A Hyperbolic Model of Granular Flow

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Abstract

In this paper we review some recent results for a model for granular flow that was proposed by Hadeler & Kuttler in [18], which has recently raised a lot of attention.

In one space dimension, this model can be written as a 2×2 hyperbolic system of balance laws, in which the unknowns represent the thickness of the moving layer and the one of the resting layer. The known theory applies to the Cauchy problem for this system, as for instance in the context of small C^1 data or small BV data. Moreover, due to the special hyperbolicity properties of the system and of the special form of the source term, it is possible to enlarge the class of initial data for which global in time solutions exist. See [2, 27].

Further, we study the "slow erosion/deposition limit", [3], where the thickness of the moving layer vanishes but the total mass of flowing down material remains positive. The limiting behavior for the slope of the mountain profile provides an entropy solution to a scalar integro-differential conservation law.

A well-posedness analysis of this integro-differential equation is presented. Therefore, the solution found in the limit turns out to be unique.

Contents

1 The model of granular flow

The following model was proposed in [18] to describe granular flows

$$\begin{cases} h_t &= \operatorname{div}(h\nabla u) - (1 - |\nabla u|)h\,, \\ u_t &= \big(1 - |\nabla u|\big)h\,. \end{cases} \tag{1.1}$$

These equations describe conservation of masses. The material is divided in two parts: a moving layer with height h on top and a standing layer with height u at the bottom. The moving layer slides downhill, in the direction of steepest descent, with speed proportional to the slope of the standing layer. If the slope $|\nabla u| > 1$ then grains initially at rest are hit by rolling grains of the moving layer and start moving as well. Hence the moving layer gets bigger. On the other hand, if $|\nabla u| < 1$, grains which are rolling can be deposited on the bed. Hence the moving layer becomes smaller.

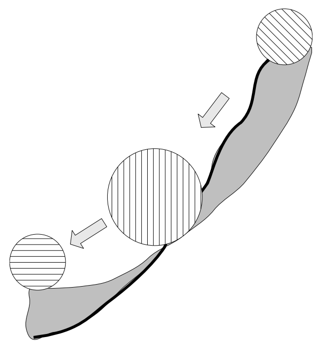

Figure 1: A mass of "snow" flowing down. The thick/thin line corresponds to the profile before/after the "snow" has flown.

This model is studied in one space dimension in the rest of the paper. Define $p \doteq u_x$, and assume $p \geq 0$, one can rewrite (1.1) into the following 2×2 system of balance laws

$$\begin{cases} h_t - (hp)_x &= (p-1)h\,, \\ p_t + \big((p-1)h\big)_x &= 0\,. \end{cases} \tag{1.2}$$

Writing the system of balance laws (1.2) in quasilinear form, the corresponding Jacobian matrix is computed as

$$A(h,p) = \begin{pmatrix} -p & -h \\ p-1 & h \end{pmatrix}.$$

For $h \geq 0$ and $p > 0$, one finds two real distinct eigenvalues $\lambda_1 < 0 \leq \lambda_2$, as

$$\lambda_{1,2} = \frac{1}{2}\left[h - p \pm \sqrt{(p-h)^2 + 4h}\right].$$

When h is small, i.e., with $h \approx 0$, we have

$$\lambda_1 = -p + \frac{p-1}{p}\,h + \mathcal{O}(h^2)\,, \qquad \lambda_2 = \frac{h}{p} + \mathcal{O}(h^2)\,.$$

Note that if $p > 0$, then the two families travel with strictly different speed. A direct computation gives

$$r_1 \bullet \lambda_1 = -\frac{2(\lambda_1 + 1)}{\lambda_2 - \lambda_1} \approx \frac{2(p-1)}{p}\,, \qquad r_2 \bullet \lambda_2 = -\frac{2\lambda_2}{\lambda_2 - \lambda_1} \approx -2\frac{h}{p^2}\,,$$

where r_1, r_2 are the corresponding eigenvectors, and the "$\bullet$" stands for the directional derivative.

This shows the fact that the first characteristic field is genuinely nonlinear away from the line $p = 1$ and the second field is genuinely nonlinear away from the line $h = 0$, therefore the system is weakly linearly degenerate at the point $(h, p) = (0, 1)$. Also, the direction of increasing eigenvalues, for the first family, changes with the sign of $p - 1$. The lines $p = 1$, $h = 0$ are characteristic curves of the first, second family respectively, along which the system becomes linear:

$$p = 1, \quad h_t - h_x = 0\,; \qquad h = 0, \quad p_t = 0\,.$$

See Figure 2 for the characteristic curves.

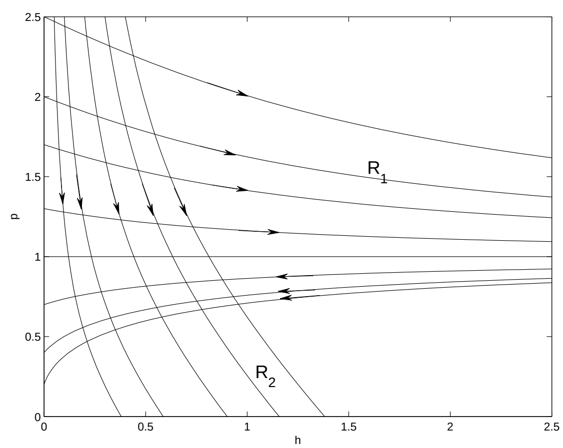

Figure 2: Characteristic curves of the two families in the h-p plane. The arrows point in the direction of increasing eigenvalues.

