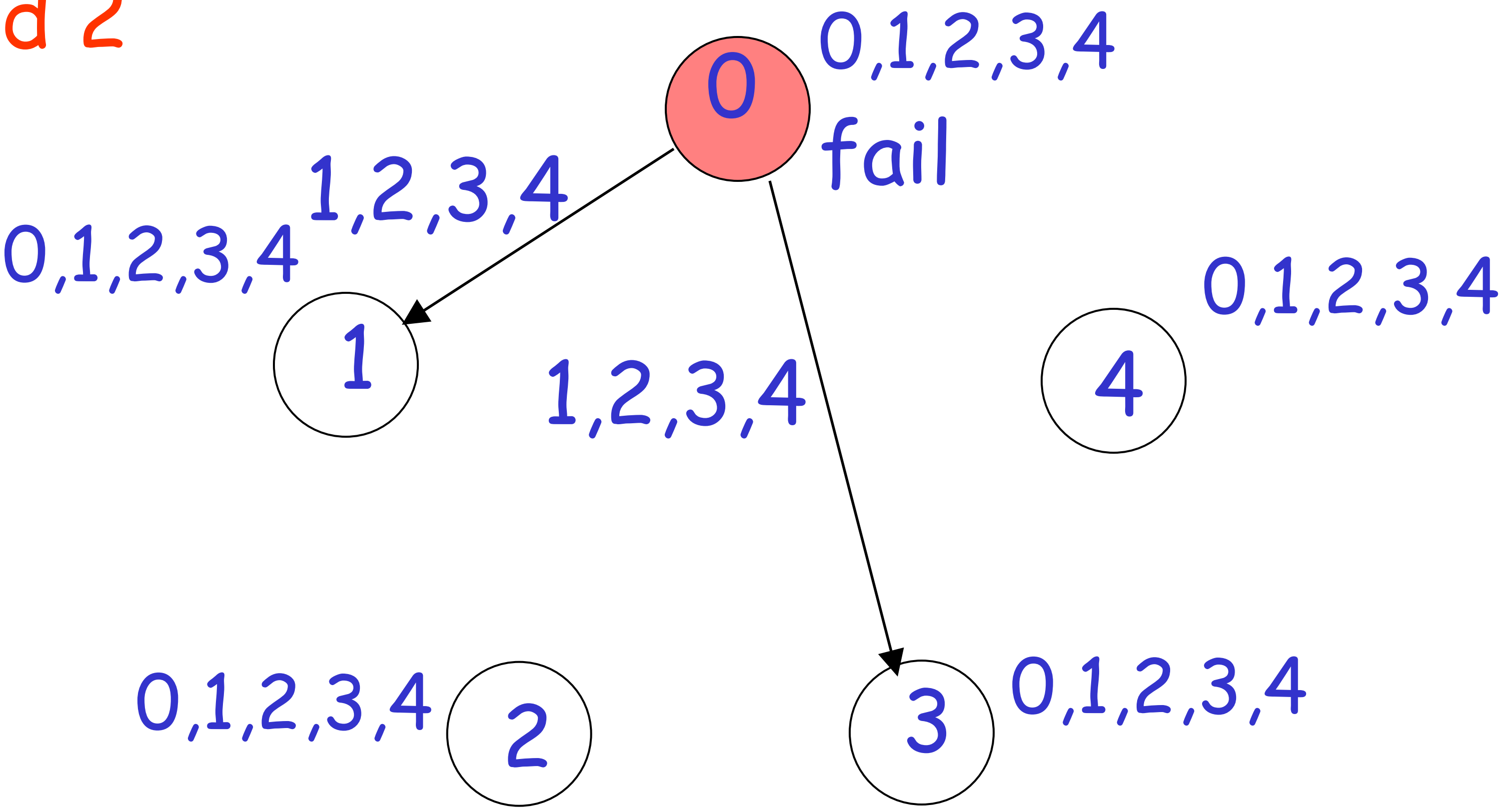
Round 1



No failures: all values are broadcasted to all

Example 2: f=1 failures, f+1 = 2 rounds needed

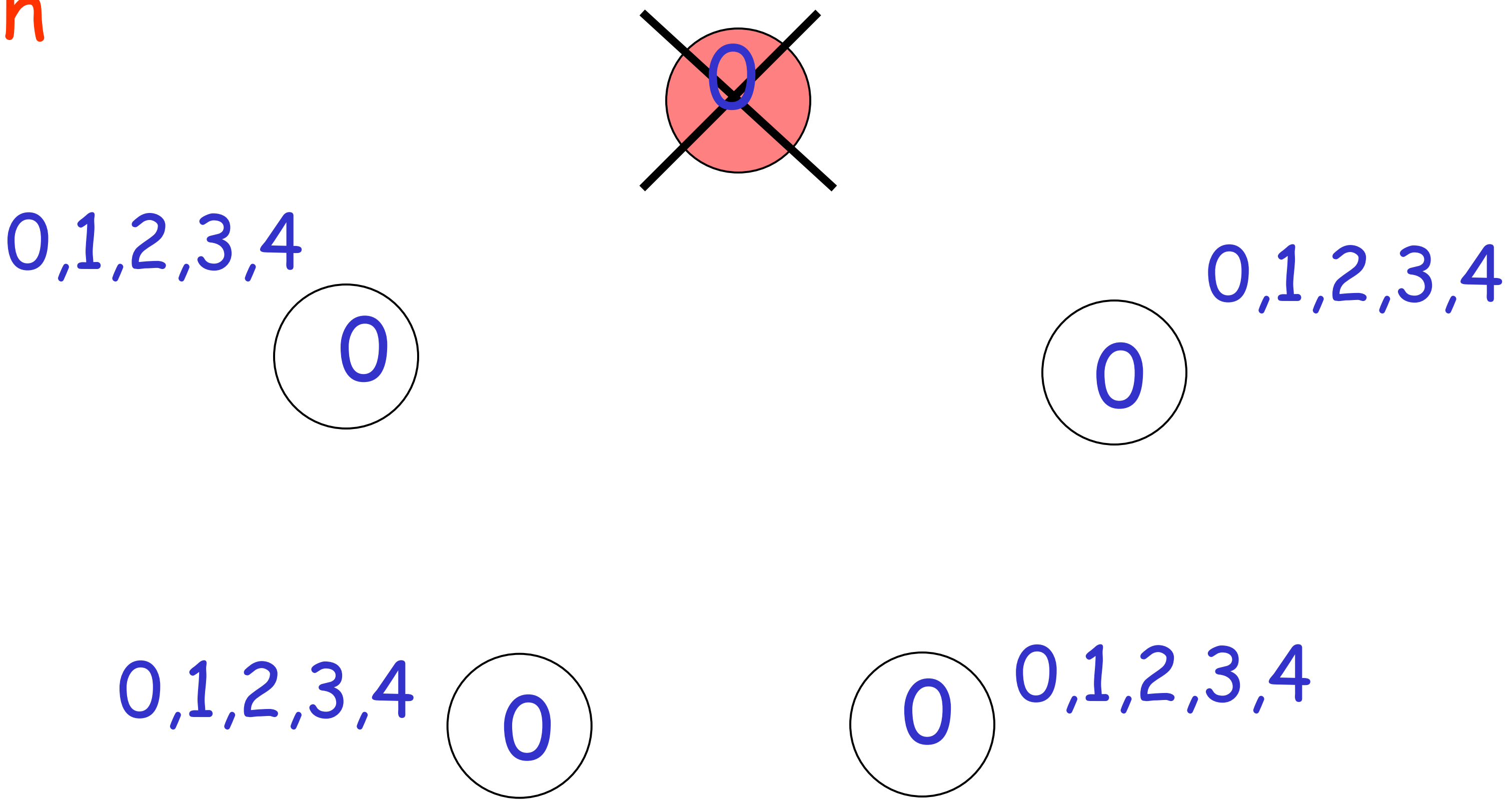
Round 2



No problems: processors "2" and "4" have already seen 1,2,3 and 4 in the previous round

Example 2: f=1 failures, f+1 = 2 rounds needed

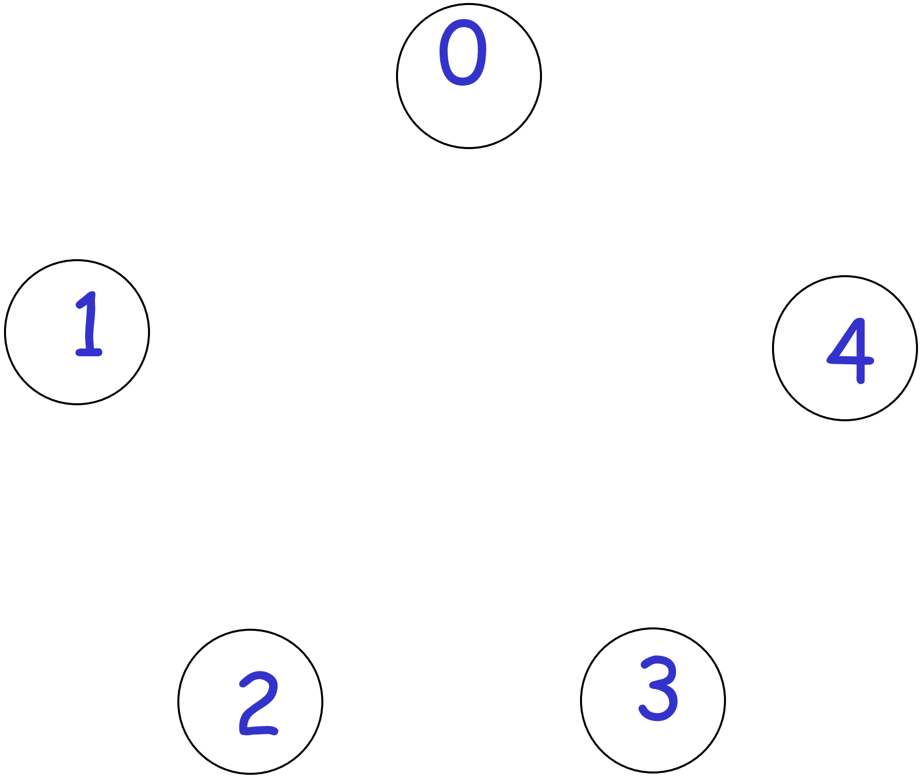
Finish



Decide on minimum value

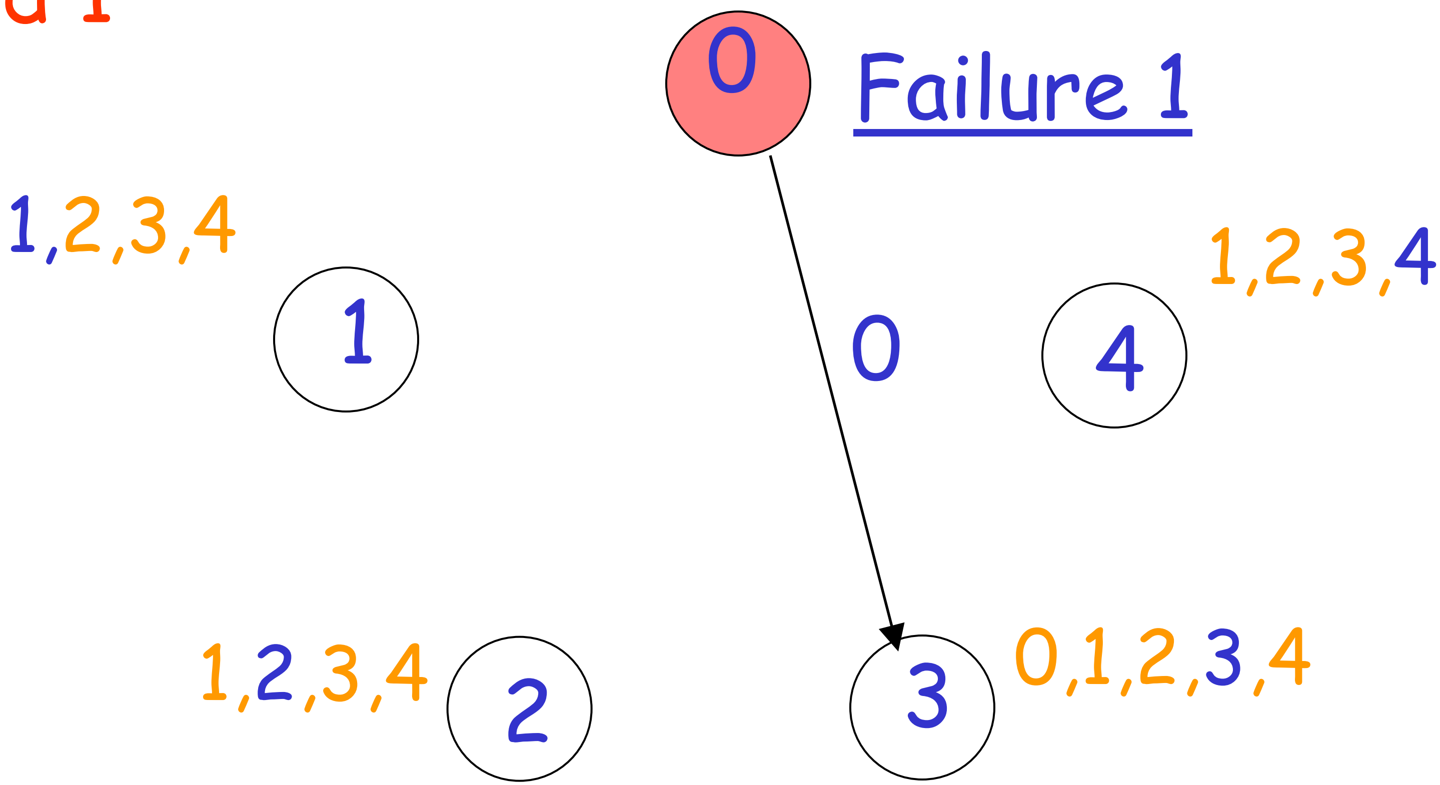
Example 3: f=2 failures, f+1 = 3 rounds needed

