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Notes on Abstract Reduction Systems

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1 Introduction

An *abstract reduction system* (from now on, ARS for short) consists of a set of elements equipped with one or more binary relations. This notion has got a level of generality such that the following concepts can be seen as particular ARSs: production rules of a grammar, rewriting of first order terms, rewriting of higher order terms, rewriting of term trees, string rewriting, graph rewriting, process rewriting, rewriting of formalized derivations.

The level of generality does not substantially decrease if, instead of a finite number of binary relations, only one relation is considered. In the following we will study ARSs with only one *reduction relation* $\longrightarrow$, that can be considered as an oriented equation or an elementary transformation.

After defining an ARS, we will introduce the following notions: reduction graphs, confluence, termination, noetherian induction and canonicity. For more details we refer to [1, 2, 4].

2 ARS

An ARS is defined as follows.

Definition 1 *An ARS (Abstract Reduction System) is a structure*

$$\mathcal{A} = \langle A, \longrightarrow \rangle$$

where A is a countable set and $\longrightarrow \subseteq A \times A$ is a binary relation called reduction.

Definition 1, differently from that in [3], requires for each ARS the countability of the set A, whose elements are written as $a, b, \ldots$.
If for $a, b \in A$ we have $(a, b) \in \longrightarrow$, then we write $a \longrightarrow b$. The element b is the result of the application of one step of reduction starting from the element a.

The reflexive-transitive closure of $\longrightarrow$ is written $\stackrel{*}{\longrightarrow}$. Thus, $a \stackrel{*}{\longrightarrow} b$ if there exists a finite sequence (possibly empty) of *reduction steps* $a \longrightarrow a_1 \longrightarrow \ldots \longrightarrow a_k \longrightarrow b$ for some $a_1, \ldots, a_k \in A$. We say that *a reduces to b* if $a \stackrel{*}{\longrightarrow} b$ and also that b is a *reduct* of a.

The transitive closure of $\longrightarrow$ is written $\stackrel{+}{\longrightarrow}$. The inverse relation of $\longrightarrow$ is written $\longleftarrow$ or $\longrightarrow^{-1}$.

The reflexive-transitive-symmetric closure of $\longrightarrow$ is written $\stackrel{*}{\longleftrightarrow}$, i.e. the equivalence relation induced by $\longrightarrow$, also called *convertibility*. In the following, convertibility will sometimes be denoted with $=$, also used to denote the syntactic identity.

An ARS $\mathcal{A}$ is *coherent* or *consistent* if $\stackrel{*}{\longleftrightarrow} \subseteq A^2$ and $\stackrel{*}{\longleftrightarrow} \neq A^2$, that is not every two elements are convertible.

An element $a \in A$ is said to be *in normal form* or is a *normal form* if there exists no element $b \in A$ such that $a \longrightarrow b$. An element *a has normal form* if there exists $b \in A$ such that $a \stackrel{*}{\longrightarrow} b$ and b is a normal form.

We say that an ARS has a property P whenever its reduction relation has the property P.

3 Reduction Graphs

For every $a \in A$, let us consider the directed graph whose root is labelled with a and whose nodes are labelled with elements $b \in A$ that are reducts of a.

Definition 2 *Let $\mathcal{A} = \langle A, \longrightarrow \rangle$ be an ARS. For every $a \in A$ we define the following sets:*

$$\begin{aligned}
\Delta(a) &= \{b \in A \mid a \longrightarrow b\} \\
\Delta^+(a) &= \{b \in A \mid a \stackrel{+}{\longrightarrow} b\} \\
\Delta^*(a) &= \{b \in A \mid a \stackrel{*}{\longrightarrow} b\}
\end{aligned}$$

Definition 3 *Let $a \in A$ be an arbitrary element of an ARS $\mathcal{A}$. The* reduction graph *of a, denoted $G(a)$, is the ARS $\langle \Delta^*(a), \longrightarrow_a \rangle$, where $\longrightarrow_a$ is the restriction of $\longrightarrow$ on $\Delta^*(a)$.*

Note that a reduction graph can be a very complex structure. A simple well-known graph is the so-called *Hindley graph* (Figure 1) that we will introduce when studying the confluence property (Section 4).

