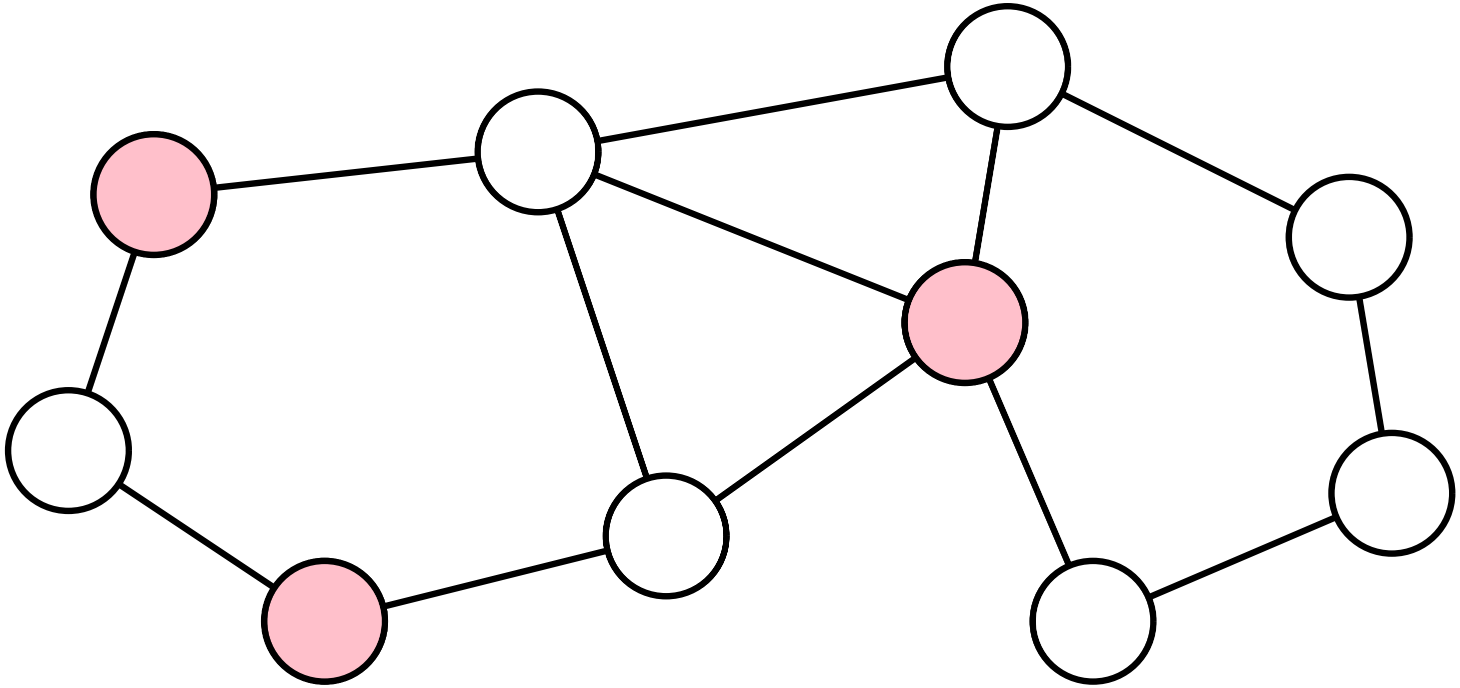
Luby's Maximal Independent Set Algorithm

Independent Sets



Definition: An *independent set* of a graph $G = (V, E)$ is a set $\mathcal{I} \subseteq V$ such that $\forall (u, v) \in E$, $u \notin \mathcal{I}$ or $v \notin \mathcal{I}$ (or both).

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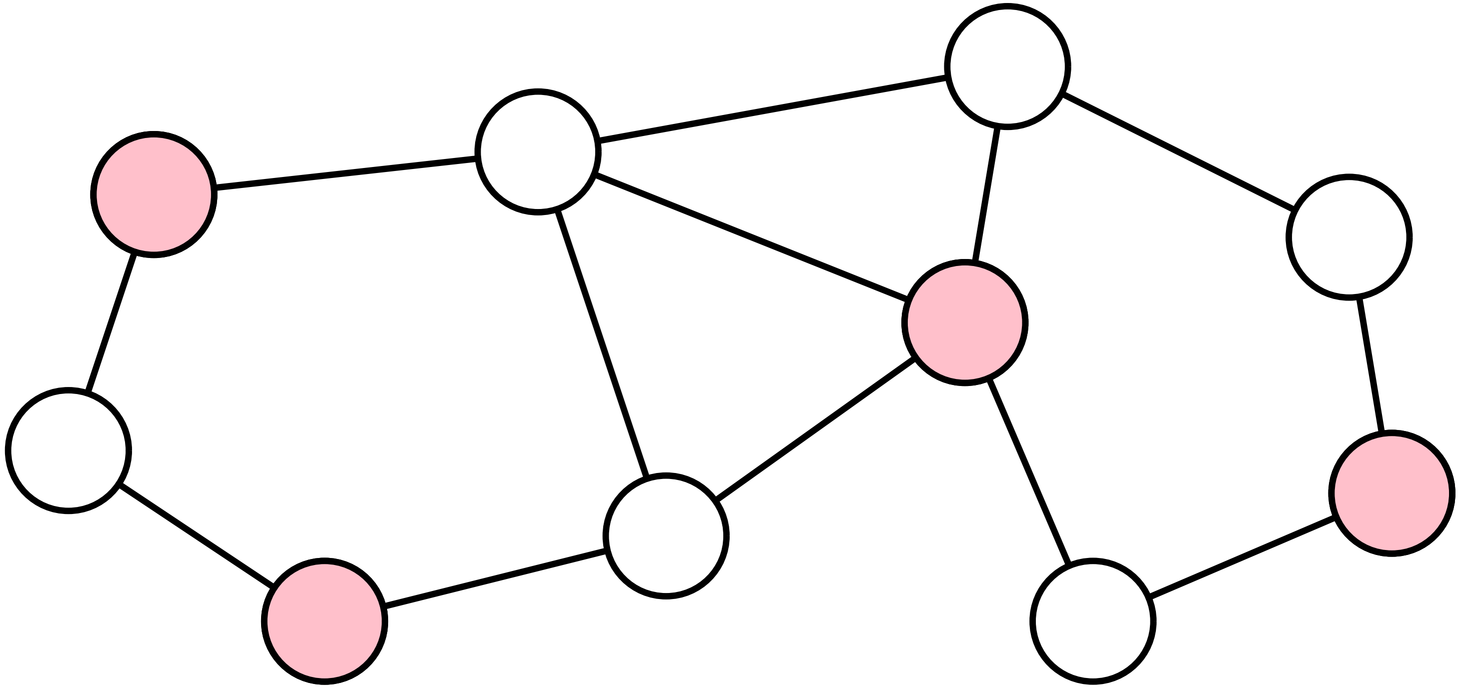