In this paper we review some recent results about the existence of solutions for system (1.2), that are shown to exist globally in time for suitable classes of initial data. For systems of conservation laws with source term, some dissipation conditions are known in the literature that ensure the global in time existence of (smooth or weak) solutions; we refer to [20], to Kawashima–Shizuta condition (see [19]) for smooth solutions and to [14, 22] for the weak solutions. These conditions exploit a suitable balance between the differential terms and the source term that enable to control the nonlinearity of the system. It is interesting to remark that system (1.2) does not satisfy none of these conditions, nevertheless it admits global in time solutions.

For a derivation of the model (1.1) of granular flow we refer to [18]. Other models can be found in [6, 15, 25]. A mathematical analysis of steady state solutions for (1.1) was carried out in [9, 10]. Note that, besides [27], the papers [2, 3] provide the first analytical study of time dependent solutions to this system.

2 Global smooth solutions

The global existence of smooth solutions is established in [27], under suitable assumptions on the initial data.

By a **decoupled initial data** we mean a set of initial conditions of the form

$$h(0,x) = \phi(x) \qquad\qquad p(0,x) = 1 + \psi(x) \tag{2.1}$$

with ϕ, ψ satisfying

$$\begin{cases} \phi(x) = 0 & \text{if } \ x \notin [a,b]\,, \\ \psi(x) = 0 & \text{if } \ x \notin [c,d]\,. \end{cases}$$

The intervals are disjoint, i.e., $a < b < c < d$. Moreover we assume $\psi(x) > -1$ for all x.

For decoupled initial data, a global solution of the Cauchy problem can be explicitly given, namely

$$h(t,x) = \phi(x+t)\,, \qquad\qquad p(t,x) = 1 + \psi(x)\,, \qquad\qquad x \in I\!R\,, \quad t \geq 0\,.$$

Our first result provides the stability of these decoupled solutions. More precisely, every sufficiently small, compactly supported perturbation of a Lipschitz continuous decoupled solution eventually becomes decoupled. Moreover, no gradient catastrophe occurs, i.e., solutions remain smooth for all time.

Theorem 1 (Global smooth solutions) *Let $a < b < c < d$ be given, together with Lipschitz continuous, decoupled initial data as in* (2.1). *Then there exists $\delta > 0$ such that the following holds. For every perturbations $\tilde\phi, \tilde\psi$, satisfying*

$$\tilde\phi(x) = \tilde\psi(x) = 0 \qquad \textit{if} \ \ x \notin [a,d]\,, \qquad |\tilde\phi'(x)| \leq \delta\,, |\tilde\psi'(x)| \leq \delta\,, \tag{2.2}$$

the Cauchy problem for (1.2) *with initial data*

$$h(0,x) = \phi(x) + \tilde\phi(x)\,, \qquad\qquad p(0,x) = 1 + \psi(x) + \tilde\psi(x)\,, \tag{2.3}$$

has a unique solution, defined for all $t \geq 0$ and globally Lipschitz continuous. Moreover, this solution becomes decoupled in finite time.

The proof replies on the method of characteristics [20]. One must bound the $\mathbf{L}^\infty$ and $\mathbf{L}^1$ norms of h_x and p_x. Since we are looking for continuous solutions, it is convenient to work in a set of Riemann coordinates. Let $(h,p) \mapsto (w,z)$ be the coordinate transformation such that

$$(w,z)(h,1) = (h,\,0) \quad \text{for all } h\,, \qquad (w,z)(0,p) = (0,\,p-1) \quad \text{for all } p > 0\,,$$

and $r_1 \bullet z \equiv 0\,,\ r_2 \bullet w = 0$. In these new variables, the system (1.2) takes the form

$$\begin{cases} w_t + \lambda_1(w,z)\,w_x = f(w,z)\,, \\ z_t + \lambda_2(w,z) z_x = g(w,z)\,, \end{cases}$$

for some functions f and g. In order to bound the gradient of the perturbed solution, showing that no shock can form, we need to study the evolution equation for w_x, z_x, namely

$$\begin{cases} (w_x)_t + \big(\lambda_1\, w_x\big)_x = f_w w_x + f_z z_x\,, \\ (z_x)_t + (\lambda_2 z_x)_x = g_w w_x + g_z z_x\,. \end{cases} \tag{2.4}$$

To establish the needed a-priori estimates, we introduce the *total strength of waves* V and the *wave interaction potential* Q at time t

$$V(t) \doteq \int_{-\infty}^{\infty} \big|w_x(t,x)\big|\,dx + \int_{-\infty}^{\infty} \big|z_x(t,x)\big|\,dx\,, \quad Q(t) \doteq \iint_{x>y} \big|w_x(t,x)\big|\,\big|z_x(t,y)\big|\,dx\,dy\,.$$

By the properties of f, g, we have

$$\begin{cases} \dfrac{d}{dt}V(t) \le 2C_0 \cdot Q\,, \\ \dfrac{d}{dt}Q = 2C_0 \cdot VQ\,, \end{cases} \qquad \begin{cases} 0 \le V(0) \le C_2 \\ 0 \le Q(0) \le C_2 C_1 (d-a)\delta\,. \end{cases}$$

for some constants C_0, C_1. By a comparison argument one can deduce

$$V(t) \le V(0) + \varepsilon\,, \qquad Q(t) \le \varepsilon \qquad \text{for all} \ \ t \ge 0\,, \tag{2.5}$$

for any $\varepsilon > 0$, provided that $\delta > 0$ is chosen small enough. The first estimate in (2.5) provides an a-priori bound on the total variation of the solution, valid as long as it remains continuous.

For a more detailed proof, we refer to [27].

3 Global existence of large BV solutions

For more general initial data, due to the nonlinearity of the flux, the solutions will develop discontinuities (shocks) in finite time. Solutions should be defined in the space of BV functions. Assuming the height of the moving layer h sufficiently small, in [2] we prove the global existence of large BV solutions, for a class of initial data with bounded but possibly large total variation.