Start



Example 3: f=2 failures, f+1 = 3 rounds needed

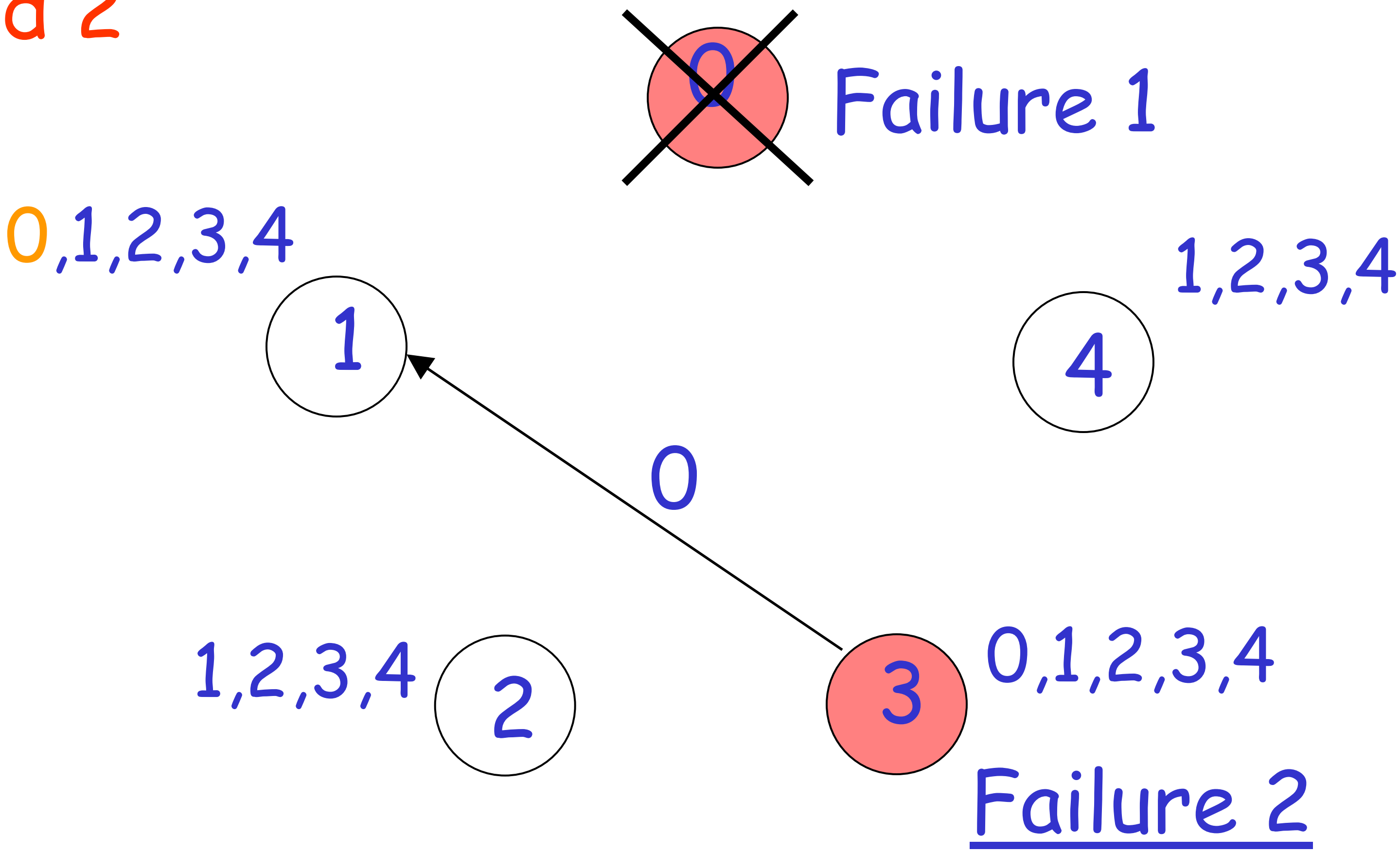
Round 1



Broadcast all values to everybody

Example 3: f=2 failures, f+1 = 3 rounds needed

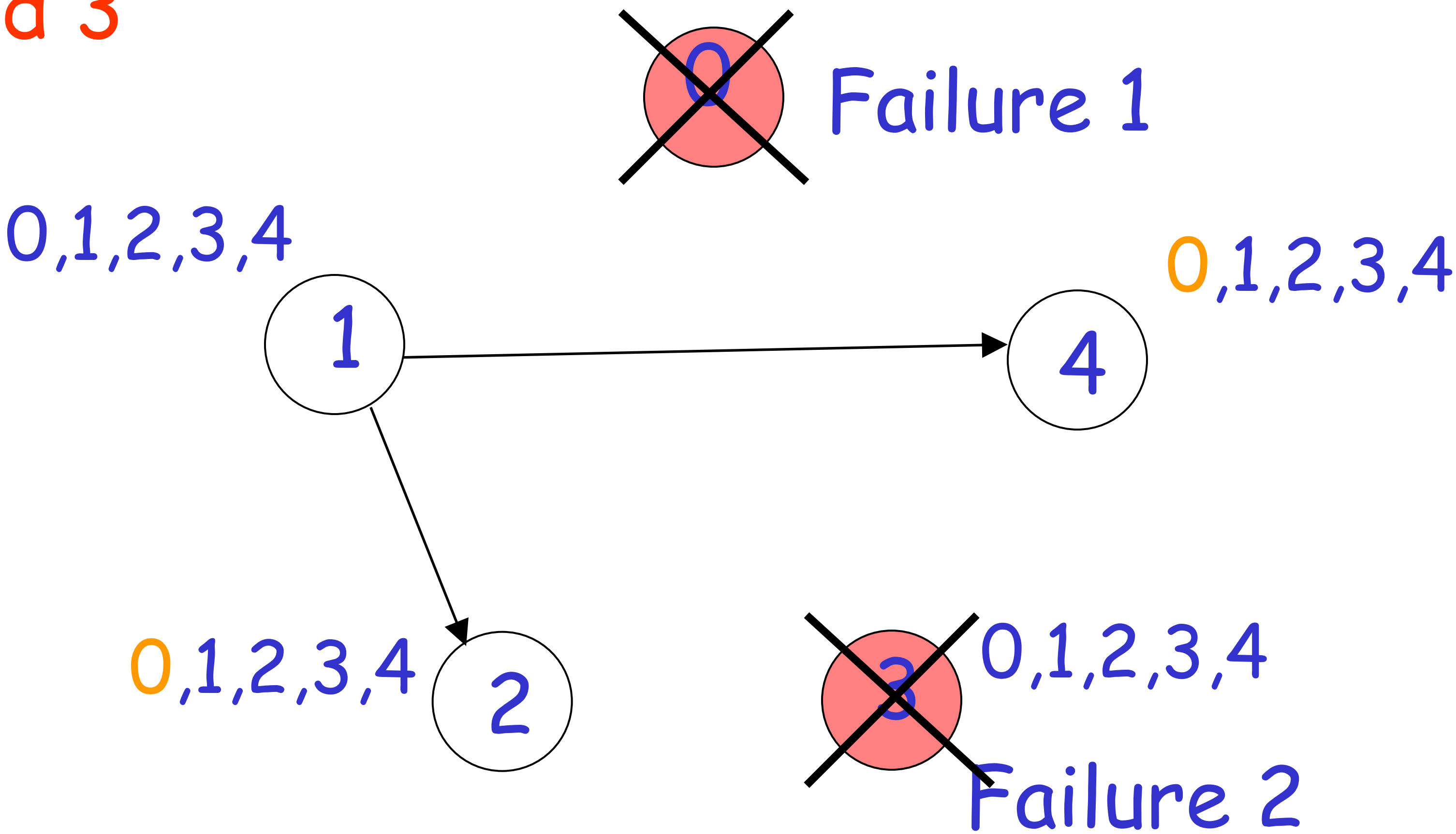
Round 2



Broadcast new values to everybody

Example 3: f=2 failures, f+1 = 3 rounds needed

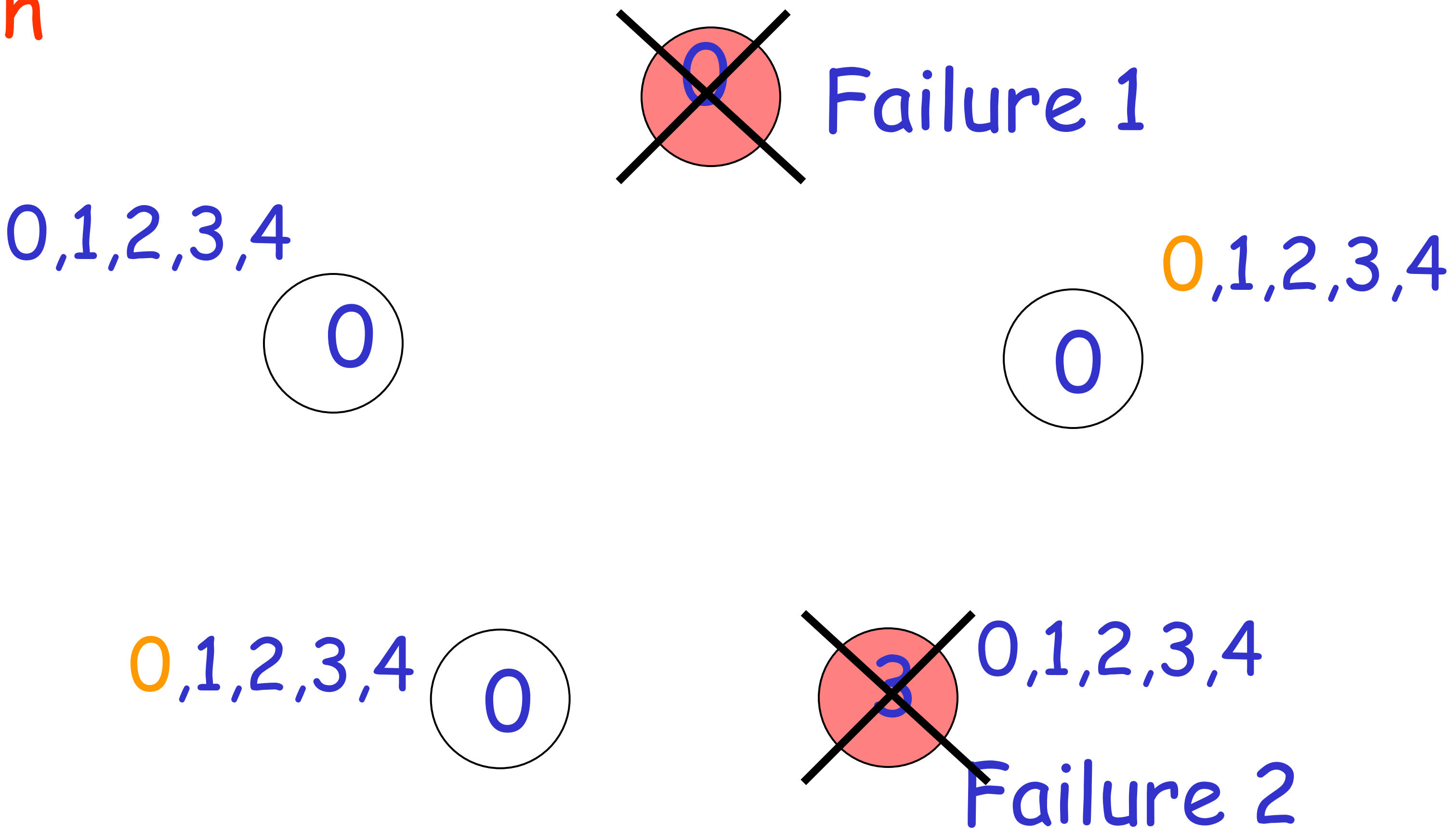
Round 3



Broadcast new values to everybody

Example 3: f=2 failures, f+1 = 3 rounds needed

Finish



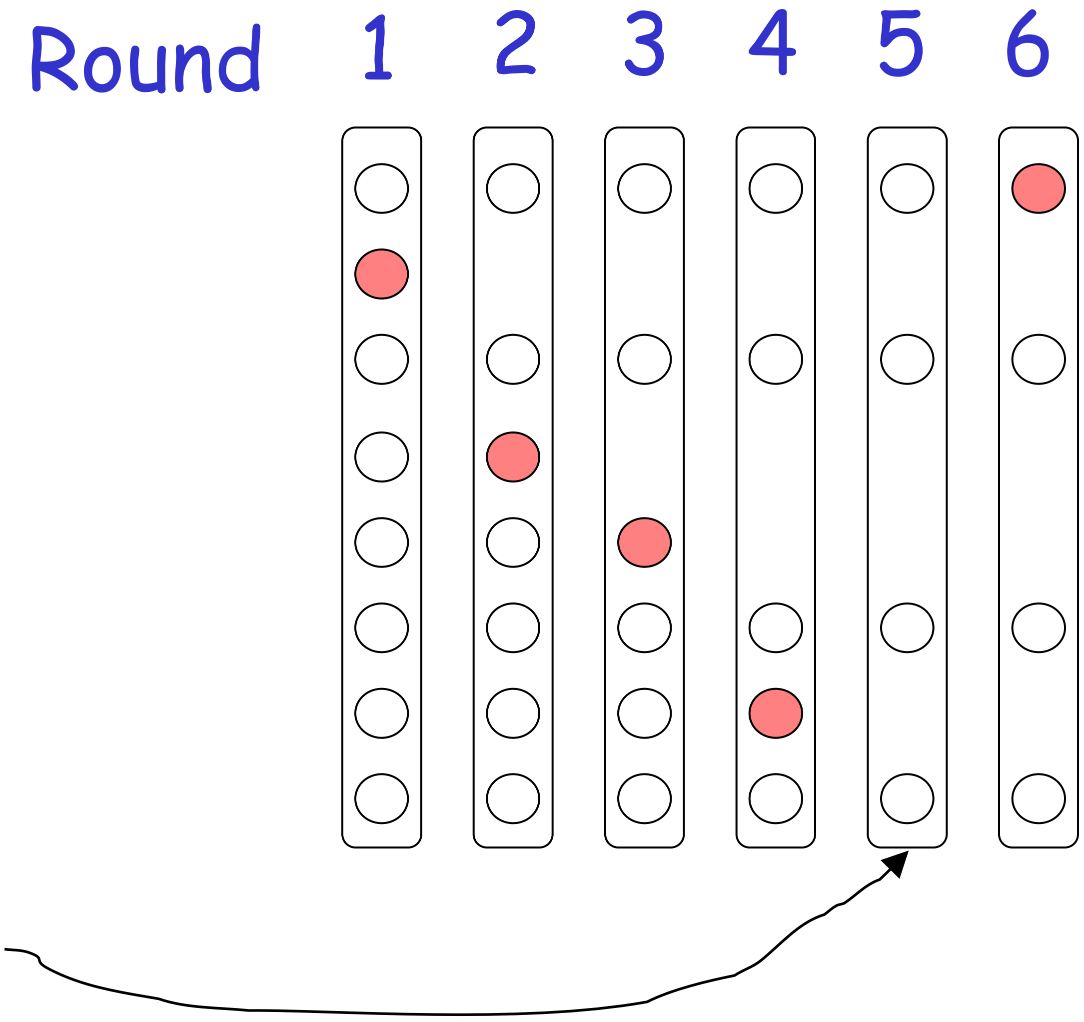
Decide on the minimum value

In general, since there are f failures and f+1 rounds, then there is at least a round with no new failed processors:

Example:

5 failures,

6 rounds



No failure

Correctness (1/2)

Lemma: In the algorithm, at the end of the round with no new failures, all the non-faulty processors know the same set of values.

Proof: For the sake of contradiction, assume the claim is false. Let x be a value which is known only to a subset of non-faulty processors at the end of the round with no failures. Observe that any such processors cannot have known x for the first time in a previous round, since otherwise it had broadcasted x to all. So, the only possibility is that it received it right in this round, otherwise all the others should know x as well. But in this round there are no failures, and so x must be received and known by all, a contradiction.

QED

Correctness (2/2)

Agreement: this holds, since at the end of the round with no failures, every (non-faulty) processor has the same knowledge, and this doesn't change until the end of the algorithm (no new values can be introduced, since we assumed synchronous start) $\Rightarrow$ eventually, everybody will decide the same value!

Remark: we don't know the exact position of the free-of-failures round, so we have to let the algorithm execute for f+1 rounds

Validity: this holds, since the value decided from each processor is some input value (no corrupted values are introduced)

Performance of Crash Consensus Algorithm

- Number of processors: $n > f$
- $f+1$ rounds
- $O(n^2 \cdot k) = O(n^3)$ messages, where $k = O(n)$ is the number of different inputs. Indeed, each processor sends $O(n)$ messages (one for each processor) containing a given seen value in X

A Lower Bound

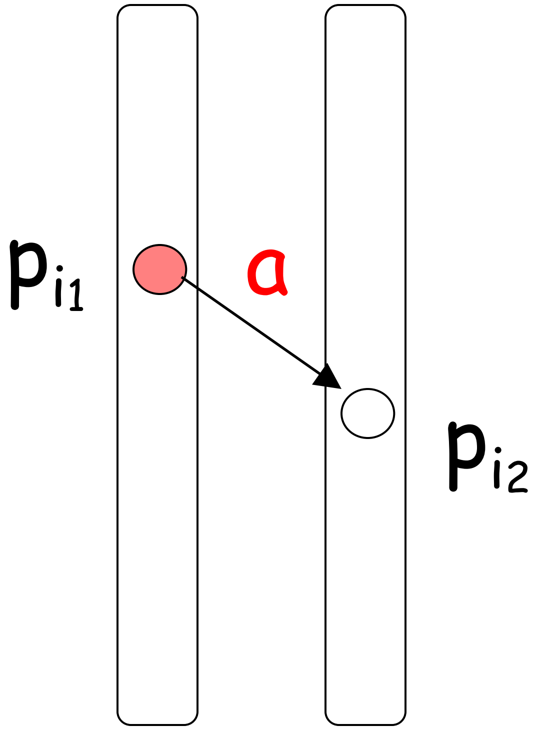
Theorem: Any f-resilient consensus algorithm with crash failures requires at least f+1 rounds

Proof sketch: Assume by contradiction that f or less rounds are enough. Clearly, every algorithm which solves consensus requires that eventually non-faulty processors have the very same knowledge

Worst case scenario:
There is a processor that fails in each round

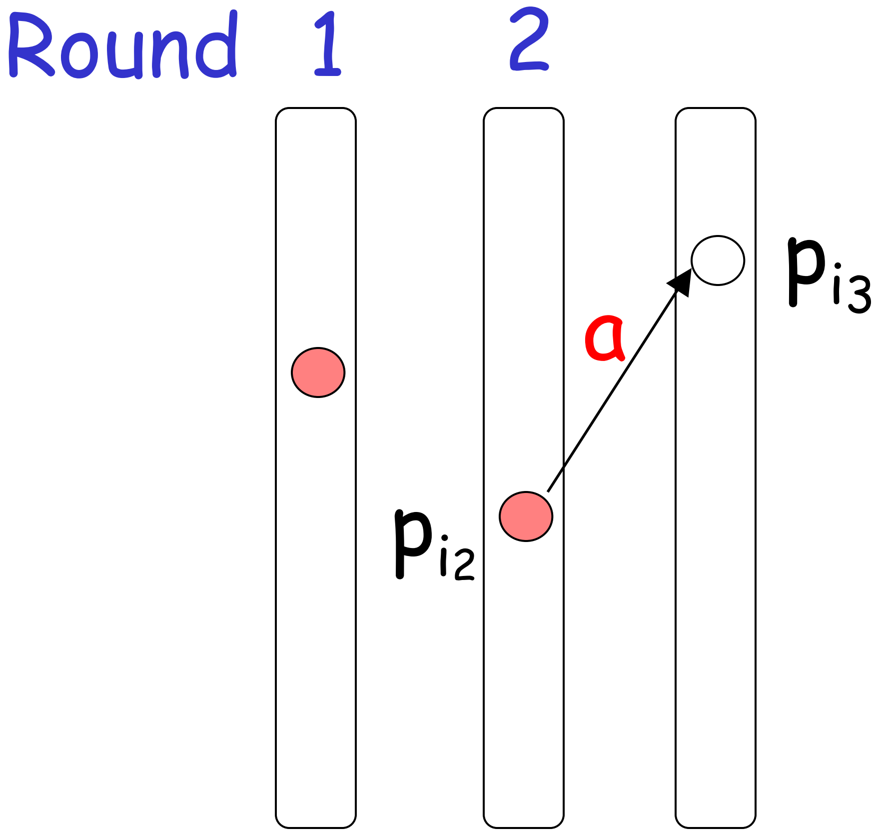
Worst case scenario

Round 1



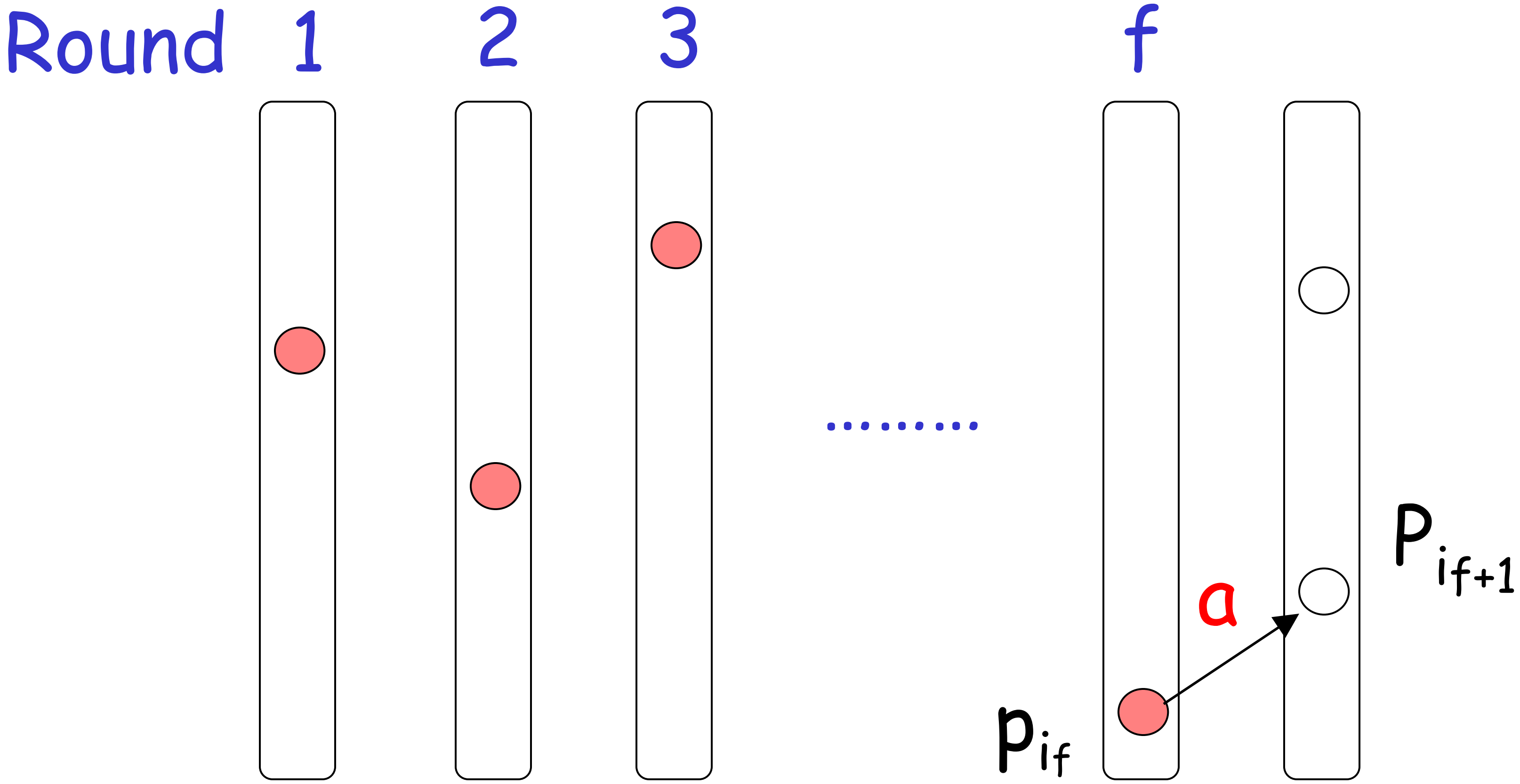
before processor p_{i_1} fails, it sends its value a to only one processor p_{i_2}