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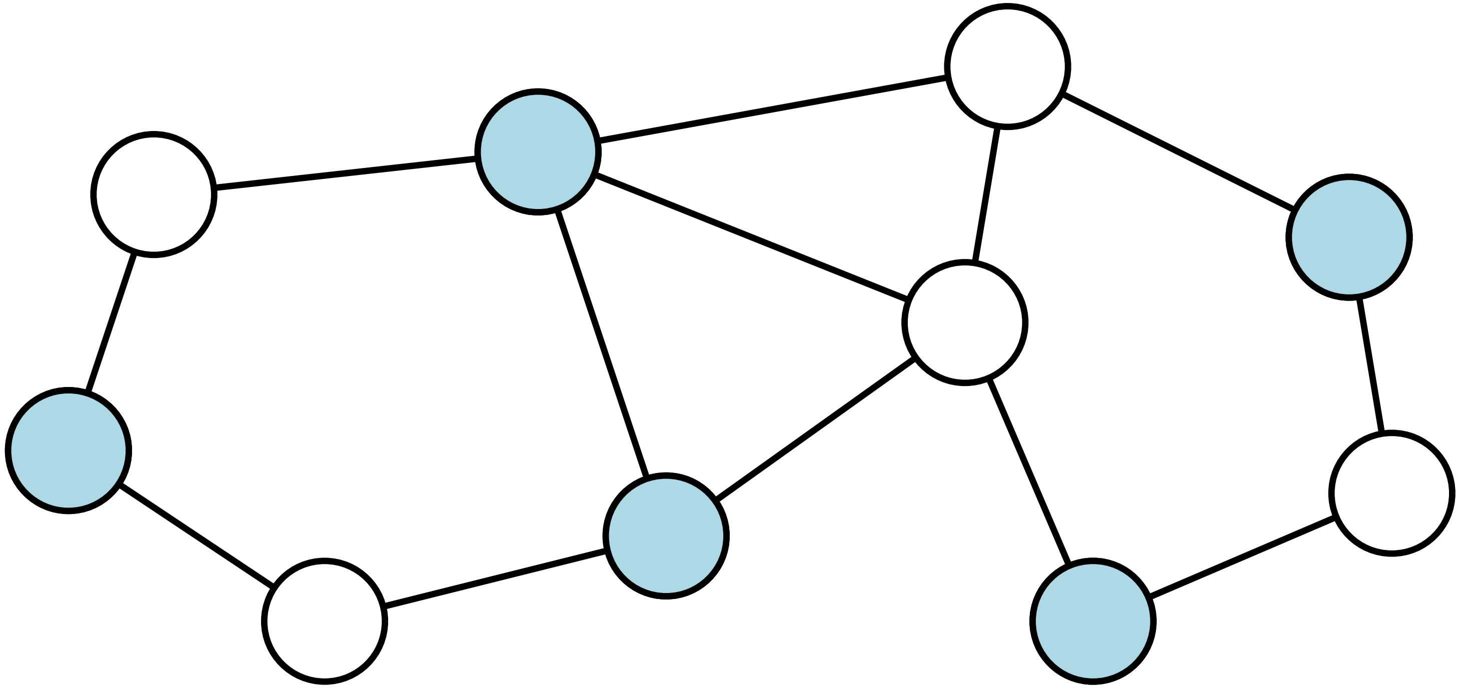

A **Maximal** Independent Set is *not necessarily* a **Maximum** Independent Set.

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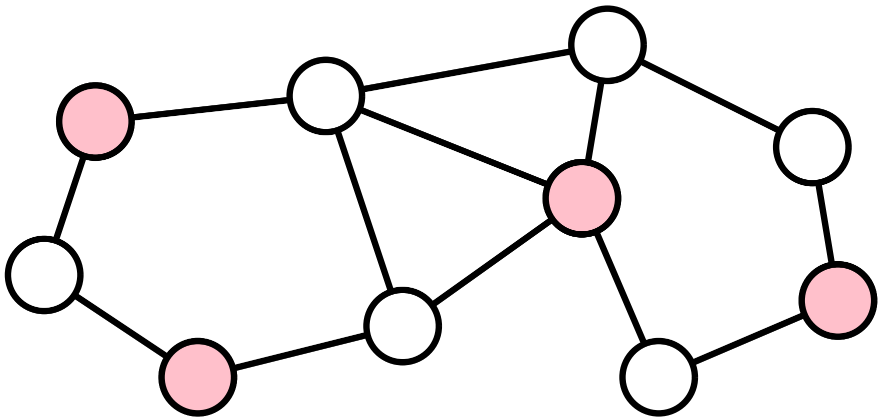
Luby's Algoritm

- A distributed algorithm to compute a Maximal Independent Set (MIS)

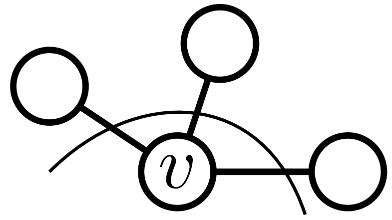
- Runs in time $O(\log d \cdot \log n)$ with high probability (w.h.p.), where d is the maximum degree of G.

- Asymptotically better than the algorithm of the previous lecture (which required $O(d \log n)$ time, w.h.p.).



Luby's Algoritm

Let $d(v)$ be the degree of vertex v in G.



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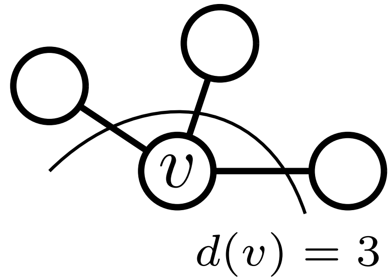

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- Each node v in G_k *elects* itself with probability $p(v) = \frac{1}{2d(v)}$.

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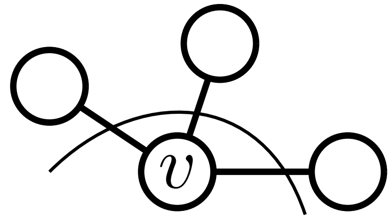

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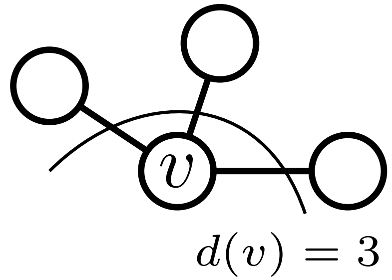

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- G_{k+1} is obtained by deleting the vertices in $\mathcal{I}_k$ and their neighbors $N(\mathcal{I}_k)$ from G_k

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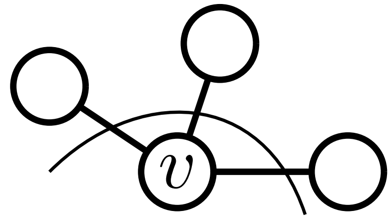

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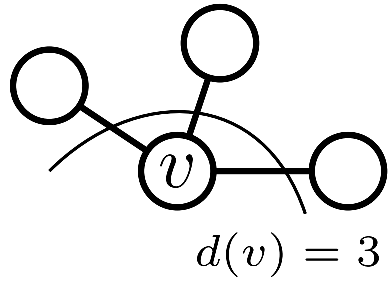

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If v is a singleton, v always elects itself.