More precisely, consider initial data of the form

$$h(0,x) = \bar h(x)\,, \qquad p(0,x) = \bar p(x)\,. \tag{3.1}$$

We assume that $\bar h : \mathbb{R} \mapsto \mathbb{R}_+^*$ and $\bar p : \mathbb{R} \mapsto \mathbb{R}_+^*$ are non-negative functions with bounded variation, such that

$$\text{Tot.Var.}\{\bar p\} \le M\,, \qquad \text{Tot.Var.}\{\bar h\} \le M\,, \tag{3.2}$$

$$\|\bar h\|_{\mathbf{L}^1} \le M\,, \qquad \|\bar p - 1\|_{\mathbf{L}^1} \le M\,, \qquad \bar p(x) \ge p_0 > 0\,, \tag{3.3}$$

for some constants M (bounded but possibly large) and p_0. The following theorem is proved in [2].

Theorem 2 (global existence of large BV solutions) *For any constants M, $p_0 > 0$, there exists $\delta > 0$ small enough such that, if* (3.2)–(3.3) *hold together with*

$$\|\bar{h}\|_{\mathbf{L}^\infty} \leq \delta\,, \tag{3.4}$$

then the Cauchy problem (1.2), (3.1) *has an entropy weak solution, defined for all $t \geq 0$, with uniformly bounded total variation.*

Compared with previous literature, the main novelty of the present result stems from the fact that

i. The system (1.2) contains source terms;

ii. We assume a small $\mathbf{L}^\infty$ bound on $h(\cdot)$, but not on the component $p(\cdot)$;

iii. We have arbitrarily large BV data;

iv. The system is strictly hyperbolic, but one of the characteristic fields is neither genuinely nonlinear nor linearly degenerate.

In the literature, for systems without source terms and small BV data, the global existence and uniqueness of entropy-weak solutions to the Cauchy problem are well known. Even in the general case of systems which are neither genuinely nonlinear nor linearly degenerate, global solutions have been constructed by the Glimm scheme [16, 21, 23, 28], by front tracking approximations [8, 4, 5], and by vanishing viscosity approximations [7]. In some special cases, existence and uniqueness of global solutions in the presence of a source term were proved in [14, 22, 11] and in [13, 1, 12], respectively.

However, global existence of solutions to hyperbolic systems with large BV data is a more difficult, still largely open problem. In addition to the special system [24], two main cases are known in the literature, where global existence of large BV solutions is achieved. One is the case of Temple class systems [26]. Here one can measure the wave strengths in terms of Riemann invariants, so that the total strength of all wave fronts does not increase in time, across each interaction. A second major result [17] refers to general 2×2 systems, where again we can measure wave strengths in terms of Riemann coordinates. To see what happens at an interaction, let σ_1 and σ_2 be the strengths of the incoming waves of different families, and let σ_1' and σ_2' be the strengths of the outgoing waves. One then has a cubic interaction estimate of the form

$$|\sigma_1' - \sigma_1| + |\sigma_2' - \sigma_2| = \mathcal{O}(1) \cdot |\sigma_1| \cdot |\sigma_2| \cdot (|\sigma_1| + |\sigma_2|)\,. \tag{3.5}$$

Thanks to the additional term $|\sigma_1| + |\sigma_2|$, if the $\mathbf{L}^\infty$ norm of the solution is sufficiently small, the increase of total variation produced by the interaction is very small, and a global existence result of large BV solutions can then be established.

We remark that the cubic estimate in (3.5) is useless in our case, since the $\mathbf{L}^\infty$ norm of the component p in (1.2) can be large, and so is the additional term $|\sigma_1| + |\sigma_2|$ in (3.5).

The validity of Theorem 2 relies heavily on some special properties of the hyperbolic system (1.2). First, the system is linearly degenerate along the straight line where $h = 0$. In the region where h is very small, the system is "almost-Temple class" and almost linearly degenerate. Rarefaction curve and shock curve through the same point are very close to each other. More precisely, let U_o be a point on the rarefaction curve of the first family through the point $U = (h,p)$. Then, there exists a point U^* on the shock curve through U, which is very close to U_o, such that

$$|U^* - U_o| = \mathcal{O}(1) \cdot h^2.$$

This allows us to replace the estimate (3.5) with

$$|\sigma_1' - \sigma_1| + |\sigma_2' - \sigma_2| = \mathcal{O}(1) \cdot |\sigma_1| \cdot |\sigma_2| \cdot \|h\|_{\mathbf{L}^\infty}\,. \tag{3.6}$$

Besides (3.6), interaction estimates of waves from the same family are also improved as follows. If two 2-waves of strength σ_2 and $\tilde\sigma_2$ interact, then the strengths σ_1^+ and σ_2^+ of the outgoing waves satisfy

$$|\sigma_1^+| + \big|\sigma_2^+ - (\sigma_2 + \tilde\sigma_2)\big| = \mathcal{O}(1) \cdot h_l \cdot |\sigma_2\,\tilde\sigma_2|\,. \tag{3.7}$$

If two 1-waves of size σ_1 and $\tilde\sigma_1$ interact, then the strengths σ_1^+ and σ_2^+ of the outgoing waves satisfy

$$\big|\sigma_1^+ - (\sigma_1 + \tilde\sigma_1)\big| + |\sigma_2^+| = \mathcal{O}(1) \cdot |p_l - 1|\,(|\sigma_1| + |\tilde\sigma_1|) \cdot |\sigma_1\,\tilde\sigma_1|\,. \tag{3.8}$$

Here h_l and p_l denote the left state of interaction.