Worst case scenario



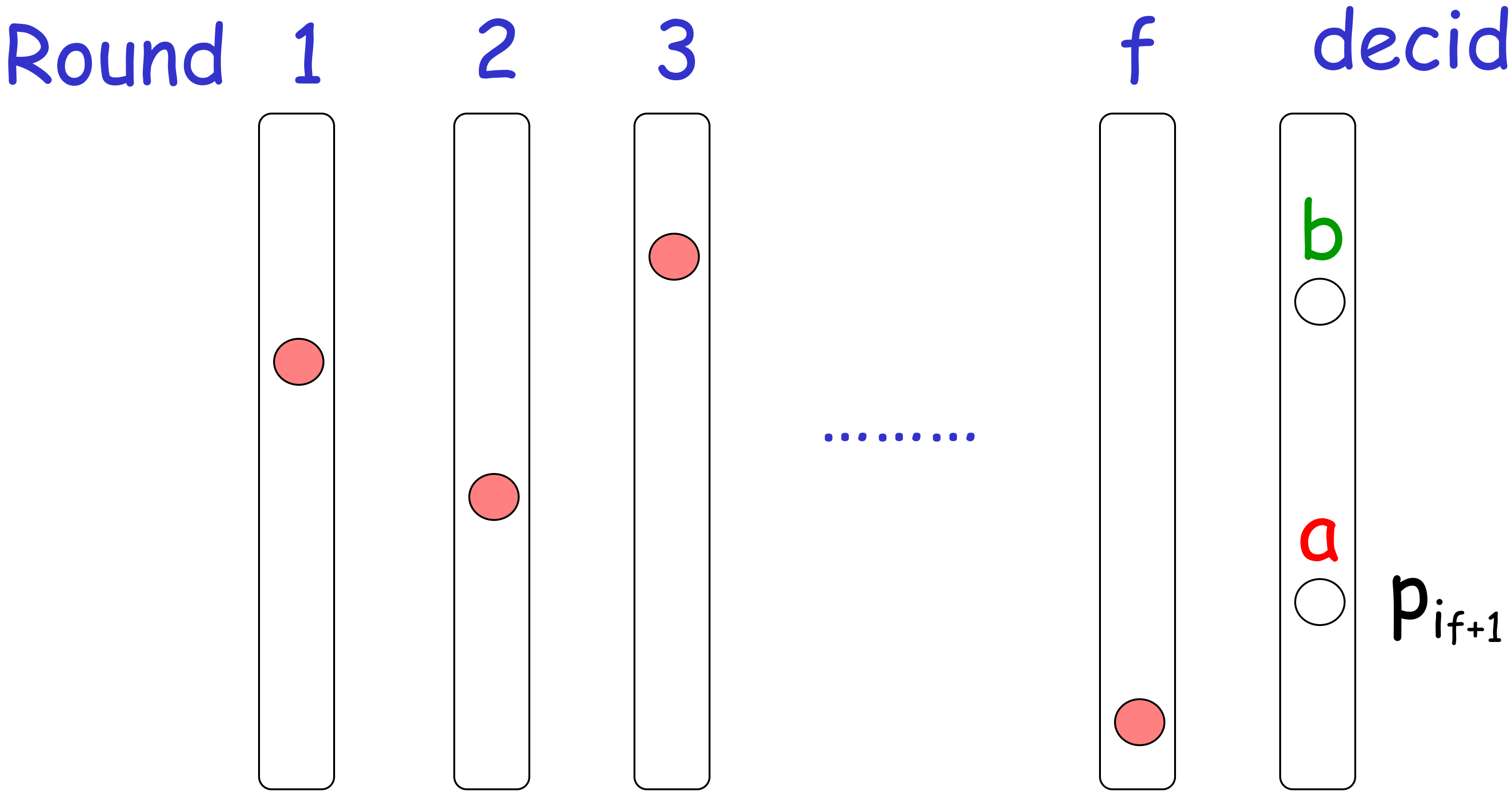
before processor p_{i_2} fails, it sends its value a to only one processor p_{i_3}

Worst case scenario



Before processor p_{i_f} fails, it sends its value a to only one processor $p_{i_{f+1}}$. Thus, at the end of round f only one processor knows about a

Worst case scenario



No agreement: Processor $p_{i_{f+1}}$ has a different knowledge, i.e., it may decide a, and all other processors may decide another value, say b>a $\Rightarrow$ contradiction, f rounds are not enough. QED

Consensus with Byzantine Failures

f-resilient to byzantine failures consensus algorithm:

solves consensus for **at most** f byzantine processors

Lower bound on number of rounds

Theorem: Any f-resilient consensus algorithm with byzantine failures requires at least f+1 rounds

Proof:

follows from the crash failure lower bound

An f-resilient to byzantine failures algorithm

The **King** algorithm

Solves consensus in $2(f+1)$ rounds for n processors out of which at most $n/4$ can be byzantine, namely $f<n/4$ (i.e., $n \geq 4f+1$)

Assumption: Processors have (distinct) ids in $\{1,\ldots,n\}$ (and so the system is non anonymous), and we denote by p_i the processor with id i; this is common knowledge, i.e., processors cannot cheat about their ids (namely, p_i cannot behave like if it was p_j, $i \neq j$, even if it is byzantine!)

The **King** algorithm

There are f+1 phases; each phase has 2 rounds, used to update in each processor p_i a preferred value v_i. In the beginning, the preferred value is set to the input value

In each phase there is a different king

$\Rightarrow$ There is a king that is non-faulty!

The **King** algorithm

Phase k

Round 1, every processor p_i:

- Broadcast to all (including myself) its preferred value v_i
- Let a be the majority of received values (including v_i)
  (in case of tie pick an arbitrary value)
- Set $v_i := a$

The **King** algorithm

Phase k

Round 2, king p_k:

Broadcast (to the others) its current preferred value v_k

Round 2, processor p_i:

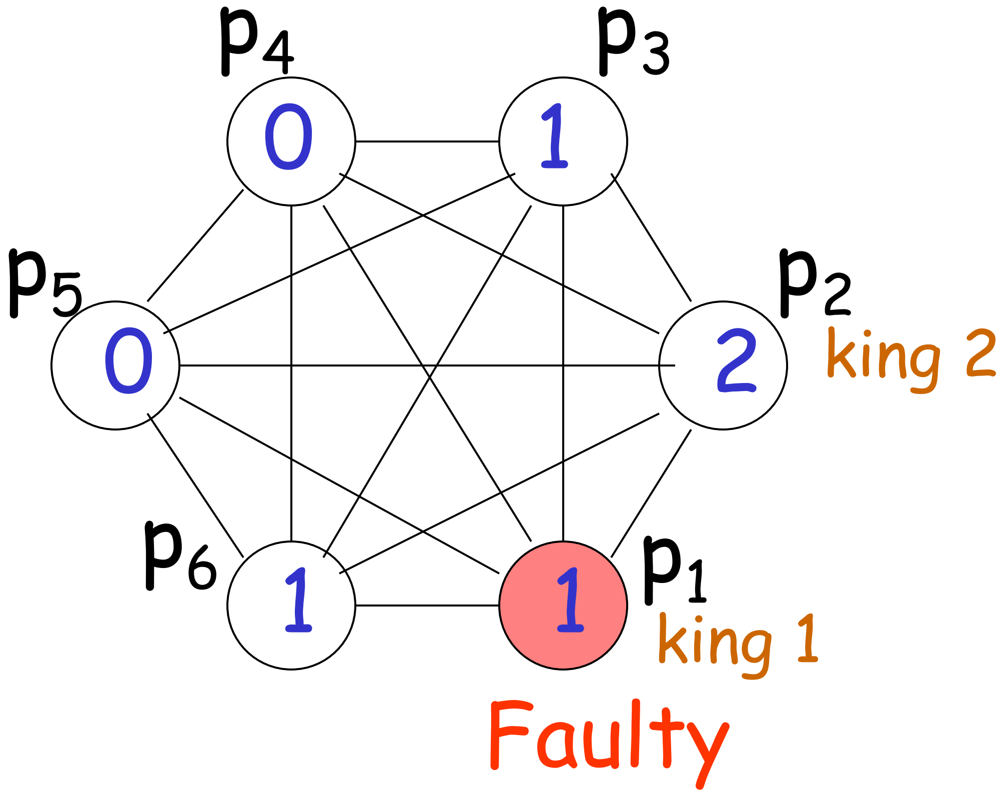
After receiving v_k, if p_i selected in Round 1 a preferred value v_i with a majority of less than $n/2+f+1$ (this is the so-called **strong majority**), then set $v_i := v_k$

The **King** algorithm

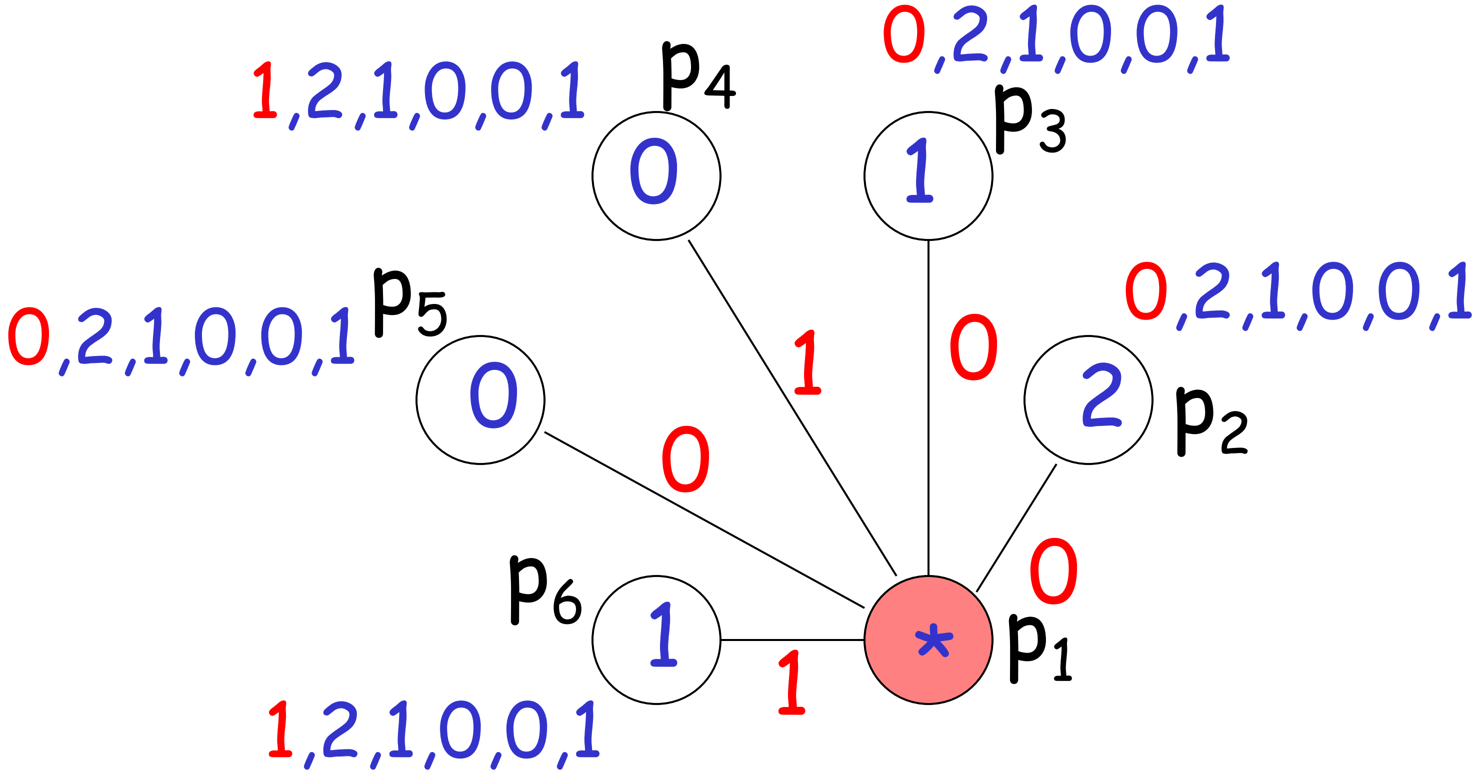
End of Phase f+1:

Each processor decides on its preferred value

Example 1: 6 processors, 1 fault, 2 phases



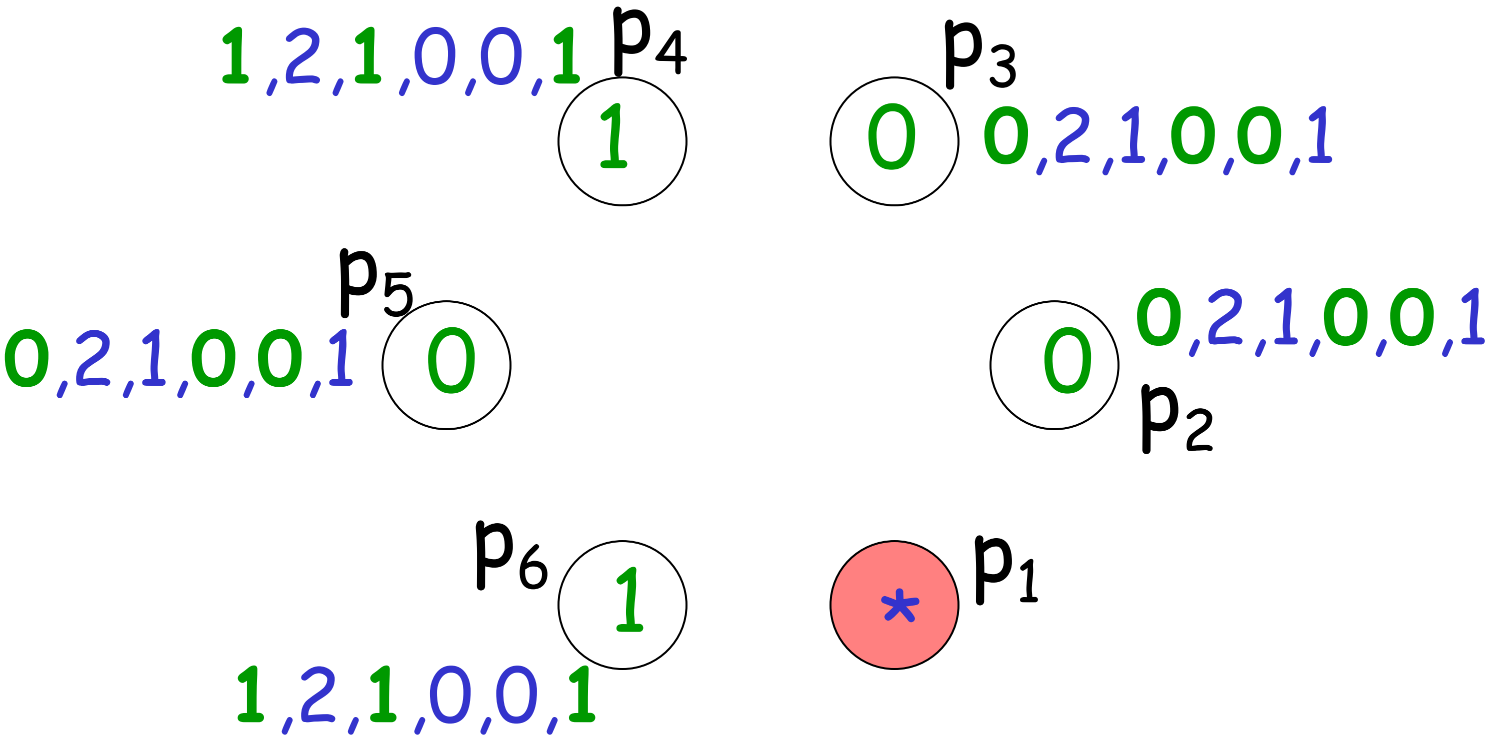
Phase 1, Round 1



Everybody broadcasts, and faulty p_1 sends arbitrary values

Phase 1, Round 1

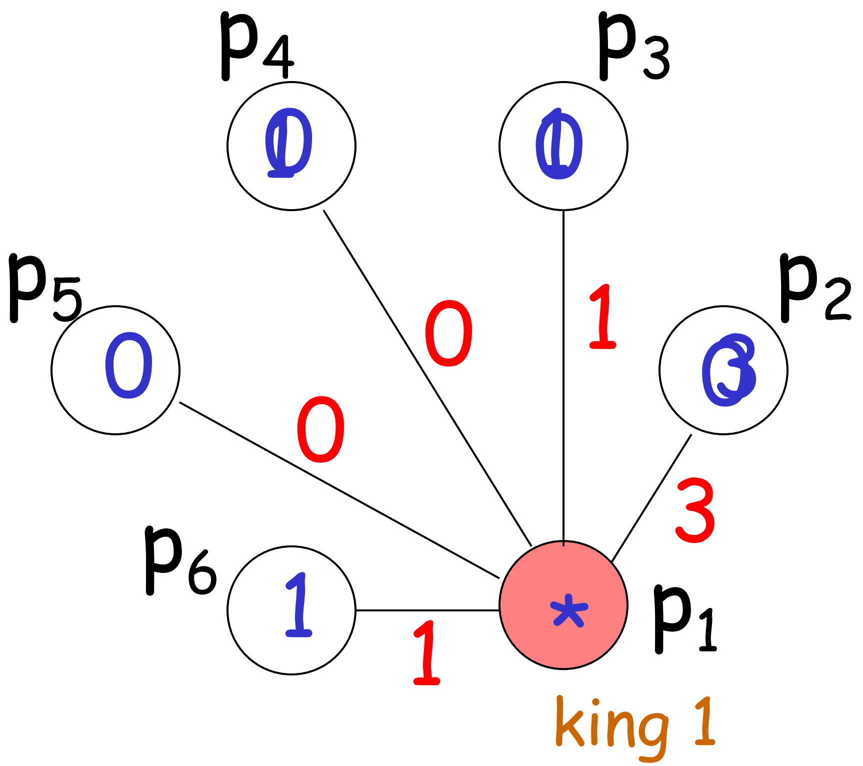
Choose the majority



Each (weak) majority is equal to $3 < \frac{n}{2} + f + 1 = 5$

$\Rightarrow$ On round 2, everybody will choose the king's value

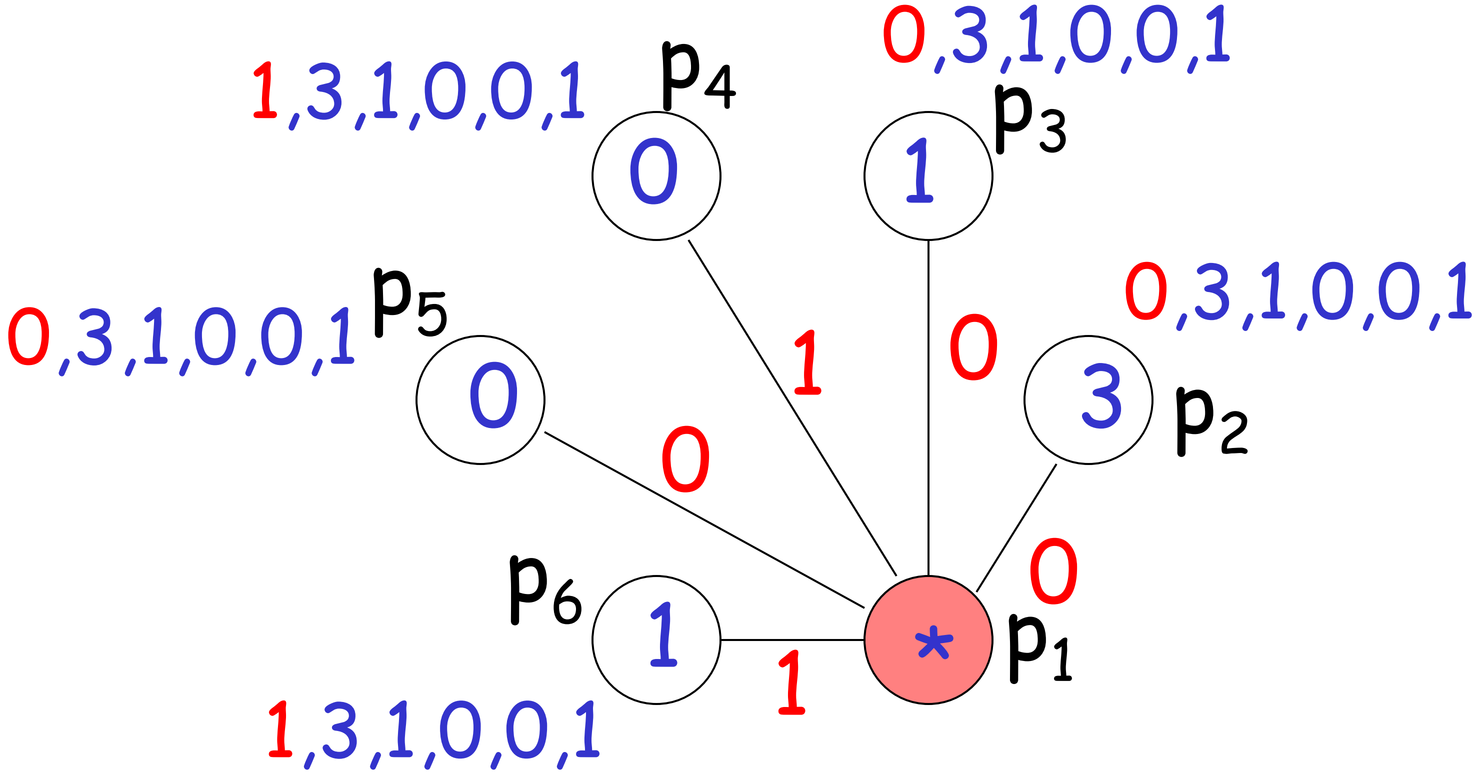
Phase 1, Round 2



The faulty king broadcasts arbitrary values

⇒ Everybody chooses the king's value

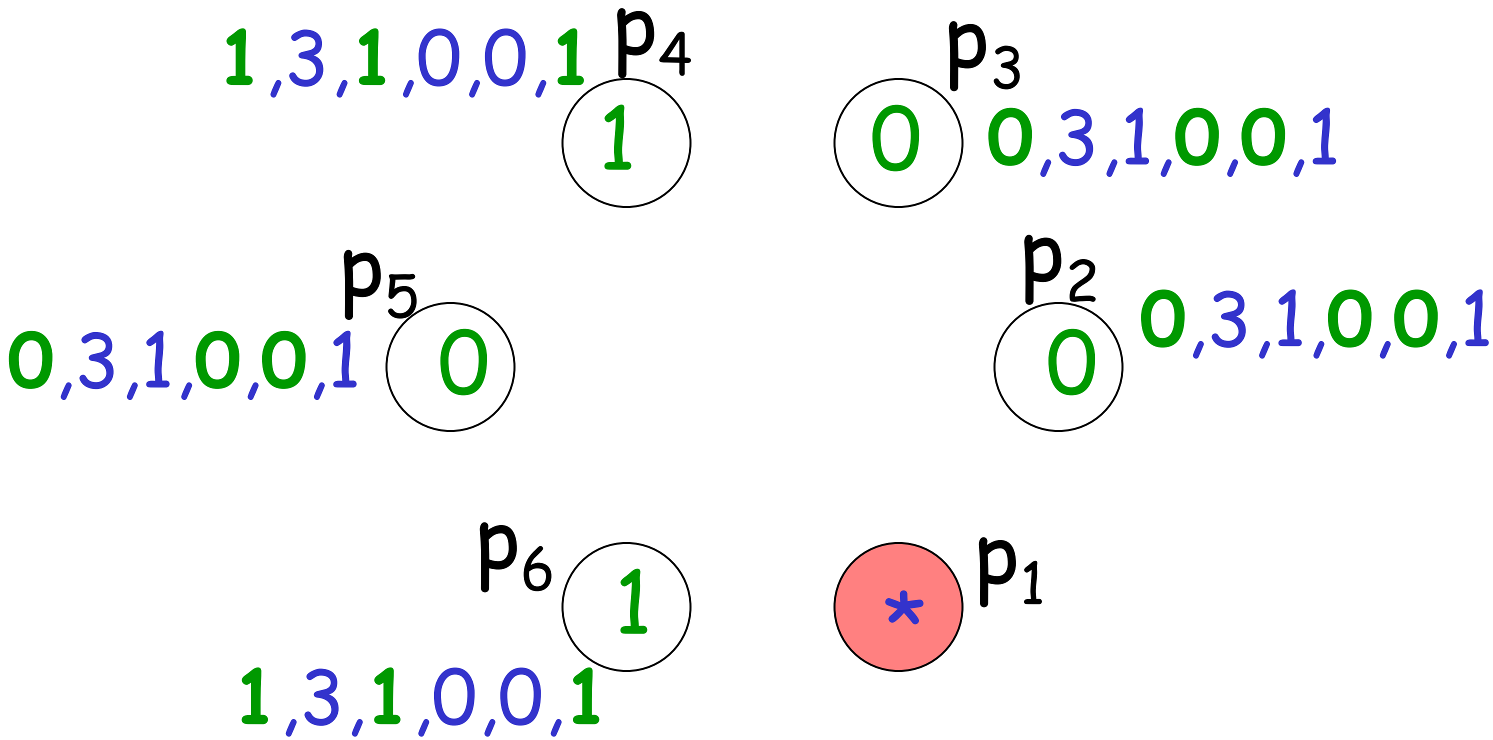
Phase 2, Round 1



Everybody broadcasts, and faulty p_1 sends arbitrary values

Phase 2, Round 1

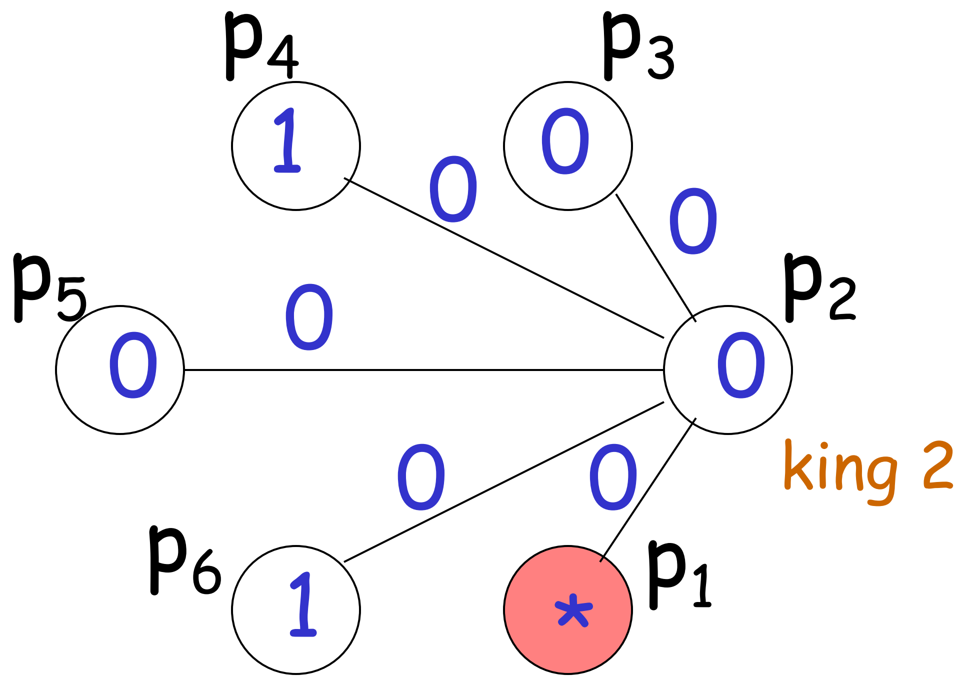
Choose the majority



Each (weak) majority is equal to $3 < \frac{n}{2} + f + 1 = 5$

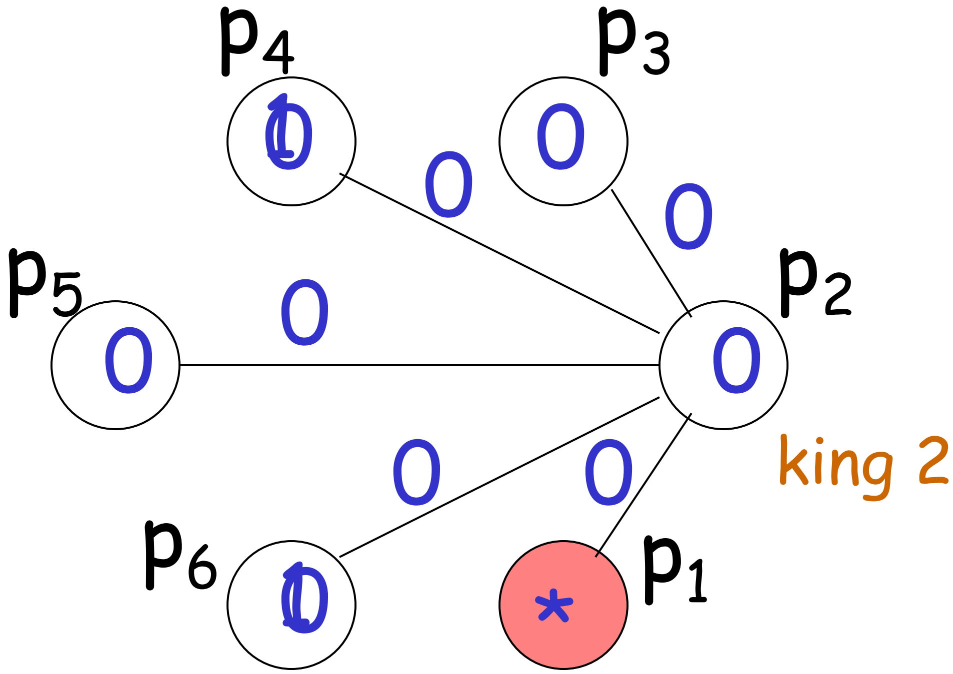
$\Rightarrow$ On round 2, everybody will choose the king's value

Phase 2, Round 2



The non-faulty king broadcasts its 0

Phase 2, Round 2

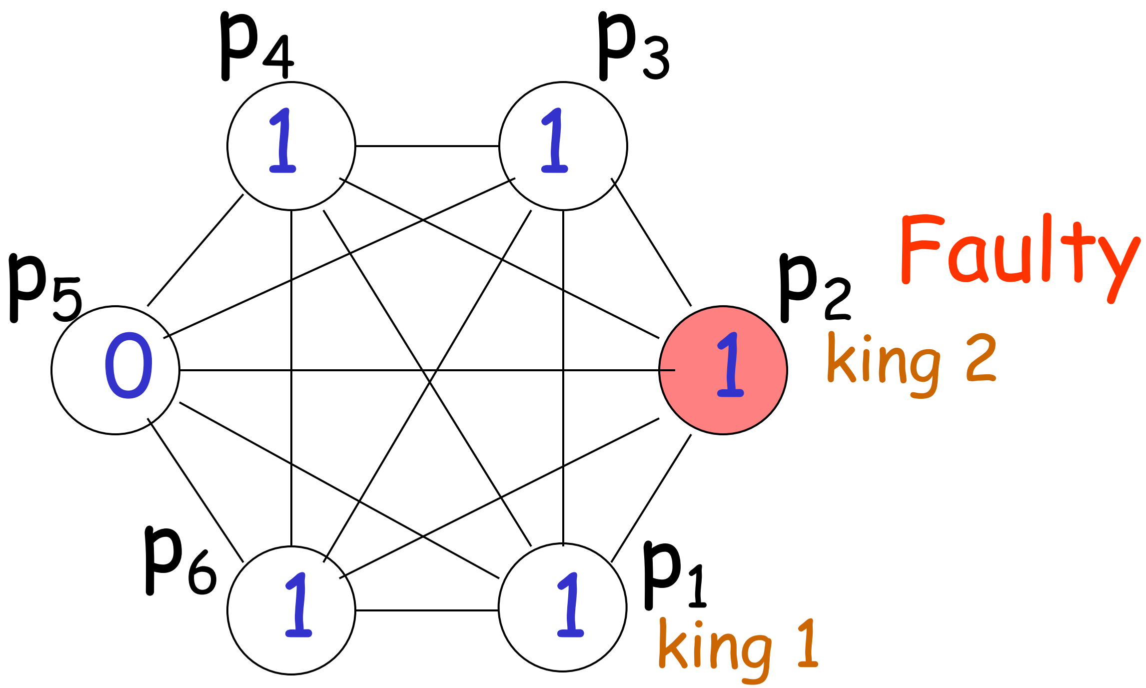


The non-faulty king broadcasts its 0

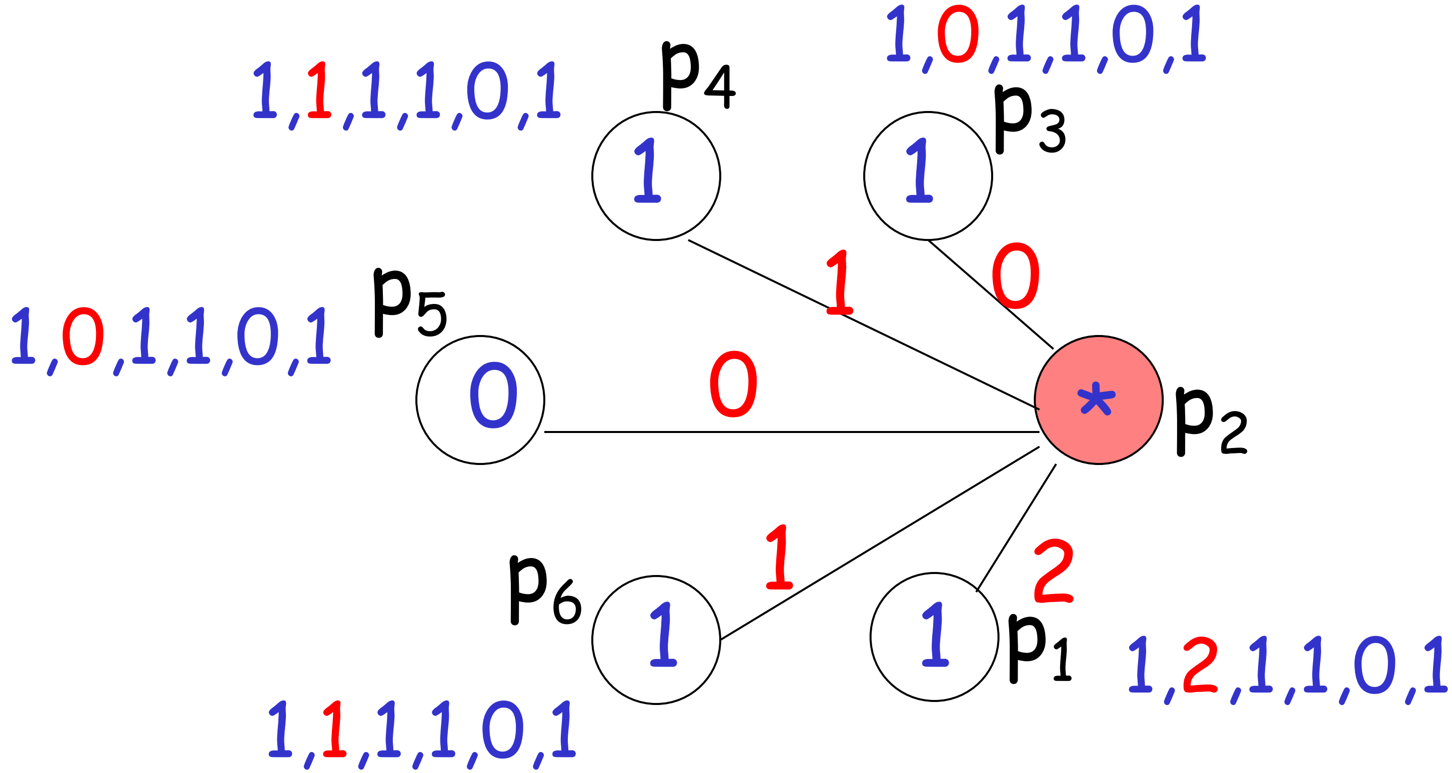
$\Rightarrow$ Everybody chooses the king's value