- Each node v in G_k *elects* itself with probability $p(v) = \frac{1}{2d(v)}$.
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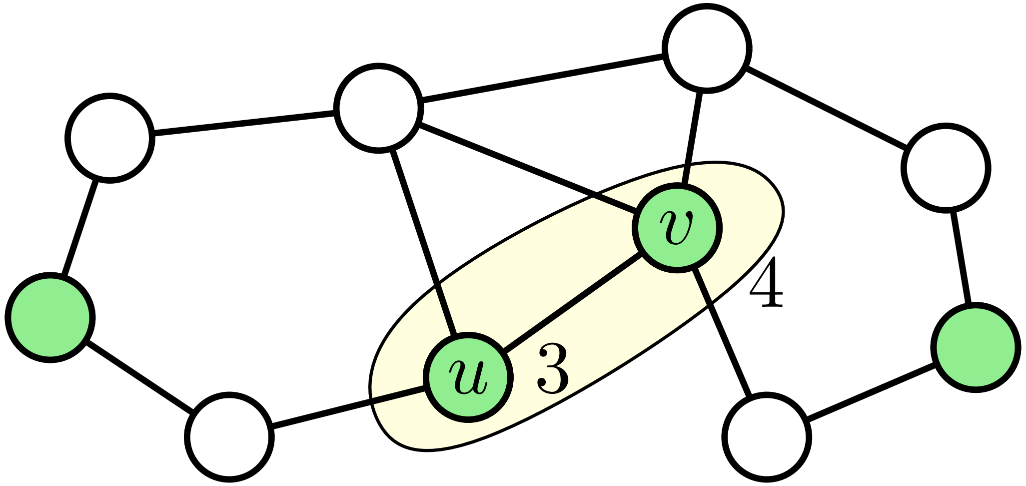

If two neighbors are elected simultaneously, the node with higher degree wins (remains in $\mathcal{I}_k$).

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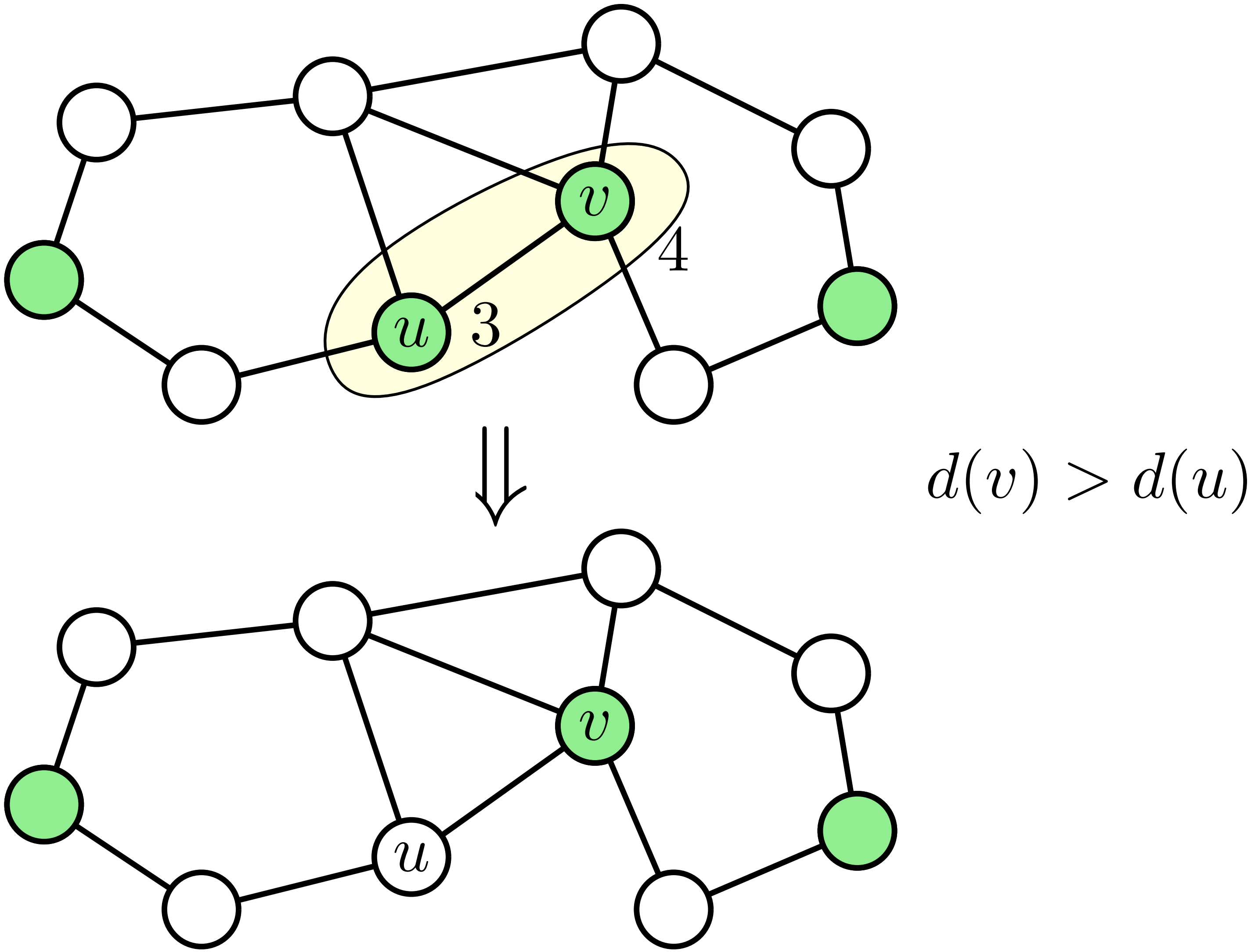