Second, the source term involves the quadratic form $h(p-1)$. Here the quantities h and $p-1$ have large, but bounded $\mathbf{L}^1$ norms. Moreover, they are transported with strictly different speeds. The total strength of the source term is thus expected to be $\mathcal{O}(1) \cdot \|h\|_{\mathbf{L}^1} \cdot \|p-1\|_{\mathbf{L}^1}$. In addition, since h itself is a factor in the source term, one can obtain a uniform bound on the norm $\|h\|_{\mathbf{L}^\infty}$, valid for all times $t \geq 0$.

Approximate solutions (h^Δ, p^Δ) are constructed by a operator splitting technique. Fix a time step $\Delta t \geq 0$ and consider the sequence of times $t_k = k\Delta t$. On each subinterval $[t_{k-1}\,,\ t_k[$ the functions (h^Δ, p^Δ) provide an approximate solution to the system of conservation laws

$$\begin{cases} h_t - (hp)_x &= 0\,, \\ p_t + \big((p-1)h\big)_x &= 0\,, \end{cases} \tag{3.9}$$

constructed by a wave-front tracking algorithm [8, 4, 5]. Moreover, in order to account for the source term, at each time t_k the functions are redefined in the following time step

$$\begin{cases} h^\Delta(t_k) &= h^\Delta(t_k-) + \Delta t\,\left[p^\Delta(t_k-) - 1\right] h^\Delta(t_k-)\,, \\ p^\Delta(t_k) &= p^\Delta(t_k-)\,. \end{cases} \tag{3.10}$$

Consider a wave front located at a point x. After the time step (3.10) is accomplished, the Riemann problem determined by the jump at x will be solved by two waves, say of strengths σ_1^+, σ_2^+. If the initial jump is of the first family, with strength σ_1, and with (h_l, p_l) as the left state, we have

$$\sigma_1 \qquad \Longrightarrow \qquad \begin{cases} \sigma_1^+ &= \sigma_1 + \mathcal{O}(1) \cdot \Delta t \cdot |p_l - 1|\,\sigma_1\,, \\ \sigma_2^+ &= \mathcal{O}(1) \cdot \Delta t \cdot |p_l - 1|\,\sigma_1\,. \end{cases} \tag{3.11}$$

On the other hand, if the initial jump is of the second family, with strength σ_2, we have

$$\sigma_2 \qquad \Longrightarrow \qquad \begin{cases} \sigma_1^+ \;=\; \mathcal{O}(1)\cdot\Delta t\cdot h_l\,\sigma_2\,, \\ \sigma_2^+ \;=\; \sigma_2+\mathcal{O}(1)\cdot\Delta t\cdot h_l\,\sigma_2\,. \end{cases} \tag{3.12}$$

The global existence of large BV solutions is established by deriving the following global a priori bounds for the approximate solutions:

- the norms $\|h^\Delta(t,\cdot)\|_{\mathbf{L}^1}$ and $\|p^\Delta(t,\cdot)-1\|_{\mathbf{L}^1}$;
- the lower bound on p^Δ, i.e., the quantity $\inf_x p^\Delta(t,x)$;
- the uniform bounds $\|h^\Delta(t,\cdot)\|_{\mathbf{L}^\infty}$ and $\|p^\Delta(t,\cdot)\|_{\mathbf{L}^\infty}$;
- the total variations Tot.Var.$\{h^\Delta(t,\cdot)\}$ and Tot.Var.$\{p^\Delta(t,\cdot)\}$.

Most of the a priori bounds are obtained by defining suitable weighted functionals that are non-increasing in time. The weights are chosen in a way such that they account for the mass to be crossed in future. Due to the strictly different wave speeds, the weights will be non-increasing in time, achieving the desired estimates.

We now study in more detail the first point above: how to obtain a priori bounds on $\|h^\Delta(t,\cdot)\|_{\mathbf{L}^1}$, $\|p^\Delta(t,\cdot)-1\|_{\mathbf{L}^1}$. We derive them in a formal way, assuming that $(h,p)(t,x)$ is a weak solution of the system (1.2), satisfying $h(t,x)\geq 0$ and $p(t,x)>0$.

Bound on the $\mathbf{L^1}$ norm of $p-1$. We rewrite the second equation of (1.2) as

$$(p-1)_t+\big((p-1)h\big)_x \;=\; 0\,.$$

Then the following inequality is satisfied in $\mathcal{D}'$:

$$|p-1|_t+\big(|p-1|h\big)_x \;\leq\; 0\,, \tag{3.13}$$

hence the $\mathbf{L^1}$-norm of $(p-1)$ is non-increasing in time:

$$\|p(t,\cdot)-1\|_{\mathbf{L}^1} \;\leq\; \|\bar p-1\|_{\mathbf{L}^1}\,, \qquad \text{for all } t\geq 0\,.$$

Bound on the $\mathbf{L^1}$ norm of h. From the first equation $h_t-(hp)_x=(p-1)h$, we get

$$\|h(t)\|_{\mathbf{L}^1}\leq \int_0^t\int \Big(p(\tau,x)-1\Big)\,h(\tau,x)\,dxd\tau$$

that may increase in time. To control the possible increment, we introduce the quantity

$$W(t,x)=\exp\left\{\int_{-\infty}^x |p(t,y)-1|\,dy\right\}\,.$$

This term accounts for the erosion/deposition that the mass h at the point x will encounter in the future. Note that W is globally bounded, thanks to the bound on $\|p(t,\cdot)-1\|_{\mathbf{L}^1}$:

$$1 \;\leq\; W(t,x) \;\leq\; \exp\left\{\|\bar p-1\|_{\mathbf{L}^1}\right\}\,.$$

Moreover, using the inequality (3.13), we get

$$\begin{aligned}\frac{d}{dt}\left[\int_{-\infty}^{x} |p(t,y)-1|\,dy\right] &\leq -|p(t,x)-1|\,h(t,x) \\ &= -\frac{d}{dx}\left[\int_{-\infty}^{x} |p(t,y)-1|\,dy\right]\,h(t,x)\,,\end{aligned}$$

so that

$$W_t \;+\; hW_x \;\leq\; 0\,.$$

Then we introduce the quantity

$$\mathcal{I}^h(t) \;\doteq\; \int W(t,x)h(t,x)\,dx\,.$$

We want to give an a-priori bound on $\mathcal{I}^h(t)$. This will lead to an a-priori bound on the $\mathbf{L^1}$ norm of h: indeed, recalling that $W \geq 1$, $h \geq 0$ one gets

$$\|h(t,\cdot)\|_{\mathbf{L^1}} \;=\; \int h(t,x)\,dx \;\;\leq\;\; \mathcal{I}^h(t)\,.$$

Lemma 1 *The quantity $\mathcal{I}^h(t)$ is non-increasing in time.*

This property leads to an a-priori bound on the $\mathbf{L^1}$ norm of h:

$$\|h(t,\cdot)\|_{\mathbf{L^1}} \;\leq\; \mathcal{I}^h(t) \;\leq\; \mathcal{I}^h(0) \;\leq\; \exp\left(\|\bar p-1\|_{\mathbf{L^1}}\right)\cdot\|\bar h\|_{\mathbf{L^1}}\,.$$

Proof of Lemma 1. We are going to prove that

$$(Wh)_t \;-\; (Whp)_x \;\;\leq\;\; 0\,. \tag{3.14}$$

As a consequence, the integral of Wh is non-increasing in time, hence the conclusion. Start from

$$W\cdot[h_t-(hp)_x] \;\;=\;\; W\cdot(p-1)h\,. \tag{3.15}$$

Now observe that

$$W_t\cdot h \;-\; W_x\cdot(hp) \;\;\leq\;\; -W_x h^2 \;-\; W_x(hp)$$

and that $W_x = W\cdot|p-1|$. Hence we are lead to

$$W_t\cdot h \;-\; W_x\cdot(hp) \;\;\leq\;\; -W\cdot|p-1|\cdot h\cdot(h+p) \tag{3.16}$$

Summing up (3.15) and (3.16), one gets

$$(Wh)_t \;-\; (Whp)_x \;\;\leq\;\; Wh\cdot[\;(p-1) \;-\; (h+p)\cdot|p-1|\;]$$

Note that the right hand side of the previous inequality is negative for all $h \geq 0$ and $p > 0$. Then we achieve the desired inequality (3.14). □

The proof of the a priori bounds on the total variation rely on the following key observation: If all wave strengths are measured in terms of Riemann coordinates, then all the interaction estimates (3.6)-(3.8) contain the additional factor $\|h\|_{\mathbf{L}^\infty}$. Therefore, if the norm $\|h\|_{\mathbf{L}^\infty}$ remains sufficiently small, we can assume that the total strength of all new waves produced by interactions is as small as we like. In essence, the change in the total variation and in the $\mathbf{L}^\infty$ norms of h, p is thus determined only by the source term in the first equation of (1.2).

For details of the proof of Theorem 2 we refer to [2].

4 Global BV solutions of an initial boundary value problem

Next we study how the mountain profile evolves when the thickness of the moving layer approaches zero, but the total mass of sliding material remains positive. The limiting behavior of the slope $p(\cdot)$, when the norms $\|h\|_{\mathbf{L}^\infty}$ and $\|F\|_{\mathbf{L}^\infty}$ approach zero, is of practical interests. This describes how the mountain profile evolves, when the granular material is poured down at a very slow rate.

This result is best formulated in connection with an initial-boundary value problem. By a translation of coordinates, it is not restrictive to consider the domain $\mathbb{R}_- \doteq \{x < 0\}$. On $\mathbb{R}_-$, consider the initial-boundary value problem for (1.2), with initial data (3.1) and the following boundary condition at $x = 0$

$$p(t,0)h(t,0) \; = \; F(t)\,. \tag{4.1}$$

The condition (4.1) prescribes the incoming flux $F(t)$ of granular material through the point $x = 0$, see the first equation in (1.2). We assume that

$$F(t) \geq 0\,, \qquad \text{Tot.Var.}\{F\} \leq M\,, \qquad 0 < M' < \int_0^\infty F(\tau)\,d\tau \leq M\,. \tag{4.2}$$

As a partial step toward the slow erosion limit, we prove in [3] next theorem on the global existence of large BV solutions to this initial-boundary value problem, provided that $\|\bar{h}\|_{\mathbf{L}^\infty}$ and $\|F\|_{\mathbf{L}^\infty}$ are sufficiently small.

Theorem 3 (global existence of large BV solutions for the initial-boundary value problem). *Given M, $p_0 > 0$ (and for $M' > 0$ arbitrary) there exists $\delta > 0$ such that the assumptions* (3.2), (3.3) *and* (4.2)*, together with*

$$\|\bar{h}\|_{\mathbf{L}^\infty} \leq \delta\,, \qquad\qquad \|F\|_{\mathbf{L}^\infty} \leq \delta\,, \tag{4.3}$$

imply that the initial-boundary value problem (1.2), (3.1), (4.1) *has a global in time solution, with uniformly bounded total variation for all $t \geq 0$.*

The main steps in the proof of Theorem 3 are the same as for Theorem 2. We approximate the initial data and the boundary data with piecewise constant functions. The approximated flux on the boundary is set to be constant on each time interval (t_{k-1}, t_k). On the time intervals (t_{k-1}, t_k) an approximate solution of the conservation laws (3.9) is constructed by front tracking, with constant flow at the boundary $x = 0$. At time $t = t_k$ the solution is updated by means of (3.10) and due to the possibly new value for F.