$\Rightarrow$ Final decision and agreement on 0

Example 2: 6 processors, 1 fault, 2 phases



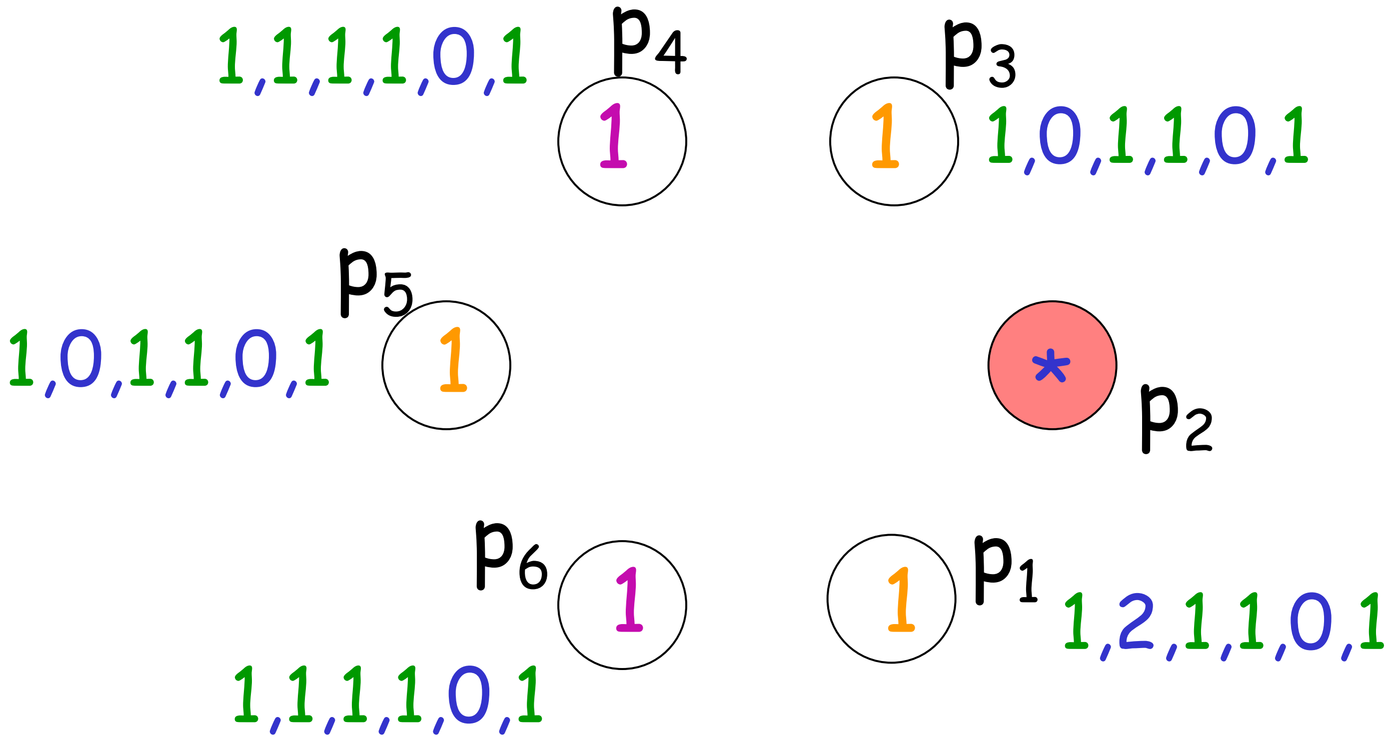
Phase 1, Round 1



Everybody broadcasts, and faulty p_2 sends arbitrary values

Phase 1, Round 1

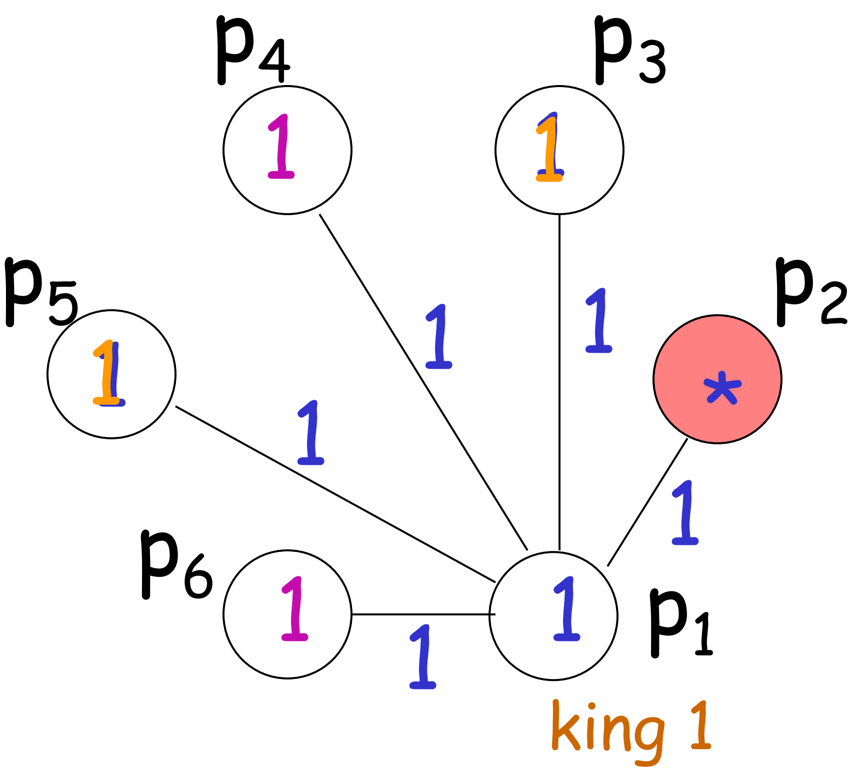
Choose the majority



Some majorities are strong (at least 5 votes), others are weak (less than 5 votes)

$\Rightarrow$ On round 2, somebody will choose the king's value, someone else will keep its own value

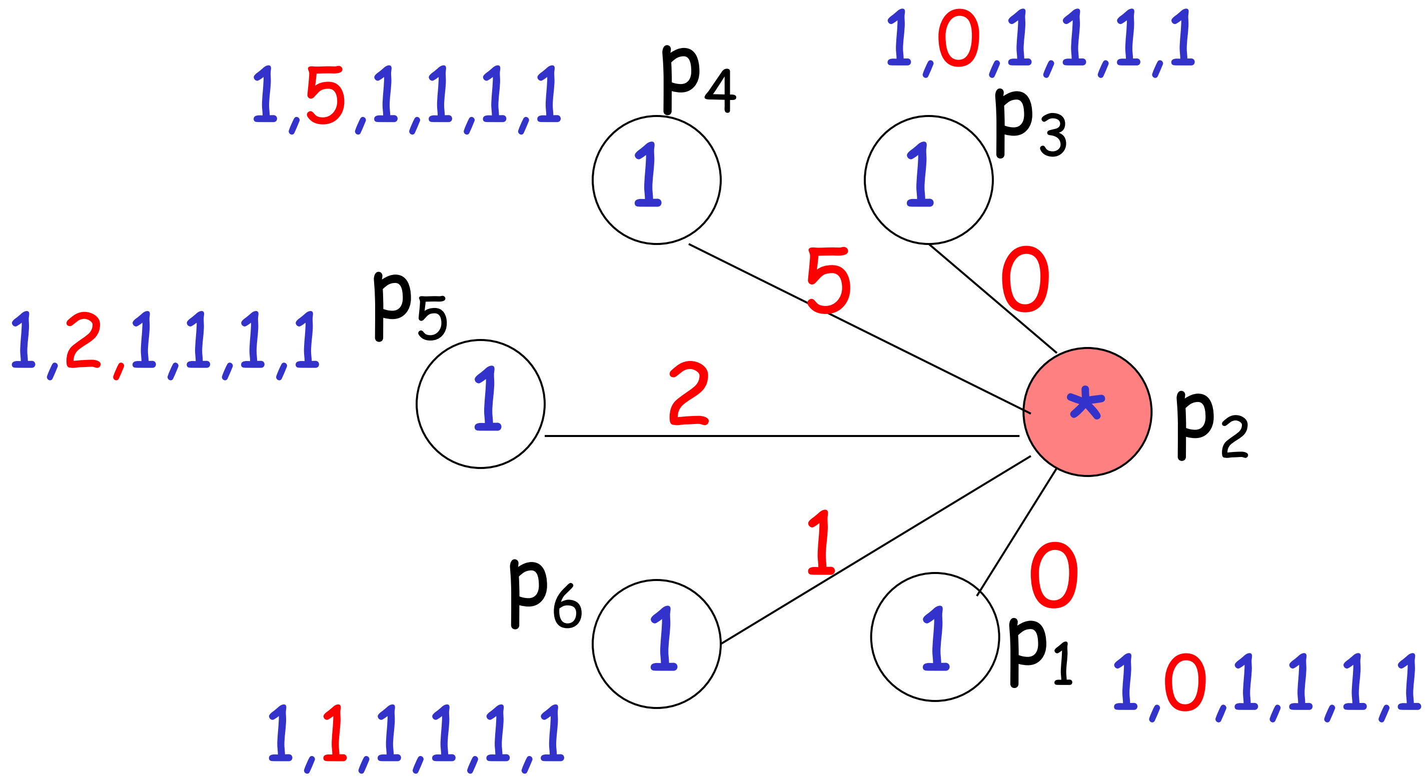
Phase 1, Round 2



The non-faulty king broadcasts its 1

⇒ Some processors switch to the king's value, but they will still selects 1!

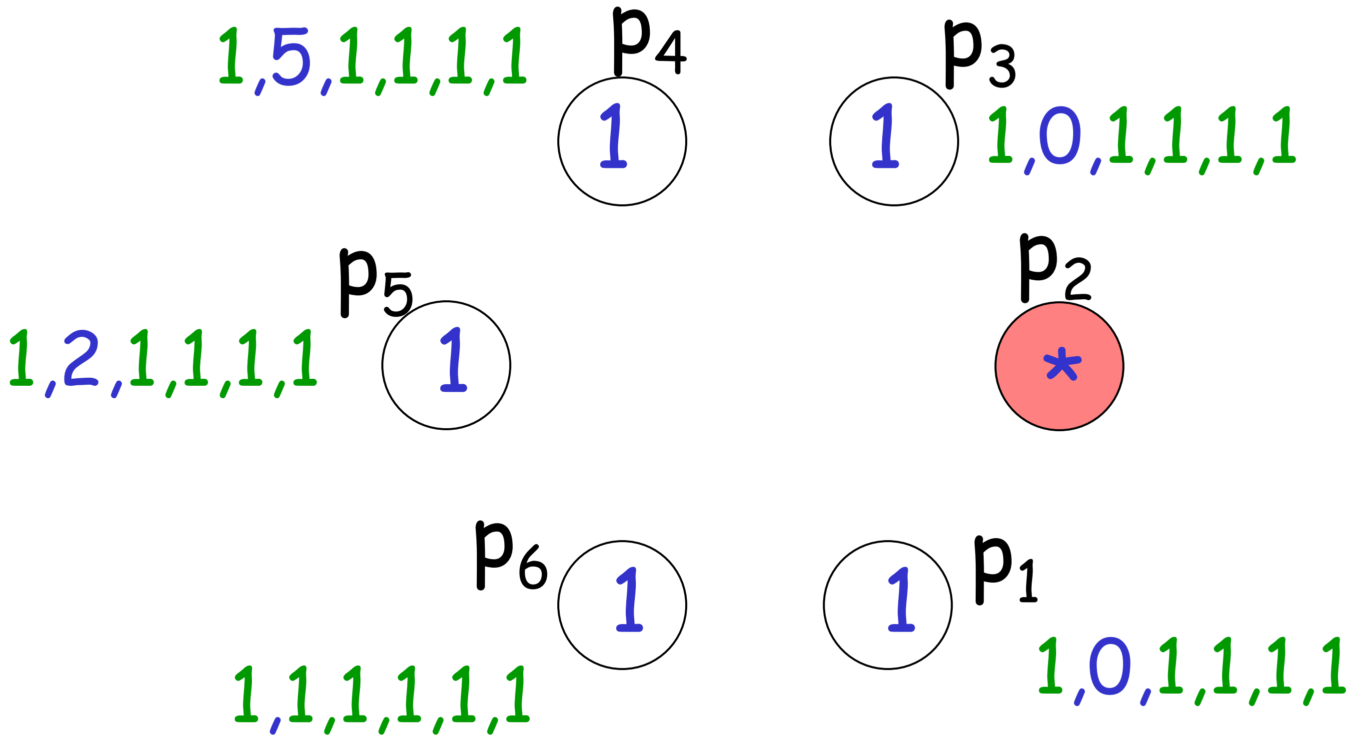
Phase 2, Round 1



Everybody broadcasts, and faulty p_2 sends arbitrary values

Phase 2, Round 1

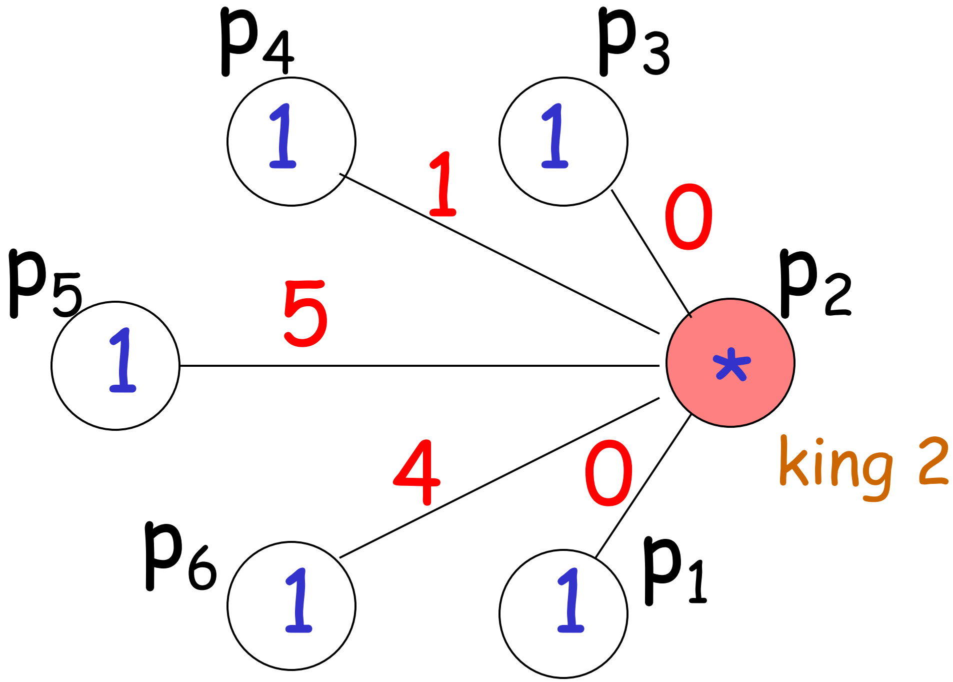
Choose the majority



Each majority is at least $5 = \frac{n}{2} + f + 1$ i.e., it's **strong**!

$\Rightarrow$ On round 2, nobody will choose the king's value

Phase 2, Round 2



The faulty king broadcasts arbitrary values, but nobody changes its preferred value

⇒ Final decision and agreement on 1

Correctness of the King algorithm

Lemma 1: At the end of a phase ϕ where the king is non-faulty, every non-faulty processor decides the same value

Proof: Consider the end of round 1 of phase ϕ.

There are two cases:

Case 1: All non-faulty processors have chosen their preferred value with **weak** majority (i.e., $< n/2+f+1$ votes) [see phase 2 of Example 1]

Case 2: Some non-faulty processor has chosen its preferred value with **strong** majority (i.e., $\geq n/2+f+1$ votes) [see phase 1 of Example 2]

Case 1: All non-faulty processors have chosen their preferred value at the end of round 1 of phase ϕ with **weak** majority (i.e., $< n/2+f+1$ votes)

$\Rightarrow$ Every non-faulty processor will adopt the value broadcasted by the king during the second round of phase ϕ, thus all of them will decide on the same value

Case 2: Suppose a non-faulty processor p_i has chosen its preferred value a at the end of round 1 of phase ϕ with **strong** majority ($\geq n/2+f+1$ votes)

$\Rightarrow$ This implies that at least $n/2+1$ non-faulty processors must have broadcasted a at start of round 1 of phase ϕ, and then at the end of that round, every other non-faulty processor must have received value a (including the king) with an absolute majority of at least $n/2+1$ votes, and so such a value becomes preferred in at least $n/2+1$ non-faulty processors

At end of round 2, there are 2 cases:

1. If a non-faulty processor keeps its own value, then it decides a
2. Otherwise, if a non-faulty processor adopts the value of the non-faulty king, then it decides a as well, since the king has decided a

Therefore: Every non-faulty processors decides a

END of PROOF

Lemma 2: Let a be a common value decided by non-faulty processors at the end of a phase ϕ. Then, a will be preferred until the end.

Proof: First of all, notice that the system contains at most f byzantine processors, and then at least $n-f$ non-faulty processors. But since $f<n/4$, it follows that $n-f>n/2+f$, since

$$f < \frac{n}{4} \Rightarrow 2f < \frac{n}{2} \Rightarrow 2f < n - \frac{n}{2} \Rightarrow n - 2f > \frac{n}{2} \Rightarrow n - f > \frac{n}{2} + f$$

This means, after ϕ, a will always be preferred with strong majority (i.e., $>n/2+f$), and so, until the end of phase $f+1$, every non-faulty processor will keep on deciding a. QED

Agreement in the King algorithm

Follows from Lemma 1 and 2, observing that since there are f+1 phases and at most f failures, there is al least one phase in which the king is non-faulty (and thus from Lemma 1 at the end of that phase all non-faulty processors decide the same, and from Lemma 2 this decision will be maintained until the end).