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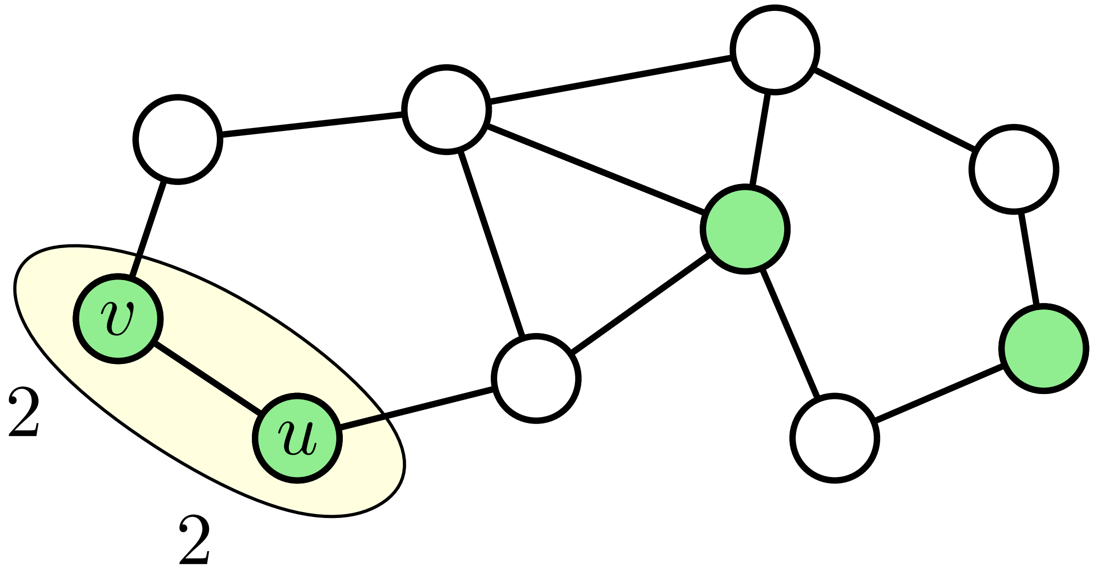
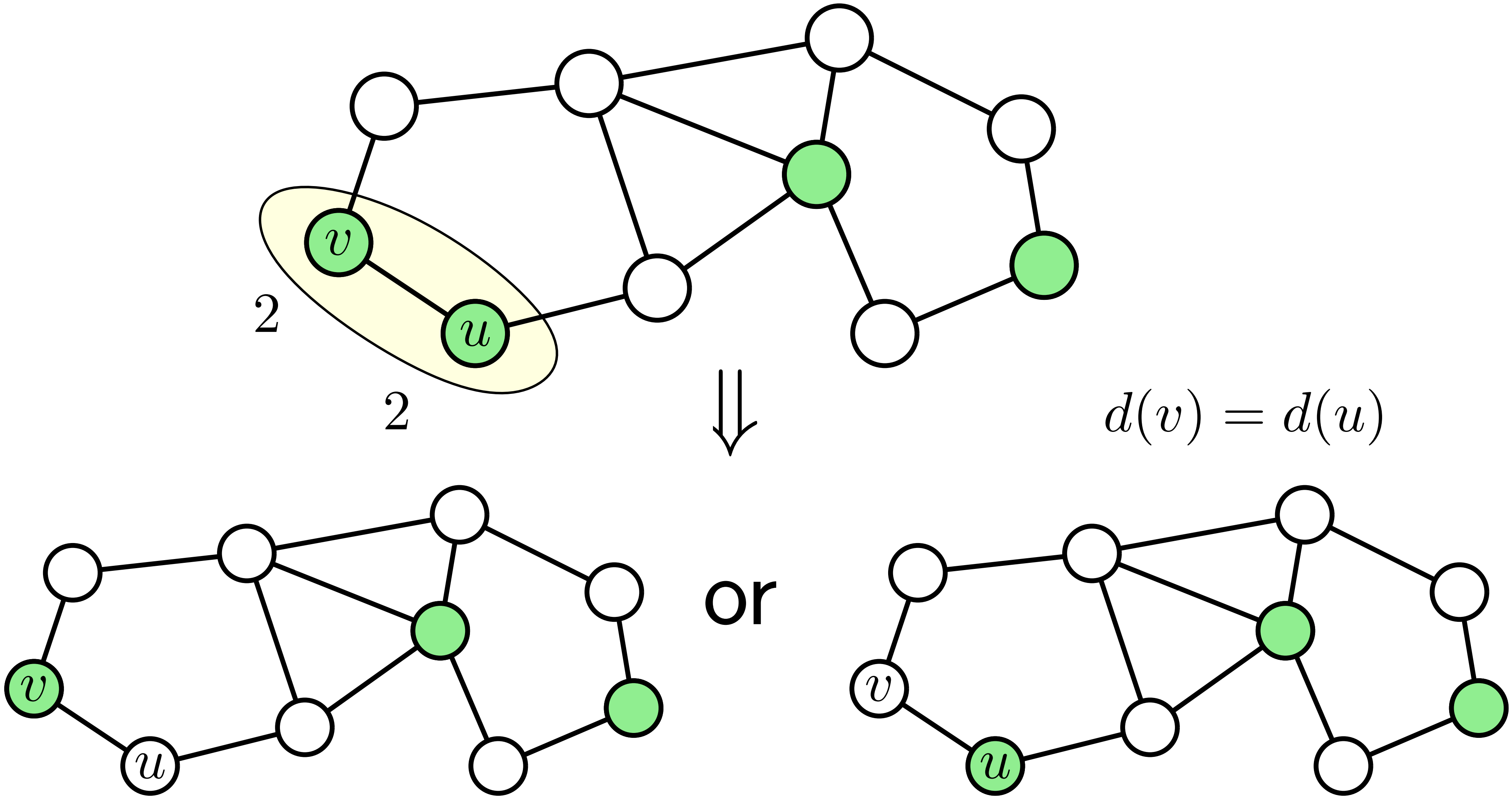


$d(v) = d(u)$

If both nodes have the same degree, choose using an arbitrary tie-breaking scheme (e.g., by vertex ID).