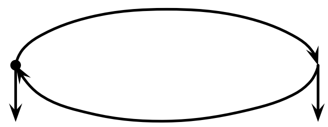


Figure 1: Hindley graph

Note that, in this simplified representation of a reduction graph, only the root node has been put in evidence and node labels have been omitted.

A reduction graph $G(a)$ can be defined as the smallest sub-ARS that contains all reducts of a, according to the following definition of sub-ARS.

Definition 4 *Let $\langle A_1, \longrightarrow_1 \rangle$ and $\langle A_2, \longrightarrow_2 \rangle$ be two ARSs.*
$\langle A_1, \longrightarrow_1 \rangle$ is a sub-ARS *of $\langle A_2, \longrightarrow_2 \rangle$, or $\langle A_2, \longrightarrow_2 \rangle$ is an* extension *of $\langle A_1, \longrightarrow_1 \rangle$, if:*

1. $A_1 \subseteq A_2$;

2. $\longrightarrow_1$ is the restriction of $\longrightarrow_2$ on A_1 (that is, for all $a, a' \in A_1$ we have that $a \longrightarrow_2 a'$ if and only if $a \longrightarrow_1 a'$);

3. A_1 is closed *with respect to $\longrightarrow_2$ (that is, for all $a \in A_1$, $a \longrightarrow_2 a'$ implies $a' \in A_1$).*

It can be shown that every property of an ARS is preserved in its sub-ARSs.

4 Confluence

The *confluence property*, also referred to as *Church-Rosser property*, is important for both the coherence of rewriting, implying (as we will see later) the uniqueness of the normal form whenever such a form exists, and the rewriting strategies, in the sense that due to confluence there is no need of backtracking. Moreover, whenever convertibility is decidable, the confluence property implies that checking convertibility of any two elements can be carried out by simply reducing both elements to a common reduct.

Definition 5 *Let $\mathcal{A} = \langle A, \longrightarrow \rangle$ be an ARS. The reduction relation $\longrightarrow$ is* confluent *(or* Church-Rosser*, CR for short) if for all $a, b, c \in A$ there exists $d \in A$ such that $c \overset{*}{\longleftarrow} a \overset{*}{\longrightarrow} b$ implies $c \overset{*}{\longrightarrow} d \overset{*}{\longleftarrow} b$.*

Definition 6 *The reduction relation $\overset{*}{\longleftrightarrow}$ is* Church-Rosser *if for all $a, b \in A$ there exists $c \in A$ such that $a \overset{*}{\longleftrightarrow} b$ implies $a \overset{*}{\longrightarrow} c \overset{*}{\longleftarrow} b$.*

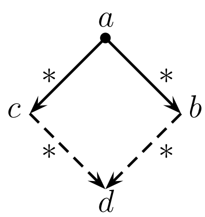


Figure 2: Confluence diagram

Proposition 1 *Let $\mathcal{A} = \langle A, \longrightarrow \rangle$ be an ARS. The reduction relation $\longrightarrow$ is confluent if and only if $\overset{*}{\longleftrightarrow}$ is Church-Rosser.*

Proof ($\Longrightarrow$) By assuming the confluence property, the proof of the Church-Rosser property is by induction on the number n of reduction steps in $a \overset{*}{\longleftrightarrow} b$ for all $a, b \in A$. If $n = 0$, then $a = b$ and the common reduct is $d = a = b$. In the induction case, let us assume that the Church-Rosser property holds for n and prove it for $n+1$. Let $a \overset{n+1}{\longleftrightarrow} b$. Hence, there exists $c \in A$ such that $a \overset{n}{\longleftrightarrow} c \longleftrightarrow b$. By induction hypothesis there exists $d \in A$ such that $a \overset{*}{\longrightarrow} d \overset{*}{\longleftarrow} c$. By case analysis on $c \longleftrightarrow b$, we have:
- if $c \longleftarrow b$, it easily follows that d is the common reduct for a and b;
- if $c \longrightarrow b$, by the confluence assumption there exists $d' \in A$ such that $d \overset{*}{\longrightarrow} d' \overset{*}{\longleftarrow} b$.