Validity in the King algorithm

Follows from the fact that if all (non-faulty) processors have a as input, then in round 1 of phase 1 each non-faulty processor will receive a at least n-f times, i.e., with strong majority, since as we observes in Lemma 2:

$$n-f > \frac{n}{2}+f$$

and so in round 2 of phase 1 this will be the preferred value of all non-faulty processors, independently of the king's broadcasted value. From Lemma 2, this will be maintained until the end, and will be exactly the decided output!

QED

Performance of King Algorithm

- Number of processors: $n > 4f$ (we will see it is not tight)
- $2(f+1)$ rounds (we will see it is not tight)
- $\Theta(n^2 \cdot f) = O(n^3)$ messages. Indeed, each non-faulty node sends n messages in the first round of each phase, each containing a given preference value, and each non-faulty king sends $n-1$ messages in the second round of each phase. Notice that we are not considering the fact that a byzantine processor could in principle generate an unbounded number of messages!

An Impossibility Result

Theorem: There is no f-resilient to byzantine failures algorithm for n processors when

$$f \geq \frac{n}{3}$$

Proof: First we prove the 3 processors case, and then the general case

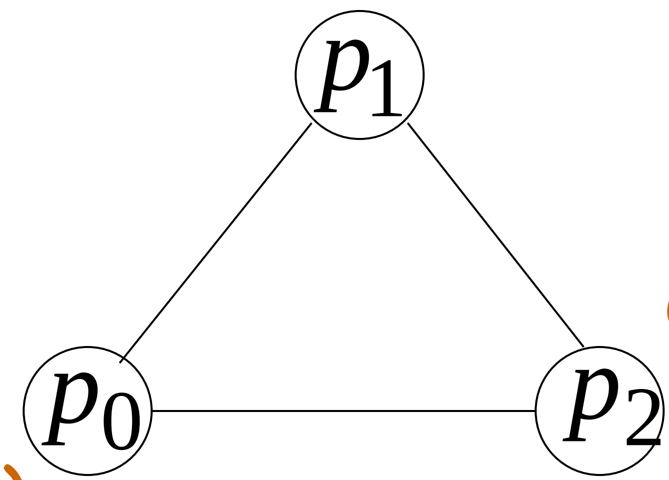
The 3 processors case

Lemma: There is no 1-resilient to byzantine failures algorithm for 3 processors

Proof: Assume by contradiction that there is a 1-resilient algorithm for 3 processors

B(1)

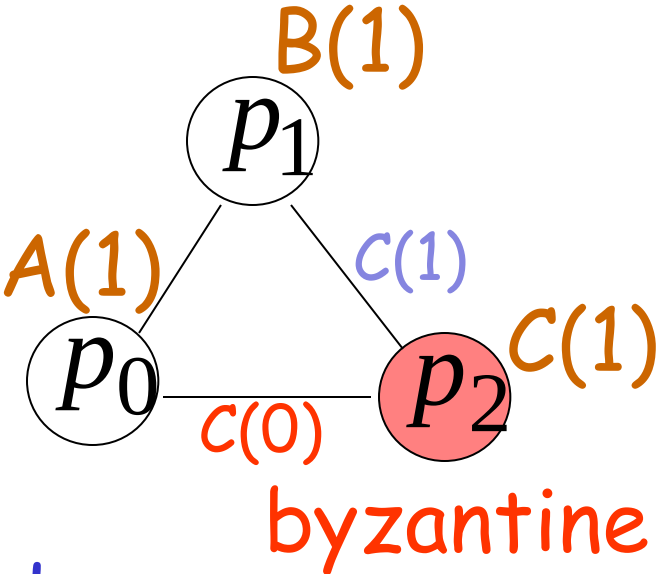
Local Algorithm (notice we admit non-homogeneity)



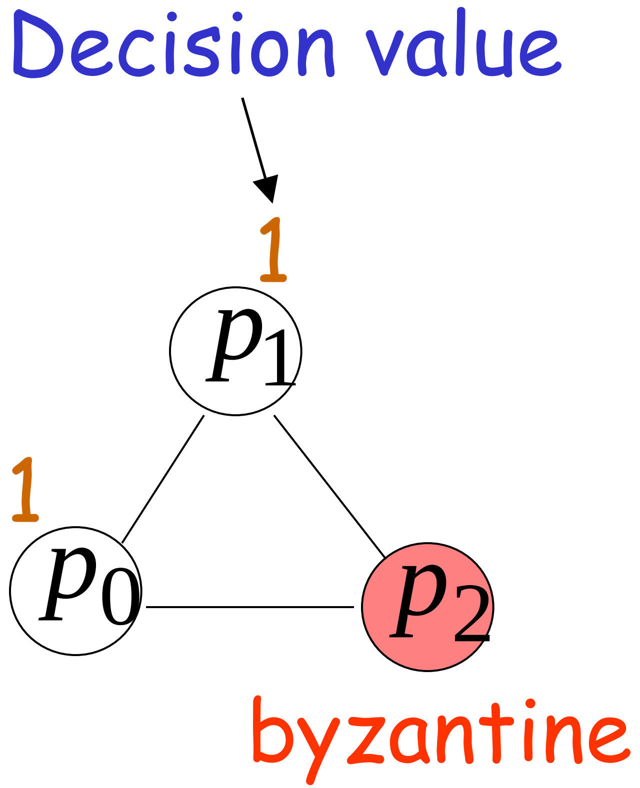
C(0)

A(0) ← Input value (either 0 or 1)

A first execution

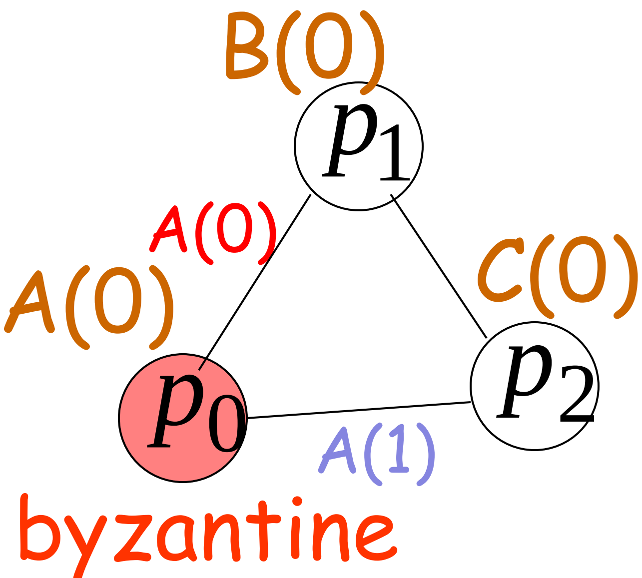


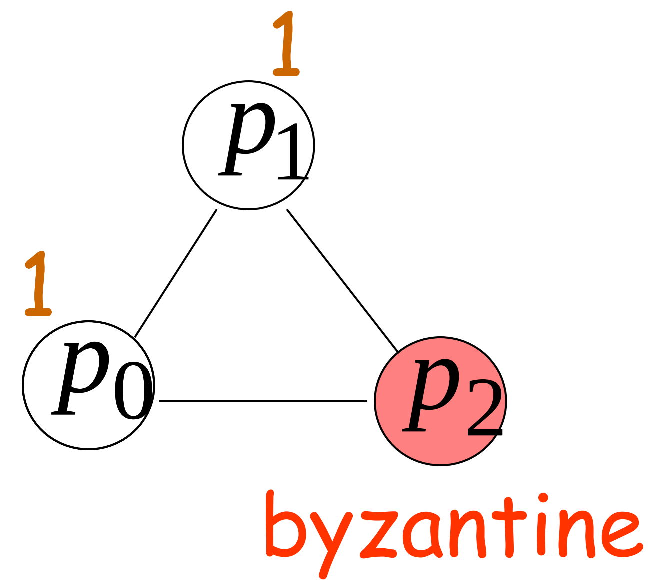
p_2 behaves towards p_0 (resp., p_1) has if it had input 0 (resp., 1)



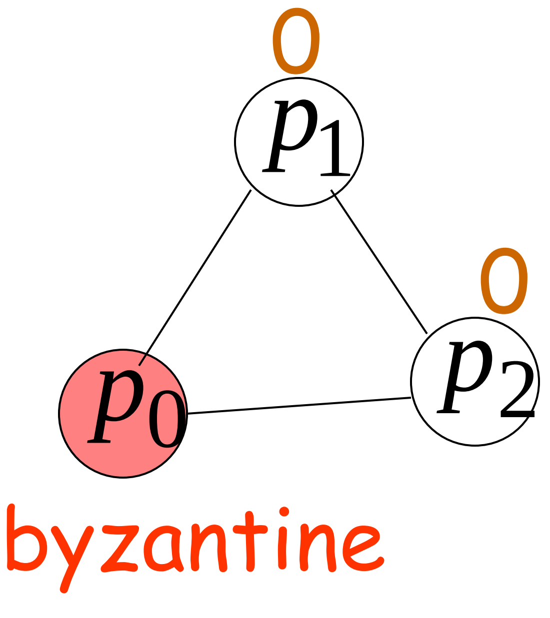
(validity condition)

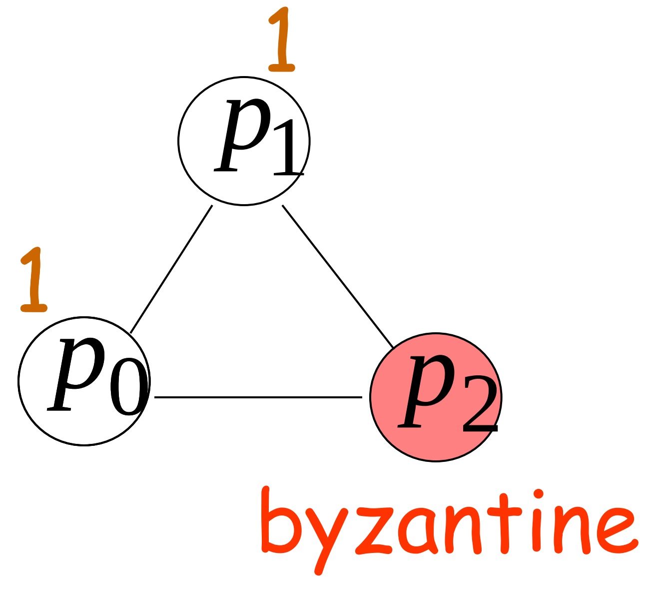
A second execution





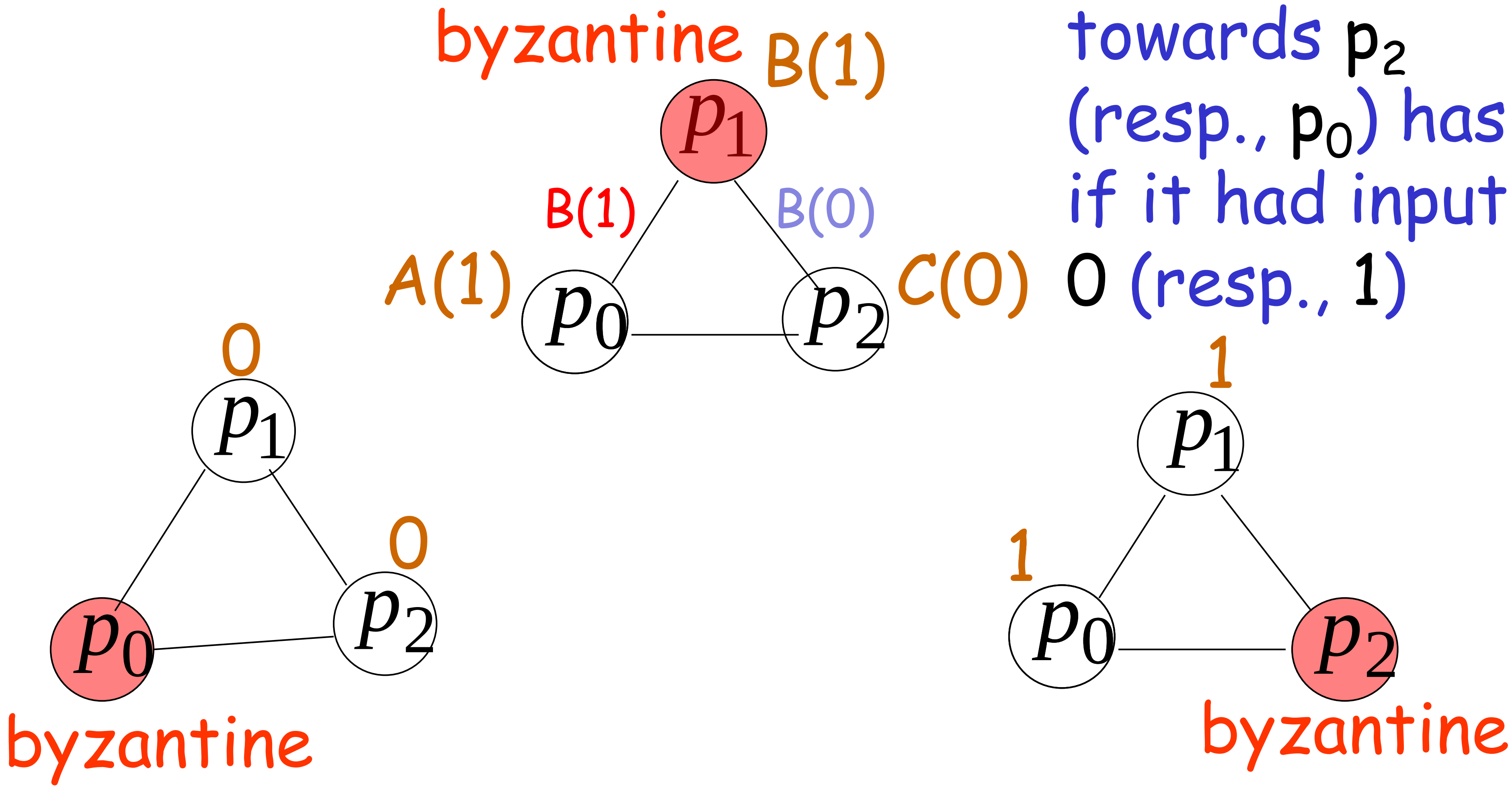
p_0 behaves towards p_1 (resp., p_2) has if it had input 0 (resp., 1)



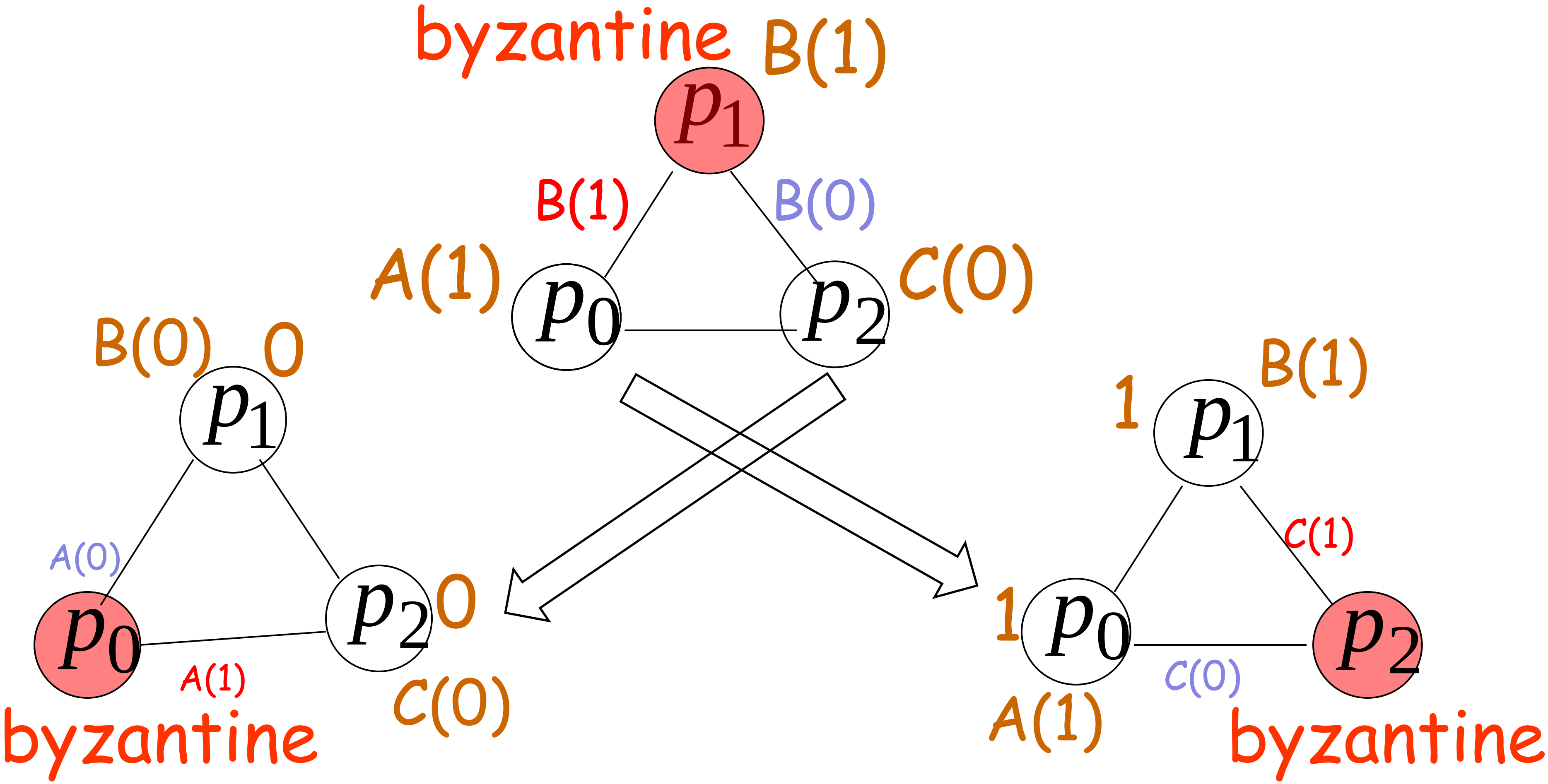


(validity condition)