The main difference is caused by the boundary condition at $x = 0$. We have the following boundary estimates:

(i) At a time τ where a 2-wave of strength σ_2 hits the boundary at $x = 0$, a new reflected front of the first family is created. Calling h_l the state to the left of the jump σ_2 and σ_1^+ the size of the new jump, we have the estimate

$$|\sigma_1^+| = \mathcal{O}(1) \cdot h_l\, |\sigma_2|\,. \tag{4.4}$$

(ii) At a time $\tau = t_k$, the inductive step (3.10) is performed and the flux $F = hp$ has a jump. Then a front of the first family, entering the domain, is created. Its strength $|\sigma_1^+|$ satisfies

$$|\sigma_1^+| \;=\; \mathcal{O}(1) \cdot \Delta t \cdot h^- \;+\; \mathcal{O}(1) \cdot |F(\tau+) - F(\tau-)| \tag{4.5}$$

where h^- is the value at $x = 0$ before the time step.

We note that the estimates (4.4), (4.5) contain either the term $\|h\|_{\mathbf{L}^\infty}$ or the term $\|F\|_{\mathbf{L}^\infty}$, which are arbitrarily small. Same global a priori estimates as for Theorem 2 can be established, proving the global existence of large BV solution. For details, see [3].

5 Slow erosion limit

We now study the slow erosion/deposition limit. For the rest of the paper, we will simply call it the "slow erosion limit".

Numerical simulations in [27] show the following observation: when the height of the moving layer h is very small, the profile of the standing layer depends only on the total mass of the avalanche flowing downhill, not on the time-law describing at which rate the material slides down. This observation is proved rigorously in [3].

In more detail, we consider the initial-boundary value problem for (1.2), with initial data $\bar{h}$ and boundary data F satisfying the assumptions of Theorem 3. We are interested in the limit as $\|h\|_{\mathbf{L}^\infty}$, $\|F\|_{\mathbf{L}^\infty}$ tend to zero.

We define a new variable which measures the total mass of avalanche flowing down, and use it in place of the time variable, as follows. Recalling that $F \geq 0$, see (4.2), the map

$$\mu(t) \doteq \int_0^t F(\tau)\, d\tau \tag{5.1}$$

is monotone non-decreasing and then admits the generalized inverse

$$t(\mu) = \min\ \{\tau \geq 0\,;\quad \mu_\nu(\tau) = \mu\}\,,$$

which is well defined for $\mu \in [0, M']$. We will use the new variable μ in place of t and re-parametrize the profile p as follows:

$$\widetilde{p}(\mu, x) \;\doteq\; p(t(\mu), x)\,. \tag{5.2}$$

The formal limit. Let us derive the formal slow erosion limit with initial data $\bar{h} \equiv 0$ and the following boundary data:

$$(ph)(t,0) \;=\; F^\varepsilon(t) = \varepsilon \bar{F}(\varepsilon t) > 0$$

for some function $\bar{F} \in \mathbf{L^1}(R_+)$. Then $\|F^\varepsilon\|_{\mathbf{L^1}}$ is constant, while

$$\|F^\varepsilon\|_{\mathbf{L}^\infty} \to 0 \qquad \text{as } \varepsilon \to 0\,.$$

Define

$$\mu^\varepsilon(t) \doteq \int_0^t F^\varepsilon(s)\,ds \;=\; \int_0^{\varepsilon t} \bar{F}(\tau)\,d\tau \doteq \bar{\mu}(\varepsilon t)\,.$$

Note that μ^ε, $\bar{\mu}$ have the same image, therefore are defined on the common interval $[0, M']$ for any $0 < M' < \|\bar{F}\|_{\mathbf{L^1}}$. The inverse functions of μ^ε, $\bar{\mu}$ satisfy the relation

$$F^\varepsilon\left(t^\varepsilon(\mu)\right) \;=\; \varepsilon\,\widetilde{F}(\mu), \qquad \text{with } \widetilde{F}(\mu) \doteq \bar{F}\left(\bar{t}(\mu)\right)\,, \tag{5.3}$$

with $\mu \in [0, M']$. Rescaling the time, one has

$$\frac{\partial}{\partial t} = \varepsilon \widetilde{F}(\mu)\frac{\partial}{\partial \mu}$$

and the system can be rewritten as

$$\begin{cases} \varepsilon\,\widetilde{F}(\mu)\,h^\varepsilon_\mu - (h^\varepsilon p^\varepsilon)_x & = (p^\varepsilon - 1)h^\varepsilon \\ \varepsilon\,\widetilde{F}(\mu)\,p^\varepsilon_\mu + \big((p^\varepsilon - 1)h^\varepsilon\big)_x & = 0\,. \end{cases} \tag{5.4}$$

As $\varepsilon \to 0$, one expects that $p^\varepsilon = \mathcal{O}(1)$ while

$$h^\varepsilon = \mathcal{O}(1)\varepsilon\,, \qquad \frac{\partial}{\partial \mu} h^\varepsilon = \mathcal{O}(1)\varepsilon\,.$$

These last relations are justified by the corresponding relations for F^ε.