Luby's Algoritm: Finding $\mathcal{I}_k$



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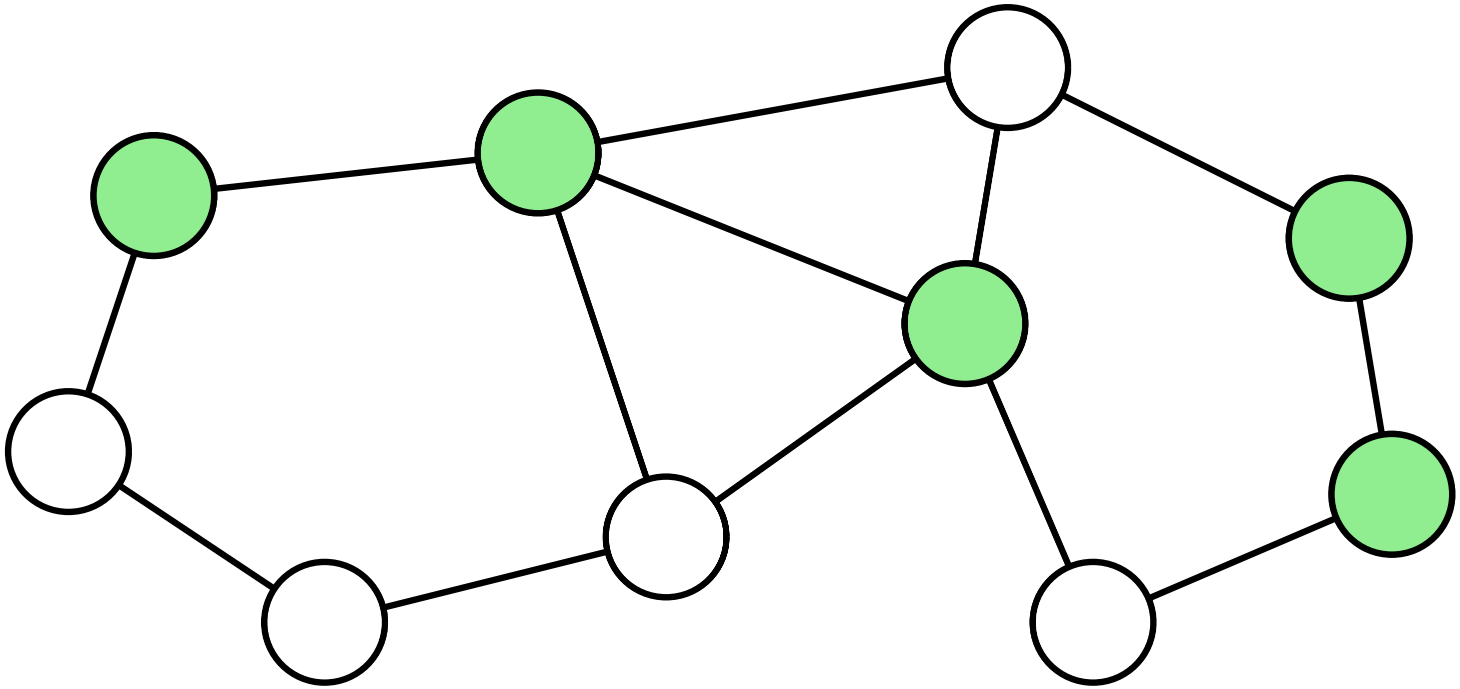
Luby's Algoritm

Previous rules are used to remove "problematic" nodes from the candidate nodes.

G_k



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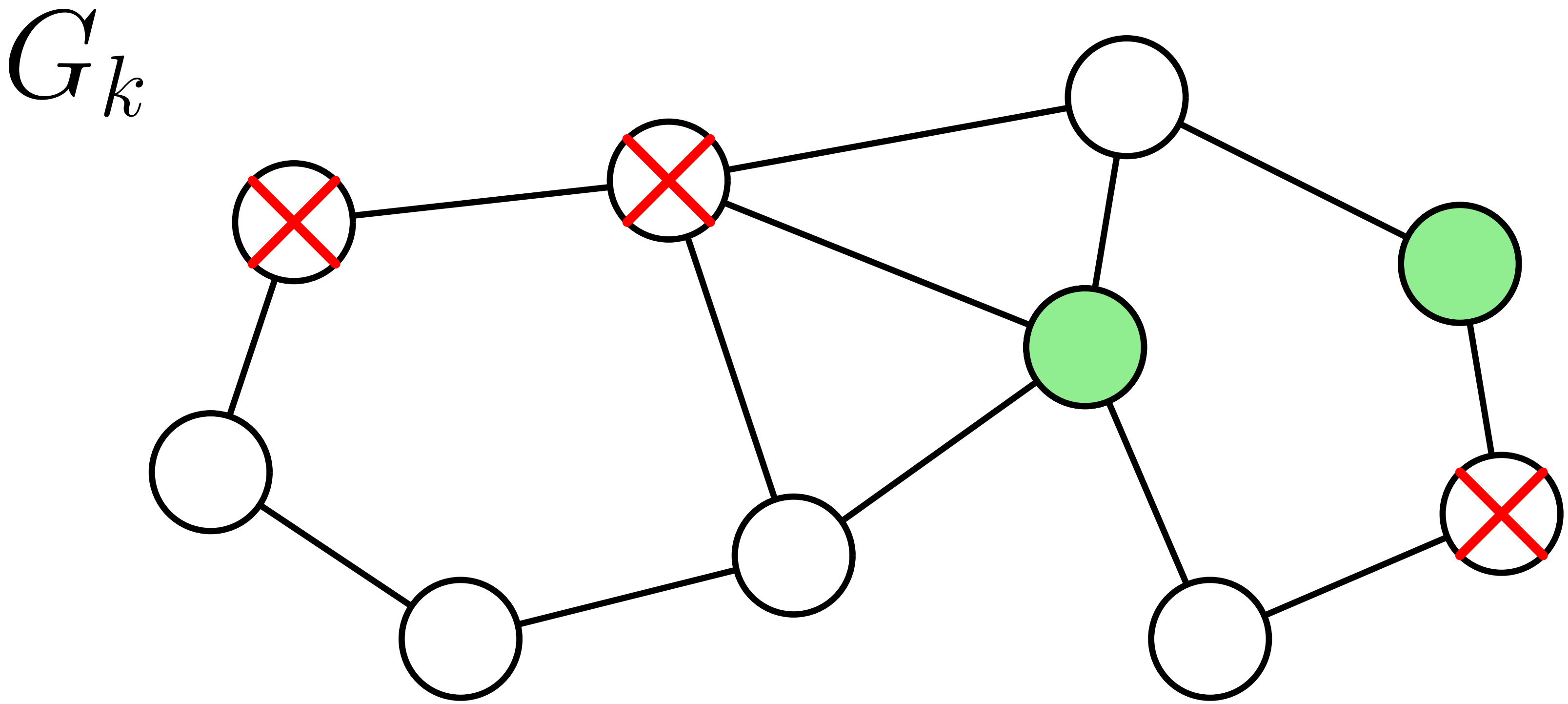