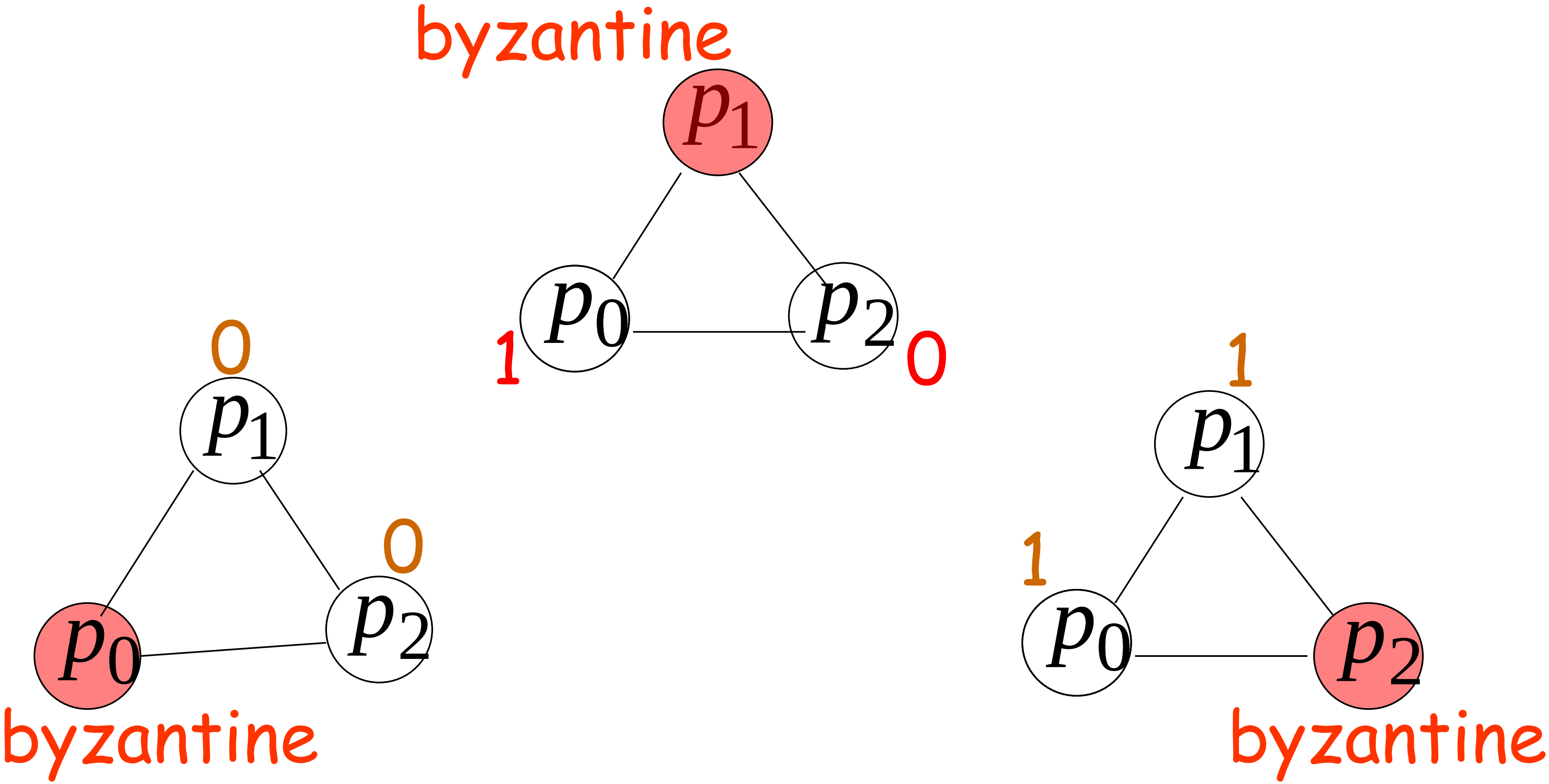
A third execution



p_1 behaves towards p_2 (resp., p_0) has if it had input 0 (resp., 1)



The view of p_2 (resp., p_0) in the third execution is exactly the same as in the second (resp., the first) execution, so it must take the same decision as before!



No agreement!!! Contradiction, since the algorithm was supposed to be 1-resilient

Therefore:

There is no algorithm that solves consensus for 3 processors in which 1 is a byzantine!

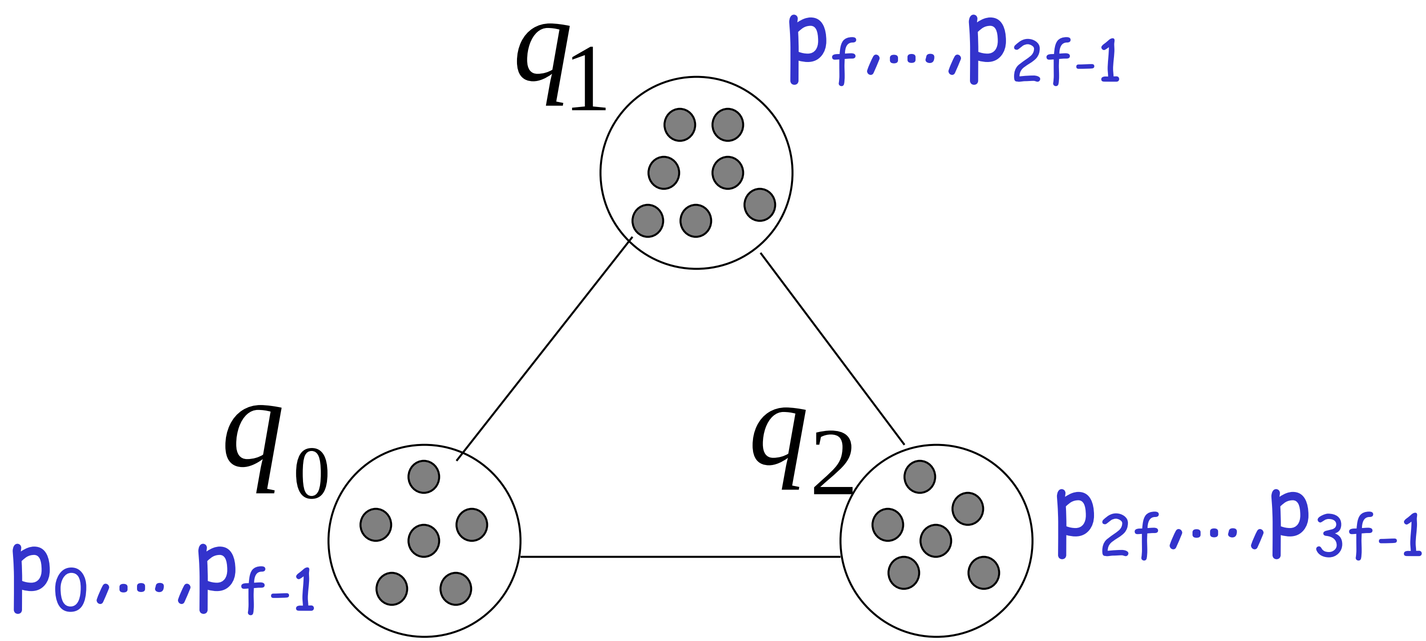
The n processors case

Assume by contradiction that there is an f-resilient distributed algorithm *A* for n>3 processors for $f \geq \frac{n}{3}$

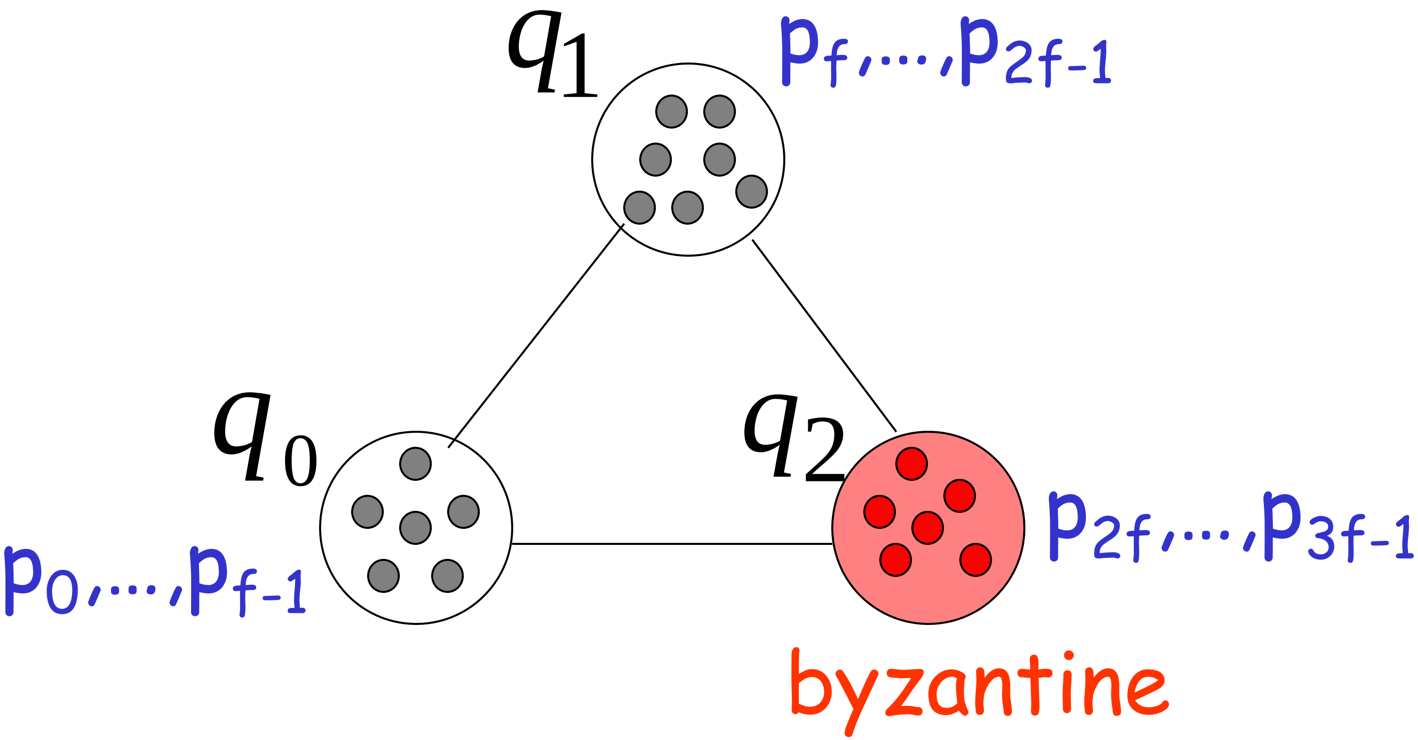
We will use *A* to solve consensus for 3 processors and 1 byzantine failure

(contradiction)

W.l.o.g. let $n=3f$, and let $P=\langle p_0,p_1,\dots,p_{3f-1}\rangle$ be the n-processor system. We partition arbitrarily the n processors in 3 sets P_0,P_1,P_2, each containing $n/3$ processors; then, given a 3-processor system $Q=\langle q_0,q_1,q_2\rangle$, we associate each q_i with P_i

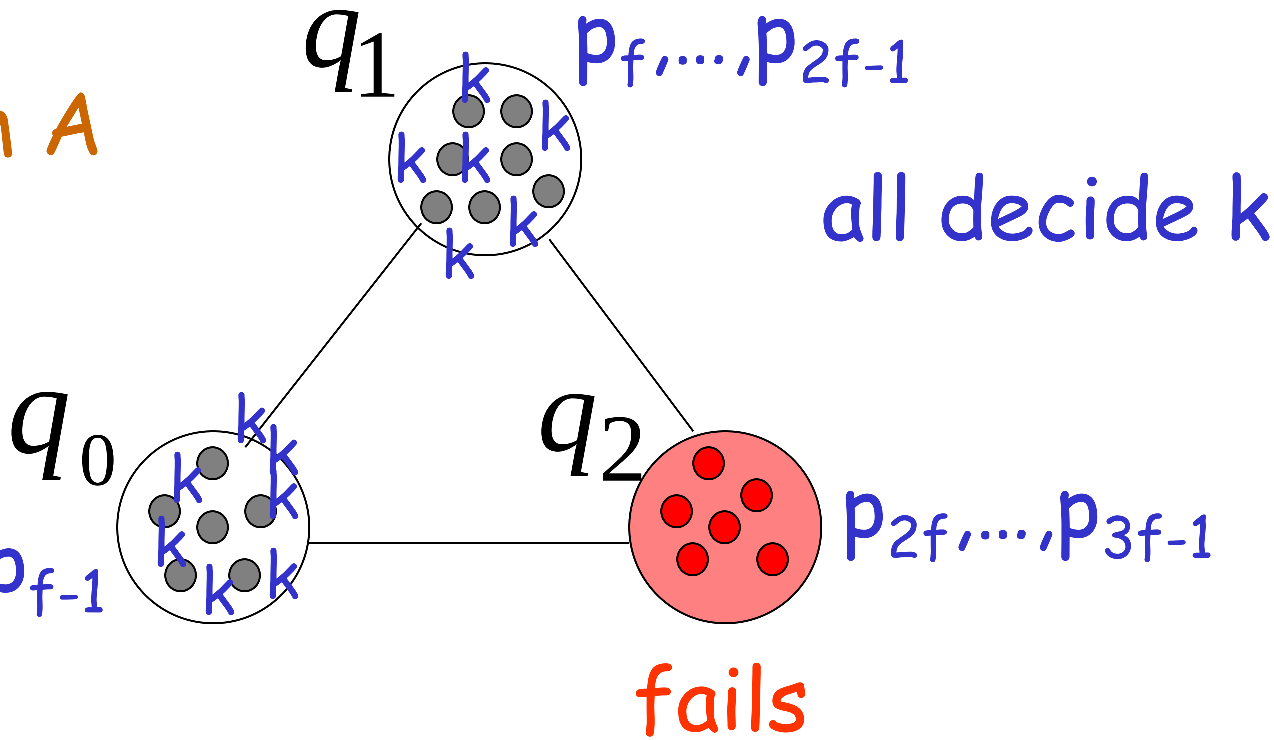


Each processor q_i simulates the execution of algorithm *A* once restricted to the set P_i of $n/3$ processors



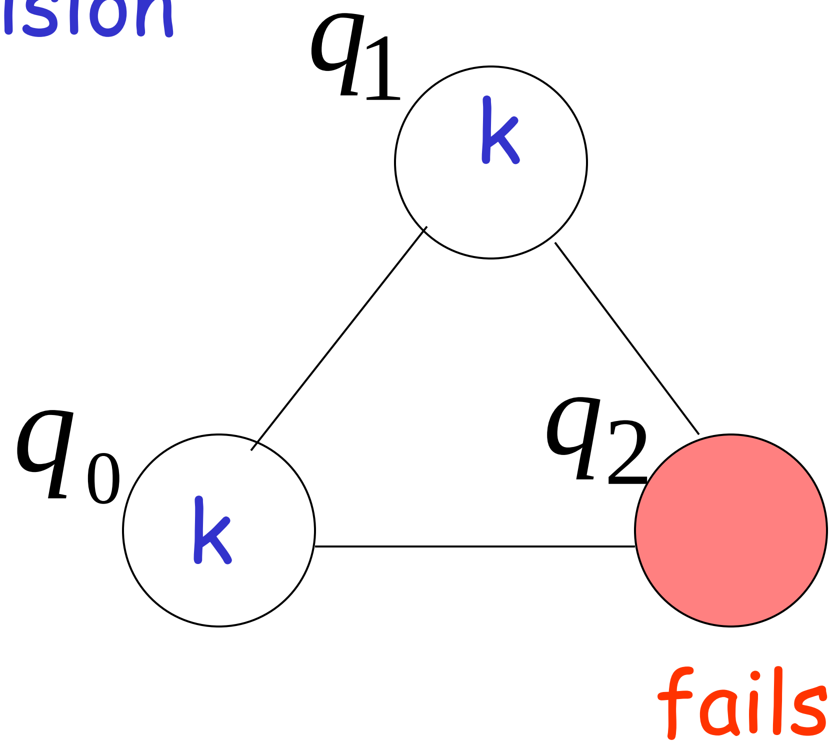
When a processor in Q fails, then at most n/3 original processors in the original n-processor system P are affected

Finish of
algorithm A



But we were assuming that the original algorithm A tolerates at most f=n/3 failures, so the remaining 2f processors must agree!

Final decision



We reached consensus with 1 failure

Impossible!!!

Therefore:

There is no f-resilient to byzantine failures algorithm for n processors in case

$$f \geq \frac{n}{3}$$

Question:

Is there an f-resilient to byzantine failures algorithm for n processors if f=(n-1)/3, namely for n=3f+1?

Exponential Tree Algorithm

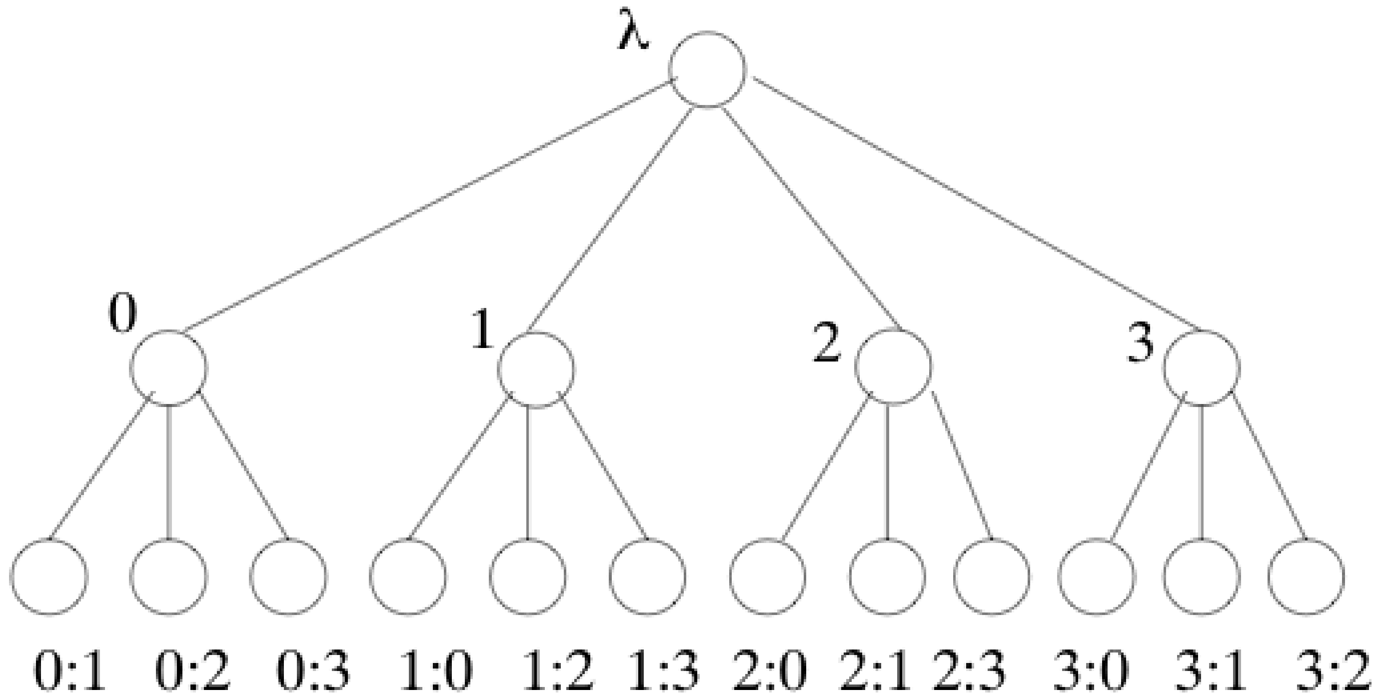
- This algorithm uses
  - $n=3f+1$ processors (optimal)
  - $f+1$ rounds (optimal)
  - exponential number of messages (sub-optimal, the King algorithm was using only $O(n^3)$ msgs)
- Each processor keeps a rooted tree data structure in its local state
- From a topological point of view, all the trees are identical: they have height $f+1$, each root has n children, the number of children decreases by 1 at each level, and all the leaves are at the same level
- Values are filled top-down in the tree during the $f+1$ rounds; more precisely, during round i, level i of the tree is filled
- At the end of round $f+1$, the values in the tree are used to compute bottom-up the decision.