Note that all the terms in (5.4) are at least of order ε. By setting

$$m^\varepsilon = \frac{h^\varepsilon p^\varepsilon}{\varepsilon}\,,$$

which is $\mathcal{O}(1)$, the system can be rewritten as

$$\begin{cases} \widetilde{F}(\mu)\,h^\varepsilon_\mu - (m^\varepsilon)_x & = \dfrac{p^\varepsilon - 1}{p^\varepsilon} m^\varepsilon\,, \\ \widetilde{F}(\mu)\,p^\varepsilon_\mu + \left(\dfrac{p^\varepsilon - 1}{p^\varepsilon} m^\varepsilon\right)_x & = 0\,. \end{cases}$$

As $\varepsilon \to 0$, one has $\quad m^\varepsilon \to m, \quad p^\varepsilon \to p$ and hence

$$\begin{cases} -m_x & = \dfrac{p-1}{p} m \\ \widetilde{F}(\mu) p_\mu + \left(\dfrac{p-1}{p} m\right)_x & = 0 \end{cases} \tag{5.5}$$

The first equation in (5.5) is then reduced to a linear differential equation for m, with p as a coefficient. At $x = 0$ it satisfies the boundary condition

$$m(0,\mu) = \widetilde{F}(\mu)\,,$$

with $\widetilde{F}(\mu)$ as in (5.3). Solving the equation on $x < 0$, we find

$$m(x,\mu) = \widetilde{F}(\mu) \cdot \exp \int_x^0 \frac{p(\mu,y)-1}{p(\mu,y)}\,dy\,.$$

We then substitute it into the second equation of (5.5) and get

$$\widetilde{F}(\mu) \cdot \left[p_\mu + \left(\frac{p-1}{p} \cdot \exp \int_x^0 \frac{p(\mu,y)-1}{p(\mu,y)}\,dy \right)_x \right] \;=\; 0\,.$$

This finally leads to a closed, scalar equation for p:

$$p_\mu + \left(\frac{p-1}{p} \cdot \exp \int_x^0 \frac{p(\mu,y)-1}{p(\mu,y)}\,dy \right)_x \;=\; 0\,.$$

5.1 The limit equation

Let us focus on the limit problem for p. It consists of a scalar, integro-differential equation on the domain $[0, M'] \times \mathbb{R}_-$. In terms of

$$q \doteq p - 1$$

and after renaming the time variable, it can be rewritten as

$$q_t + \left(k(t,x)\,\frac{q}{q+1} \right)_x = 0\,, \tag{5.6}$$

with

$$k(t,x) \;\doteq\; \exp\left\{ \int_x^0 \frac{q(t,\xi)}{q(t,\xi)+1}\,d\xi \right\}. \tag{5.7}$$

The flux in (5.6) is given by the product of the *nonlocal* term k and of the *local* term

$$f(q) = \frac{q}{q+1}\,, \tag{5.8}$$

that satisfies $f' > 0$, $f'' < 0$ on our domain of interest, namely $q + 1 > 0$. Note that the characteristic speed of the equation, $kf'(q)$, is positive. Since the flux function is concave down as a function of q, the admissible shocks in q will be the ones that jump up.

The equation (5.6) is supplemented by the initial data

$$q(0,x) \;=\; q_0(x) \;=\; \bar{p}(x) - 1\,, \qquad\qquad x < 0\,. \tag{5.9}$$

At the boundary $x = 0$ no condition is provided, since the characteristic speed is positive for $q + 1 > 0$.

A natural definition of an entropy weak solution of (5.6)–(5.7), (5.9) can be given as follows. Let C, p_0 be some positive constants and define $\mathcal{D} = \mathcal{D}_{C,p_0}$ as the set of functions $q(\cdot)$ satisfying the following uniform bounds

$$\inf_{x<0} q(x) + 1 \geq p_0 > 0, \qquad \text{TV } q(\cdot) \leq C, \qquad \|q\|_{\mathbf{L^1}(\mathbb{R}_-)} \leq C\,. \tag{5.10}$$

Let $q_0 \in \mathcal{D}$. We say that q provides an entropy weak solution to (5.6)–(5.7), (5.9) on the domain $[0,T] \times \mathbb{R}_-$ if the following conditions **(H1)**, **(H2)** are satisfied:

(H1) $q : [0,T] \to \mathcal{D}$; the map $[0,T] \ni t \mapsto q(t)$ is Lipschitz in $\mathbf{L^1}(\mathbb{R}_-)$;

(H2) q is a weak entropy solution of the scalar conservation law (5.6), with k defined by (5.7), and with $q(0,\cdot) = q_0$.

We remark that, thanks to **(H1)**, the function k has the following properties: (i). it is defined on $[0,T] \times \mathbb{R}_-$; (ii). it is bounded and Lipschitz continuous, with $\inf k > 0$; and (iii). the total variation of $k(t,\cdot)$ and $k_x(t,\cdot)$ are bounded uniformly in time.

Indeed, according to the assumption **(H1)**, the integral term

$$\int_x^0 \frac{q(t,\xi)}{q(t,\xi)+1}\,d\xi$$

is bounded, since

$$\left|\int_x^0 \frac{q(t,\xi)}{q(t,\xi)+1}\,d\xi\right| \;\leq\; \int_{-\infty}^0 \frac{|q(t,\xi)|}{p_0}\,d\xi \leq \frac{C}{p_0}\,.$$

Moreover, it is Lipschitz continuous as a function of t. About k_x, we have

$$k_x \;=\; -k\,\frac{q}{q+1}$$

hence

$$|k_x| \;\leq\; \frac{e^{C/p_0}}{p_0}|q| \;\in\; \mathbf{L^1} \cap \mathbf{L^\infty}\,,$$

and therefore $k(t,\cdot) \in BV(\mathbb{R}_-)$. Also,

$$\begin{aligned} \text{TV}\,(k_x) \;&\leq\; \text{TV}\,(k) \cdot \frac{\|q\|_{\mathbf{L^\infty}(\mathbb{R}_-)}}{p_0} \;+\; \frac{e^{C/p_0}}{p_0^2}\text{TV}\,(q) \\ &\leq\; M\,\text{TV}\,(q) \end{aligned}$$

for a suitable $M > 0$.