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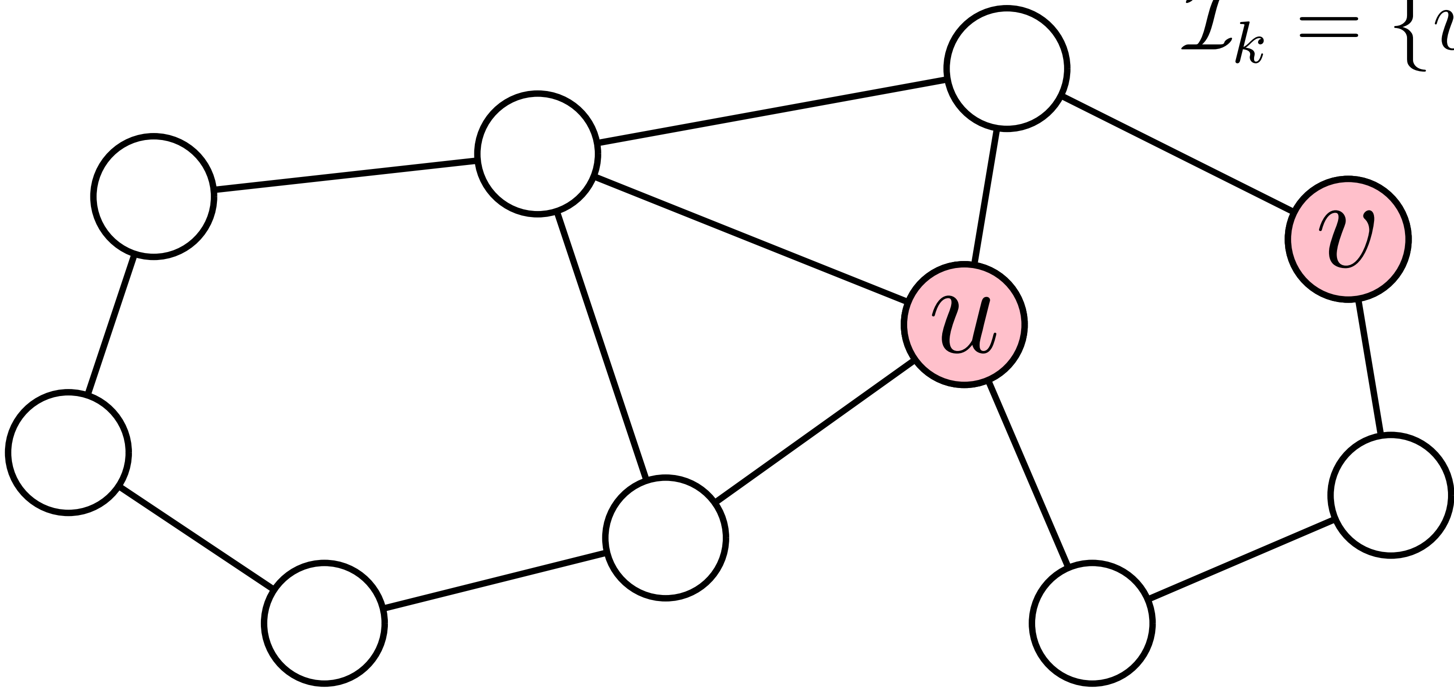
G_k $\mathcal{I}_k = \{u, v\}$



The remaning nodes form the independent set $\mathcal{I}_k$

Analysis

Consider a generic phase k

A *good event* H_v for node v is the following:

At least one neighbor of v *enters* $\mathcal{I}_k$ *(i.e.,* $\mathcal{I}_k \cap N(v) \neq \emptyset$*)*

Analysis

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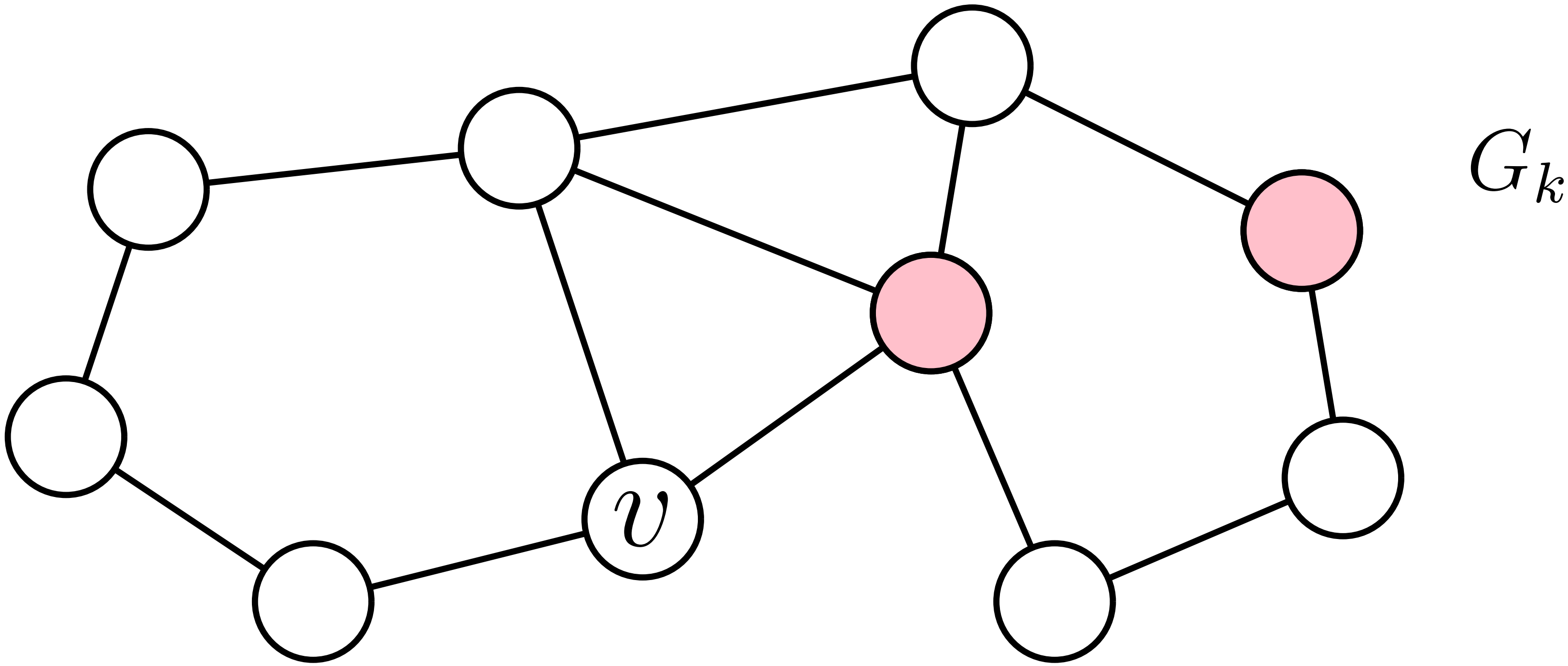
At least one neighbor of v enters $\mathcal{I}_k$ (i.e., $\mathcal{I}_k \cap N(v) \neq \emptyset$)

If H_v happens, then $v \in N(I_k) \implies v$ does not belong to G_{k+1}.