($\Longleftarrow$) By definition of the convertibility relation $\overset{*}{\longleftrightarrow}$ we have that, for all $a, b, c \in A$, any confluence peak $c \overset{*}{\longleftarrow} a \overset{*}{\longrightarrow} b$ implies $c \overset{*}{\longleftrightarrow} b$. The confluence property then follows from the assumption of the Church-Rosser property. ∎

In the following we will indifferently use ‘confluent’ or ’Church-Rosser’.

Due to the confluence property, we can assert that, starting from a given element, a reduction relation cannot obtain different results, i.e. different normal forms.

Corollary 1 *Let $\mathcal{A} = \langle A, \longrightarrow \rangle$ be an ARS such that $\longrightarrow$ is confluent. Each element in A has at most a normal form.*

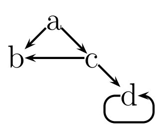
Proof By contradiction: let us suppose that there exists an element a such that $a \xrightarrow{*} b$ and $a \xrightarrow{*} c$ for some distinct normal forms b and c. By the Church-Rosser property, elements b and c should have a common reduct, but by assumption they are both in normal form. This contradicts the Church-Rosser property, thus b and c must be equal. ■

The uniqueness normal form property (UN for short) does not necessarily imply confluence.

Example 1: Let $\mathcal{A} = \langle \{a, b, c, d\}, \longrightarrow \rangle$ be an ARS with the reduction relation $\longrightarrow$ defined as follows:

$$\begin{array}{ccc} a & \longrightarrow & b \\ a & \longrightarrow & c \\ c & \longrightarrow & b \\ c & \longrightarrow & d \\ d & \longrightarrow & d \end{array}$$

or graphically:



The relation $\longrightarrow$ has the uniqueness normal form property, but is not confluent. •

Definition 7 *Let $\mathcal{A} = \langle A, \longrightarrow \rangle$ be an ARS. The reduction relation $\longrightarrow$ is* weakly confluent *or* locally confluent *or* weakly Church-Rosser *(WCR for short) if for all $a, b, c \in A$ there exists $d \in A$ such that $c \longleftarrow a \longrightarrow b$ implies $c \xrightarrow{*} d \xleftarrow{*} b$.*

It can be easily seen that local confluence is a particular case of confluence.

Corollary 2 *Let $\mathcal{A} = \langle A, \longrightarrow \rangle$ be an ARS. If the relation $\longrightarrow$ is confluent, then is locally confluent.*

The opposite inplication is not true in general, that is local confluence does not imply confluence. A counter-example is the following:

Example 2: Let $a, b, c, d \in A$ and the reduction relation $\longrightarrow$ be defined as follows:

$$\begin{array}{ccc} a & \longrightarrow & b \\ a & \longrightarrow & c \\ b & \longrightarrow & a \\ b & \longrightarrow & d \end{array}$$

It is enough to consider the possible reductions starting from the elements a and b and we can show that the relation is locally confluent, but is not confluent. The reduction graph is the Hindley graph that we have already seen (Section 3). •

Another example of ARS that is locally confluent, non-confluent and also acyclic is given in Figure 3.

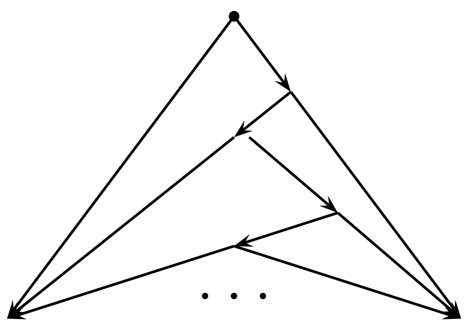


Figure 3: Acyclic WCR graph

Corollary 3 *Let $\mathcal{A} = \langle A, \longrightarrow \rangle$ be an ARS. The relation $\longrightarrow$ is confluent if and only if $\stackrel{*}{\longrightarrow}$ is weakly confluent.*

Note that a confluent ARS that has at least two distinct normal forms is coherent, as not all elements can be reduced to the same element.