Local Tree Data Structure

- **Assumption**: Similarly to the King algorithm, processors have (distinct) ids (now in {0,1,…,n-1}), and we denote by p_i the processor with id i; this is common knowledge, i.e., processors cannot cheat about their ids;
- Each tree node is labeled with a sequence of unique processor ids in 0,1,…,n-1:
  - Root's label is empty sequence λ (the root has level 0 and height f+1);
  - Root has n children, labeled 0 through n-1
  - Child node of the root (level 1) labeled i has n-1 children, labeled i:0 through i:n-1 (skipping i:i);
  - Node at level d>1 labeled $i_1:i_2:\ldots:i_d$ (these indexes are **distinct** values in 0,1,…,n-1) has n-d children, labeled $i_1:i_2:\ldots:i_d:0$ through $i_1:i_2:\ldots:i_d:n-1$ (skipping any index $i_1,i_2,\ldots,i_d$);
  - Nodes at level f+1 are leaves with label $i_1:i_2:\ldots:i_{f+1}$ and have height 0.

Example of Local Tree

The tree when n=4 and f=1:



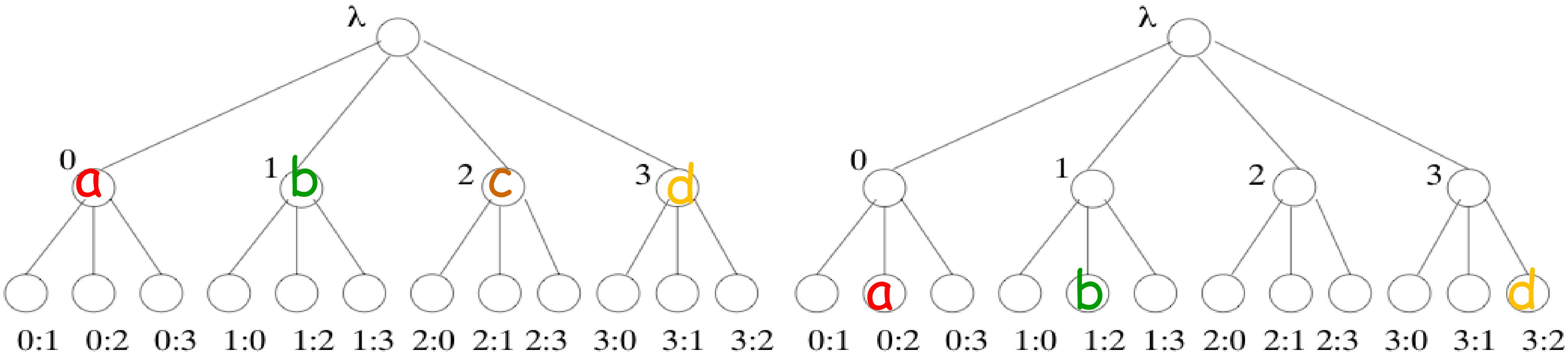
Filling-in the Tree Nodes

- **Round 1:**
  - Initially store your input in the root (level 0)
  - send level 0 of your tree (i.e., your input) to all (including yourself)
  - store value x received from p_j, $j=0,\ldots,n-1$, in tree node labeled j (level 1); use a default value "*" (known to all!) if necessary (i.e., in case a value is not received or it is unfeasible)
  - node labeled j in the tree associated with p_i now contains what "p_j told to p_i" about its input (assuming p_i is non-faulty)
- **Round 2:**
  - send level 1 of your tree to all, including yourself (this means, send n messages to each processor)
  - let $\{x_0,\ldots,x_{n-1}\}$ be the set of values that p_i receives from p_j; then, p_i discards x_j, and stores each remaining x_k in level-2 node labeled $k{:}j$ (and use default value "*" if necessary)
  - node $k{:}j$ in the tree associated with p_i now contains "p_j told to p_i that "p_k told to p_j that its input was x_k""

Example: filling the Local Tree at round #2

As before, n=4 and f=1, and assume that non-faulty p_2 tells to non-faulty p_1 that the first level of its local tree contains {a,b,c,d}; then, p_1 stores in the local tree:

Tree at p_2 at the end of round 1 Tree at p_1



⇒ The value c is not stored in the tree at p_1 since there is no node with label 2:2

Filling-in the Tree Nodes (2)

⋮

- Round d>2:
  - send level d-1 of your tree to all, including yourself (this means, send $n(n-1)\ldots(n-(d-2))$ messages to each processor, one for each node on level $d-1$)
  - Let x be the value that p_i receives from p_j for node of level d-1 labeled $i_1:i_2:\ldots:i_{d-1}$, with $i_1,i_2,\ldots,i_{d-1} \neq j$; then, p_i stores x in tree node labeled $i_1:i_2:\ldots:i_{d-1}:j$ (level d), using default value "*" if necessary
- Continue for f+1 rounds

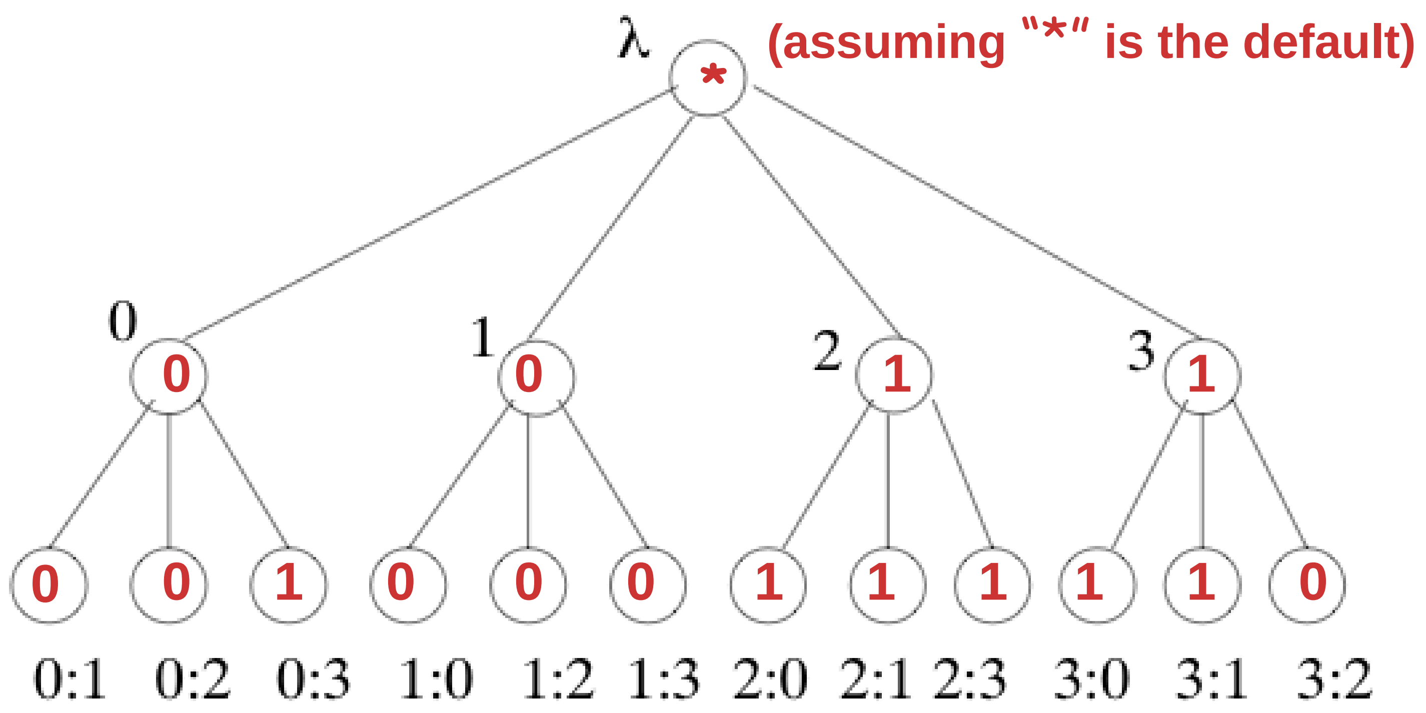
Calculating the Decision

- In round f+1, each processor uses the values in its tree to compute its final decision (output)
- Recursively compute the "resolved" value for the root of the tree, **`resolve`**(λ), based on the "resolved" values for the other tree nodes:

$$\texttt{resolve}(\pi) = \begin{cases} \text{value in tree node labeled } \pi \text{ if it is a leaf} \\ \text{majority}\{\texttt{resolve}(\pi') : \pi' \text{ is a child of } \pi\} \text{ otherwise (use default “*” if tied)} \end{cases}$$

Example of Resolving Values

The tree when n=4 and f=1:



Resolved Values are consistent

Lemma 1: If p_i and p_j are non-faulty, then p_i's resolved value for tree node labeled $\pi=\pi'j$ is equal to what p_j stores in its node π' during the filling-up of the tree (and so the value stored in π by p_i is the same value which is resolved in π by p_i, i.e., the resolved value is consistent with the stored value). (Notice this lemma does not hold for the root)

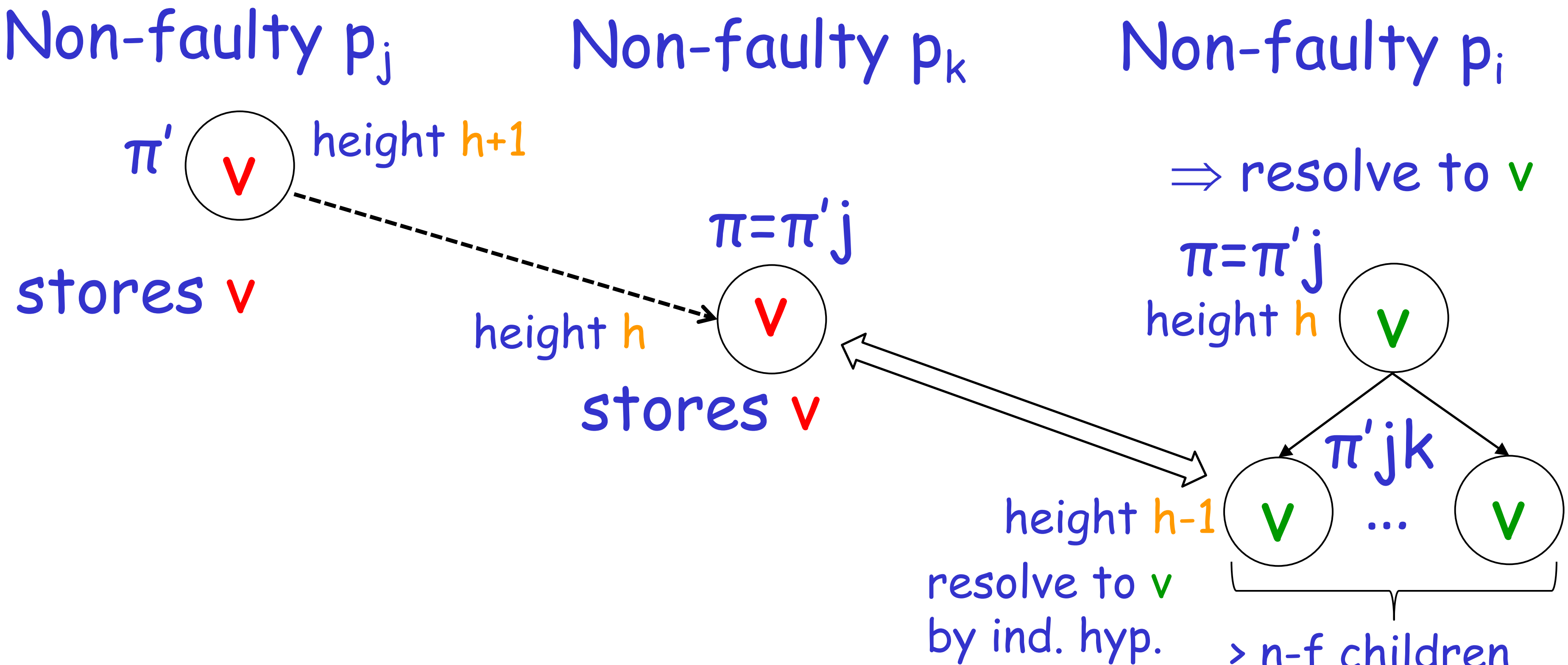
Proof: By induction on the height of the tree node.

- **Basis**: height=0 (leaf level). Then, p_i stores in node $\pi=\pi'j$ what p_j sends to it for π' in the last round. By definition, this is the resolved value by p_i for π.

- **Induction**: π is not a leaf, i.e., has height $h>0$;
  - By definition, π has at least n-f children, and since $n>3f$, this implies $n-f>2f$, i.e., it has a majority of non-faulty children (i.e., whose last digit of the label corresponds to a non-faulty processor)
  - Let $\pi k=\pi' jk$ be a child of π of height $h-1$ such that p_k is non-faulty.
  - Since p_j is non-faulty, it correctly reports a value v stored in its π' node; thus, p_k stores it in its $\pi=\pi' j$ node.
  - By induction, p_i's resolved value for πk equals the value v that p_k stored in its π node.
  - So, all of π's non-faulty children resolve to v in p_i's tree, and thus π resolves to v in p_i's tree.

END of PROOF

Inductive step by a picture



Remark: all the non-faulty processors will resolve the very same value in $\pi=\pi'j$, namely v

Validity

- Suppose all inputs of (non-faulty) processors are v
- Non-faulty processor p_i decides `resolve`(λ), which is the majority among `resolve`(j), $0 \leq j \leq n-1$, based on p_i's tree.
- Since by Lemma 1 resolved values are consistent, if p_j is non-faulty, then p_i's resolved value for tree node labeled j, i.e., `resolve`(j), is equal to what p_i stores in the tree node labeled j, which in turn is equal to what p_j stores in its root, namely p_j's input value, i.e., v.
- Since there is a majority of non-faulty processors, and their inputs are all equal to v, then p_i decides v.

Agreement: Common Nodes and Frontiers

Definition 1: A tree node π is **common** if all non-faulty processors compute the same value of `resolve`(π).

$\Rightarrow$ To prove agreement, we have now to show that the root is common

Definition 2: A tree node π has a **common frontier** if every path from π to a leaf contains at least a common node.

Lemma 2: If π has a common frontier, then π is common.

Proof: By induction on the height of π:

- **Basis** (π is a leaf): then, since the only path from π to a leaf consists solely of π, the common node of such a path can only be π, and so π is common;
- **Induction** (π is not a leaf): By contradiction, assume π has height h and is not common; then:
  - Every child π' of π has a common frontier (this is not true, in general, if π would be common);
  - Since every child π' of π has height h-1 and has a common frontier, then by the inductive hypothesis, it is common;
  - Then, all non-faulty processors resolve the same value for every child π' of π, and thus all non-faulty processors resolve the same value for π, i.e., π is common.

END of PROOF

Agreement: the root has a common frontier

- There are $f+2$ nodes on any root-leaf path
- The label of each non-root node on a root-leaf path ends in a distinct processor index: $i_1, i_2, \ldots, i_{f+1}$
- Since there are at most f faulty processors, at least one of such nodes has a label ending with a non-faulty processor index
- This node, say $i_1:i_2:,\ldots,i_{k-1}:i_k$, by Lemma 1 is **common** (more precisely, in all the trees associated with non-faulty processors, the resolved value in $i_1:i_2:,\ldots,i_{k-1}:i_k$ equals the value stored by the non-faulty processor p_{i_k} in node $i_1:i_2:,\ldots,:i_{k-1}$)

$\Rightarrow$ Thus, the root has a common frontier, since on any root-leaf path there is at least a common node, and so the root is common (by previous lemma)

$\Rightarrow$ Therefore, agreement is guaranteed!

Complexity

- Exponential tree algorithm uses $f+1$ rounds, and $n=3f+1$ processors are enough to guarantee correctness (see Lemma 1)
- Exponential number of messages:
  - In round 1, each (non-faulty) processor sends n messages $\Rightarrow O(n^2)$ total messages
  - In round $2 \le d \le f+1$, each of the $O(n)$ (non-faulty) processors broadcasts to all (i.e., n processors) the level $d-1$ of its local tree, which contains $n(n-1)(n-2)\ldots(n-(d-2))$ nodes $\Rightarrow$ this means, for round d, a total of

    $O(n \cdot n \cdot n(n-1)(n-2)\ldots(n-(d-2)))=O(n^{d+1})$ messages
  - This means a total of $O(n^2)+O(n^3)+\ldots+ O(n^{f+2})= O(n^{f+2})$ messages, and since $f=O(n)$, this number is exponential in n if f is more than a constant relative to n

Exercise 1: Show an execution with n=4 processors and f=1 for which the King algorithm fails.

Exercise 2: Show an execution with n=3 processors and f=1 for which the exp-tree algorithm fails.