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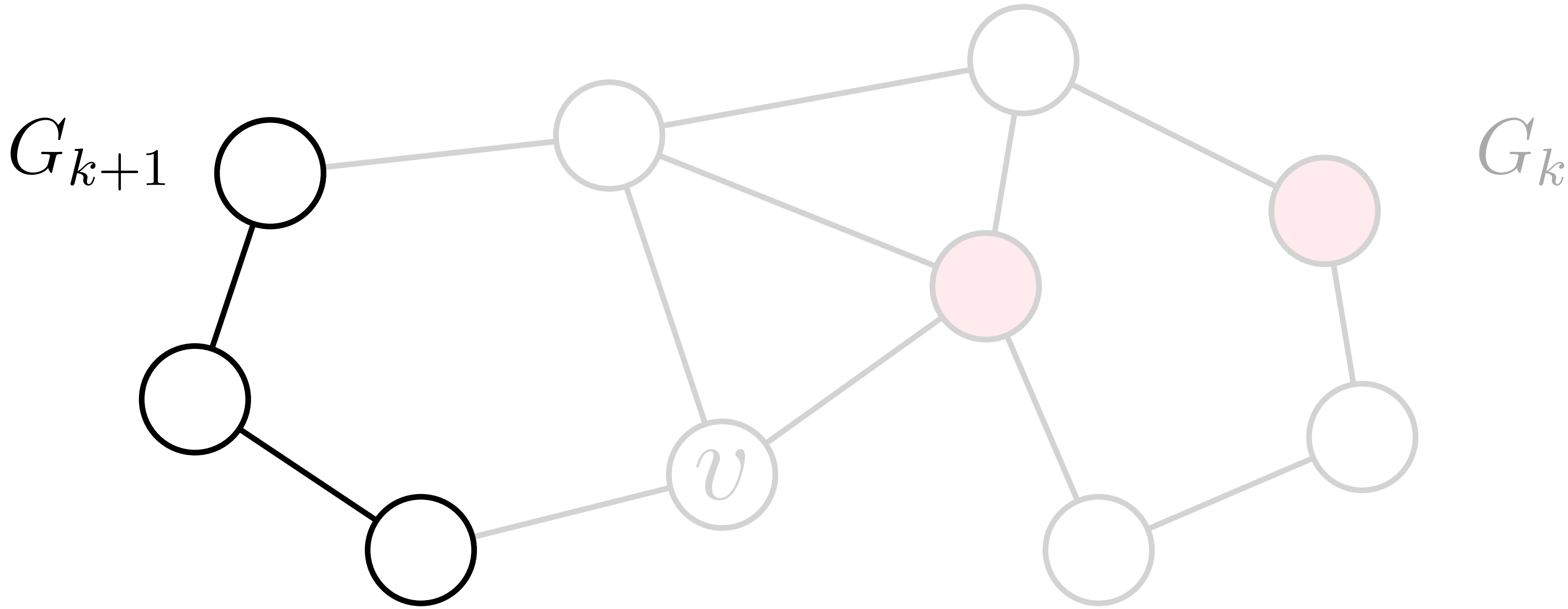

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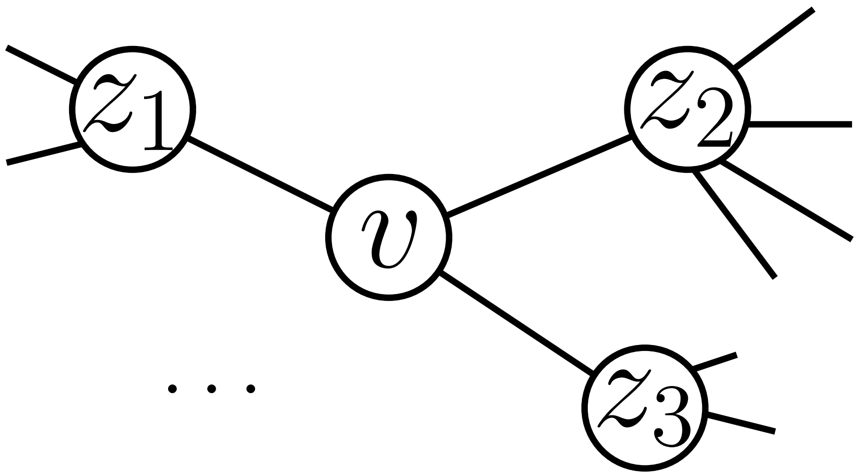
> **Lemma 1**: With probability at least $1 - e^{-\frac{d(v)}{2\widetilde{d}(v)}}$, at least one neighbor of v elects itself.
>
> Where $\widetilde{d}(v) = \max_{z_i \in N(v)} d(z_i)$ is the maximum degree among the neighbors of v.



$d(v) = 3$

$\widetilde{d}(v) = 5$

Analysis

Lemma 1: With probability at least $1 - e^{-\frac{d(v)}{2\widetilde{d}(v)}}$, at least one neighbor of v elects itself.

Proof:

The probability of the complementary event (no neighbor of v elects itself) is:

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$$\prod_{z_i \in N(v)} (1 - p(z_i))$$

(Recall that elections are independent)

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$$\leq e^{-\frac{d(v)}{2\widetilde{d}(v)}}. \qquad \square$$

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Lemma 2: If some neighbor of v elects itself, then some neighbor z of v belongs to $\mathcal{I}_k$ with probability at least $\frac{1}{2}$.

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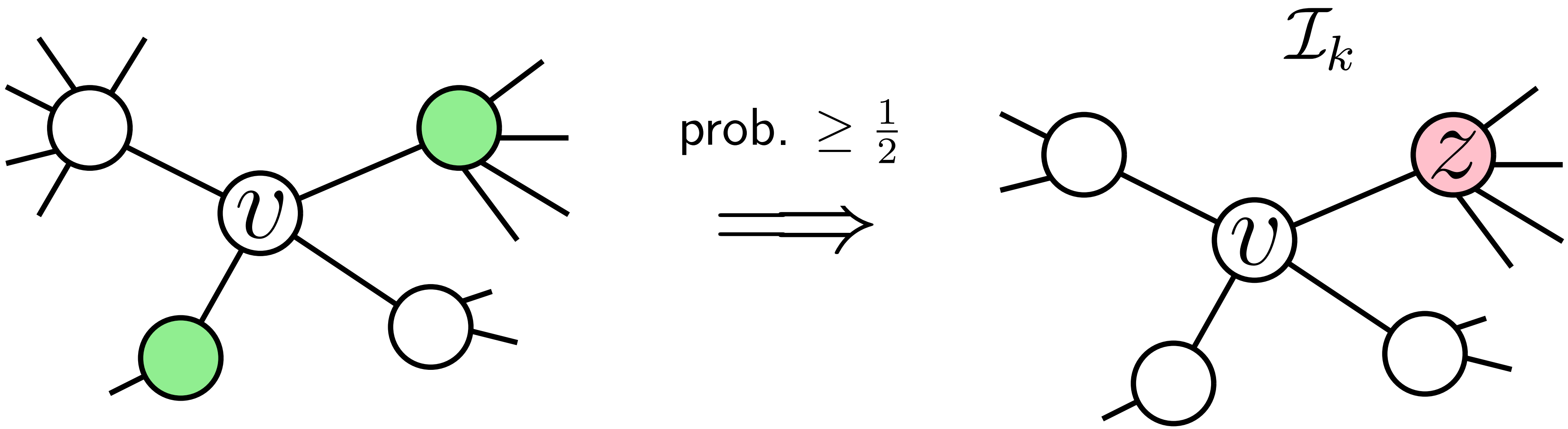