5 Termination

Typically, the notions of termination that can be considered are *weak termination* and *strong termination*.

Definition 8 *Let $\mathcal{A} = \langle A, \longrightarrow \rangle$ be an ARS. The reduction relation $\longrightarrow$ is* weakly normalizing *(WN for short) if each element $a \in A$ has a normal form.*

Definition 9 *Let $\mathcal{A} = \langle A, \longrightarrow \rangle$ be an ARS. The reduction relation $\longrightarrow$ is* strongly normalizing *or* terminating *or* noetherian *(SN for short) if there exists no infinite derivation $a_0 \longrightarrow a_1 \longrightarrow \dots a_n \longrightarrow \dots$ of reduction steps, i.e. every derivation ends with a normal form.*

Example 3: Let $a, b, c \in A$ and

$$\begin{array}{ccc} a & \longrightarrow & b \\ a & \longrightarrow & c \\ c & \longrightarrow & a \end{array}$$

Such a reduction relation is weakly normalizing (every element has a normal form, the element b), but is not terminating because there exists the infinite derivation $a \longrightarrow c \longrightarrow a \longrightarrow c \longrightarrow \dots$.
The relation defined in Example 2 is also weakly normalizing and non-terminating. •

A terminating relation is obviously weakly normalizing.

To prove that an ARS is terminating is generally difficult, as the termination property is in general undecidable, even in simple cases such as, for example, that of a rewrite system with only one rule. For proving termination several methods have been defined in the literature. A method consists in the definition of a suitable partial ordering $>$ on the elements of an ARS such that $a \longrightarrow b$ implies $a > b$. Another method consists of embedding an ARS in another ARS that is known to be terminating. Various conditions sufficient for the termination of particular ARSs have been given in the literature.

6 Noetherian Induction

In order to prove properties of an ARS, hence properties satisfied by each $a \in A$, we can make use of the principle of *noetherian induction*. Le us recall the definition of the following set of elements:

$$\Delta^+(a) = \{b \in A \mid a \xrightarrow{+} b\}.$$

Definition 10 *Let $\mathcal{A} = \langle A, \longrightarrow \rangle$ be an ARS. Let P be a unary predicate on the set A. P is said* $\longrightarrow$-complete *if*

$$\forall a \in A \ \ [\forall b \in \Delta^+(a) \ \ P(b)] \Longrightarrow P(a).$$

Note that, if an element a is a normal form, then $P(a)$ is true for each $\longrightarrow$-complete predicate P.

Theorem 1 *(Principle of Noetherian Induction) Let $\mathcal{A} = \langle A, \longrightarrow \rangle$ be an ARS such that $\longrightarrow$ is noetherian. Let P be a unary $\longrightarrow$-complete predicate on the set A. Then $\forall a \in A$ we have $P(a)$.*

Proof The proof is by contradiction. Let us assume that there exists a subset B of A such that $B = \{b \in A \mid \neg P(b)\} \neq \emptyset$. For each $b \in B$ we have that b is either in normal form or (in the sense of "exclusive or") reducible to a normal form.

- b is in normal form.
  This case is not possible because it contradicts the assumption that the predicate P is $\longrightarrow$-complete. Any $\longrightarrow$-complete predicate is indeed true on normal forms.

- b is reducible.
  In this case there exists an element $b_0 \in B$ such that for all $b' \in \Delta^+(b_0)$ we have that $P(b')$ holds. But this contradicts the assumption that P is $\longrightarrow$-complete. In fact, in order not to contradict the $\longrightarrow$-completeness of P we are forced to build an infinite derivation from b_0 whose elements do not satisfy the property P (and this would contradict the noetherian hypothesis).

Hence, we have must have $B = \emptyset$. ∎

We have seen that, in general, local confluence does not imply confluence. However, this implication is true whenever the reduction relation is terminating. An application of the Principle of Noetherian Induction is the following result.

Lemma 1 *(Newman Lemma)*
Let $\mathcal{A} = \langle A, \longrightarrow \rangle$ be an ARS such that $\longrightarrow$ is noetherian.
The relation $\longrightarrow$ is confluent if and only if is locally confluent.

Proof The implication "confluence implies locally confluence" is given by Corollary 2 without the termination hypothesis on the reduction relation. The vice versa, i.e. "locally confluence implies confluence", is proved by noetherian induction. It is enough to formalize the property to be proved, that is the confluence of the relation $\longrightarrow$, as the unary predicate

$$P(a) = \forall b\, c\, (b \overset{*}{\longleftarrow} a \overset{*}{\longrightarrow} c) \Longrightarrow \exists d.\, b \overset{*}{\longrightarrow} d \overset{*}{\longleftarrow} c$$

and then prove that $P(a)$ holds for every a (in other words, we prove that the relation $\longrightarrow$ is confluent). By the principle of noetherian induction, to prove $P(a)$ for every $a \in A$ it is sufficient to show that P is $\longrightarrow$-complete, as $\longrightarrow$ is noetherian by hypothesis. Le us suppose that $a \overset{m}{\longrightarrow} b$ and $a \overset{n}{\longrightarrow} c$.
If $m = 0$, then choose $d = c$.
If $n = 0$, then choose $d = b$.
If $m, n \neq 0$, then at least one reduction step is made, thus there exist $b_1, c_1 \in A$ such that $a \longrightarrow b_1 \overset{*}{\longrightarrow} b$ and $a \longrightarrow c_1 \overset{*}{\longrightarrow} c$. By hypothesis the relation $\longrightarrow$ is locally confluent, hence there exists $d_1 \in A$ such that $b_1 \overset{*}{\longrightarrow} d_1 \overset{*}{\longleftarrow} c_1$. Proving that the confluence predicate P is $\longrightarrow$-complete means proving that $P(a)$ is true whenever we know that $P(e)$ holds for every $e \in \Delta^+(a)$. Because $b_1, c_1 \in \Delta^+(a)$, $P(b_1)$ is true, thus there exists $d_2 \in A$ such that $b \overset{*}{\longrightarrow} d_2 \overset{*}{\longleftarrow} d_1$. $P(c_1)$ is also true, thus there exists $d \in A$ such that $d_2 \overset{*}{\longrightarrow} d \overset{*}{\longleftarrow} c$. Hence, there exists an element d such that $b \overset{*}{\longrightarrow} d \overset{*}{\longleftarrow} c$, that is $P(a)$ holds. The predicate P is thus $\longrightarrow$-complete and, by the principle of noetherian induction, $P(a)$ holds for every $a \in A$. ■

An abbreviated formulation of Newman Lemma is $SN + WCR \Longrightarrow CR$.

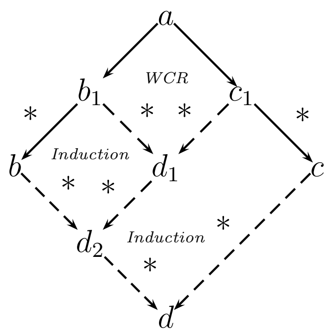


Figure 4: Newman Lemma

7 Canonicity

Definition 11 *Given an ARS* $\langle A, \longrightarrow \rangle$*, the relation* $\longrightarrow$ *is* canonical *or* complete *if it is confluent and terminating.*

From Corollary 1 we know that, if the relation $\longrightarrow$ is confluent, then the normal form of any element in A, if it exists, then is unique. Moreover, the following holds.

Corollary 4 *Given an ARS* $\langle A, \longrightarrow \rangle$*, if the relation* $\longrightarrow$ *is canonical, then the normal form of any element in* A *exists and is unique.*

We observe that canonicity makes the convertibility relation decidable: indeed, given any two elements in A, it is enough to reduce them to their normal forms (which exist and are unique) and then check their syntactic equality.

Acknowledgements

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