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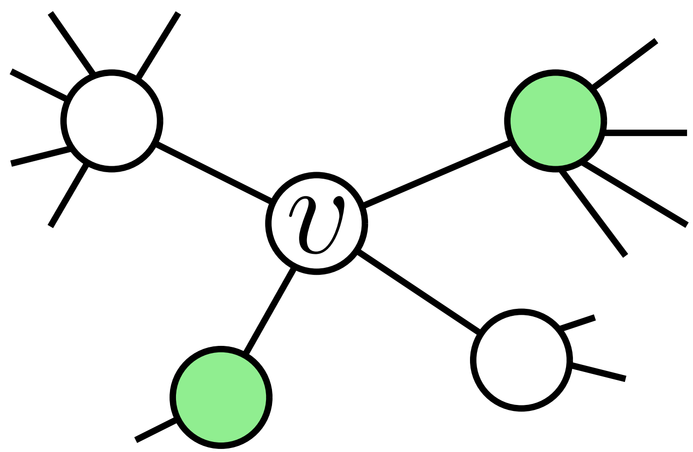

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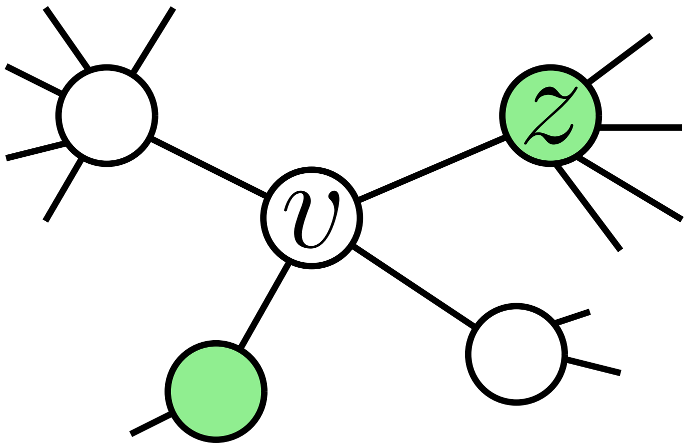

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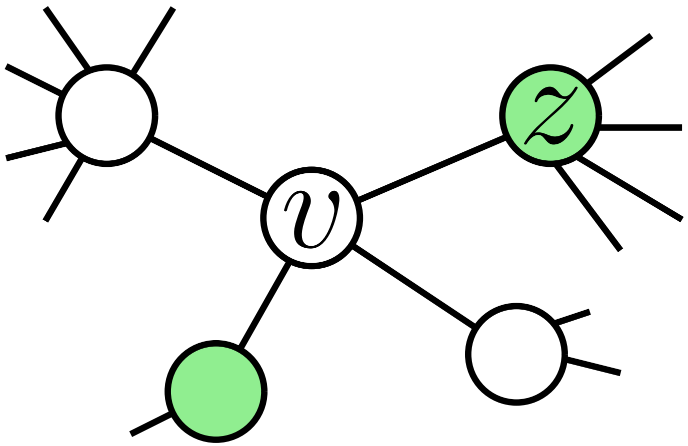
In case of ties, break them using the chosen tie-breaking scheme



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If no vertex in W elects itself, then $z \in \mathcal{I}_k$.

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Proof:

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$$= \frac{|W|}{2d(z)} \leq \frac{d(z)}{2d(z)} = \frac{1}{2}.$$

Analysis

Lemma 2: If some neighbor of v elects itself, then some neighbor z of v belongs to $\mathcal{I}_k$ with probability at least $\frac{1}{2}$.

Proof:

Let A_w be the event "w elects itself" and let $\overline{A}_w$ be the complementary event "w does not elect itself"

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This is just some constant $c \approx 0.39$

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In a generic phase k, a node with degree at least $\frac{d_k}{2}$ disappears with probability at least c.

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$$(1-c)^\phi = (1-c)^{3 \log_{1-c} \frac{1}{n}} = (1-c)^{\log_{1-c} \frac{1}{n^3}} = \frac{1}{n^3}$$

Analysis

Thus, after $3\log_{1-c}\frac{1}{n}$ phases, the probability that:

- v did not disappear; **and**
- the degree of v is still above $\frac{d_k}{2}$;

is at most $\frac{1}{n^3}$.

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The probabiltiy that after $3\log_{1-c}\frac{1}{n}$ phases there is **at least one node** with degree larger than $\frac{d_k}{2}$ is at most:

$$n \cdot \frac{1}{n^3} = \frac{1}{n^2}$$

Analysis

In other words:

Every $3 \log_{1-c} \frac{1}{n}$ phases the maximum degree of the graph **halves** with probability at least $1 - \frac{1}{n^2}$.

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Or it will be formed in the next iteration.

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Question: How many "groups" of $3\log_{1-c}\frac{1}{n}$ phases are needed for the maximum degree to reach 0 **if** each group halves the maximum degree?

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$$\approx \log d$$

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$$\approx \left(3 \log_{1-c} \frac{1}{n}\right) \cdot \log d = O(\log d \cdot \log n)$$

Analysis

Every $3\log_{1-c}\frac{1}{n}$ phases the maximum degree of the graph **halves** with probability at least $1 - \frac{1}{n^2}$.

Question: What is the probability that **all** of the $\log d$ "groups" of phases actually halve the maximum degree?

Analysis

Every $3\log_{1-c}\frac{1}{n}$ phases the maximum degree of the graph **halves** with probability at least $1-\frac{1}{n^2}$.

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- The probabilty that **one fixed group** does not halve the maximum degree is at most $\frac{1}{n^2}$.

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Every $3 \log_{1-c} \frac{1}{n}$ phases the maximum degree of the graph halves with probability at least $1 - \frac{1}{n^2}$.

Question: What is the probability that **all** of the $\log d$ "groups" of phases actually halve the maximum degree?

- The probabilty that **one fixed group** does not halve the maximum degree is at most $\frac{1}{n^2}$.
- The probabilty **at least** one group does not halve the maximum degree is at most $\log d \cdot \frac{1}{n^2} < \log n \cdot \frac{1}{n^2} < \frac{1}{n}$.

Analysis

Every $3\log_{1-c}\frac{1}{n}$ phases the maximum degree of the graph **halves** with probability at least $1-\frac{1}{n^2}$.

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MIS forms in $O(\log d \cdot \log n)$ phases with probability at least $1-\frac{1}{n}$.

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Luby's Algorithm finds a Maximal Independent Set of G.

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- Total number of phases: $O(\log d \cdot \log n)$
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- Total time: $O(\log d \cdot \log n)$
- Probability of success: $\geq 1 - \frac{1}{n}$