We can show that the integro-differential equation (5.6)–(5.7) is well posed. This is not trivial because the flux is a nonlocal function. More precisely, we prove that the flow generated by the integro-differential equation (5.6)–(5.7) is Lipschitz continuous restricted to the domain of functions satisfying the bounds (5.10).

Indeed, consider q, $\tilde{q} : [0,T] \to \mathcal{D}$ satisfying **(H1)** and **(H2)**, with corresponding initial data

$$q(0,x) = q_0(x)\,, \qquad\qquad \tilde{q}(0,x) = \tilde{q}_0(x) \qquad\qquad x<0\,.$$

One can show that

$$\|q(t,\cdot) - \tilde{q}(t,\cdot)\|_{\mathbf{L}^1(\mathbb{R}_-)} \;\; \leq \;\; \|q_0 - \tilde{q}_0\|_{\mathbf{L}^1} + L\cdot \int_0^t \|q(s,\cdot)-\tilde{q}(s,\cdot)\|_{\mathbf{L}^1}\,ds$$

for a suitable constant L, for all $t \in [0,T]$. By Gronwall's lemma, this yields the Lipschitz continuous dependence of solutions on the initial data.

5.2 Convergence to the limit profile

We now state the main result of this section.

Theorem 4 (Slow erosion limit). *Consider the initial boundary value problem for*

$$\begin{cases} h_t - (hp)_x &= (p-1)h\,,\\ p_t + \big((p-1)h\big)_x &= 0 \end{cases} \tag{5.11}$$

with initial data (3.1) *and boundary data* (4.1)*. Assume all the assumptions in Theorem 3 hold. Recalling* (5.2)*, let* $\widetilde{p}(\mu,\cdot)$ *be the time-rescaled* p *component of the solution to the initial boundary value problem above.*

Then, as $\|\bar{h}\|_{\mathbf{L}^\infty} \to 0$ *and* $\|F\|_{\mathbf{L}^\infty} \to 0$*, the functions* $\widetilde{p}$ *converge to a limit function* $\widehat{p}$ *in the distance of* $\mathbf{L}^\infty\big([0,M'];\mathbf{L}^1(\mathbb{R}_-)\big)$*. The function* $\widehat{p}$ *provides the unique entropy solution to the scalar integro-differential conservation law*

$$p_\mu + \left(\frac{p-1}{p}\cdot \exp\int_x^0 \frac{p(\mu,y)-1}{p(\mu,y)}\,dy\right)_x = 0\,, \tag{5.12}$$

with $\mu \in [0,M']$ *and initial data* $\widehat{p}(0,x) = \bar{p}(x)$ *for* $x<0$*.*

The proof of Theorem 4 is carried out with several steps. To start, let $\bar{h}_\nu$ and F_ν be sequences of initial data for h and of boundary data, such that

$$\|\bar{h}_\nu\|_{\mathbf{L}^\infty} \to 0\,, \qquad \|F_\nu\|_{\mathbf{L}^\infty} \to 0 \qquad \text{as} \quad \nu\to\infty\,. \tag{5.13}$$

Note that the initial data for p is not modified in the limit. Define

$$t_\nu(\mu) \doteq \min\left\{t\geq 0\,;\quad \int_0^t F_\nu(s)\,ds = \mu\right\} \qquad\qquad \mu\in[0,M']\,,$$

and consider the rescaled functions

$$\widetilde{p}_\nu(\mu,x) \doteq p_\nu(t_\nu(\mu),x)\,,$$

obtained by using μ as new time variable. These are well defined for $x \leq 0$, $\mu \in [0, M']$ and ν sufficiently large.

Note that, while the map $t \mapsto p_\nu(t, \cdot)$ is continuous in $\mathbf{L^1}_{loc}$, the now defined map $\mu \mapsto \widetilde{p}_\nu(\mu, \cdot)$ may fail to be continuous in $\mathbf{L^1}_{loc}$ since $t_\nu(\mu)$ is not necessarily continuous.

Step 1. We first establish a Lipschitz-type dependence of p on the rescaled time variable for the solutions of (5.11). The estimate is uniform in ν.

More precisely, let (h, p) be a solution of the initial-boundary value problem for (5.11), with boundary condition (4.1) (for brevity we omit the dependence on ν). Fix $R > 0$, assume that $t' < t''$ and set

$$g(x) \doteq \sup_{t \in [t', t'']} |p(t, x) - p(t', x)| . \tag{5.14}$$

We establish the estimate

$$\int_{-R}^{0} g(x)\, dx \quad \leq \quad L \cdot \Big(\int_{t'}^{t''} F(t)\, dt + \|h\|_{\mathbf{L}^\infty} \Big) \tag{5.15}$$

where the constant L depends on R, the total variations of p, h and the $\mathbf{L^1}$ norms of h, F.

As a consequence of (5.15), the rescaled function $\widetilde{p}$ satisfies

$$\begin{aligned} \|\widetilde{p}(\mu_2, \cdot) - \widetilde{p}(\mu_1, \cdot)\|_{\mathbf{L^1}((-R,0))} \quad &\leq \quad L \cdot \left\{ \left| \int_{t(\mu_1)}^{t(\mu_2)} F(t)\, dt \right| \; + \; \|h\|_{\mathbf{L}^\infty} \right\} \\ &= \quad L \cdot |\mu_2 - \mu_1| \; + \; \delta \, . \end{aligned} \tag{5.16}$$

Now, $\delta = L\|h\|_{\mathbf{L}^\infty} \to 0$ because of the assumptions (5.13).

To obtain compactness, notice also that the total variations of the functions $\widetilde{p}_\nu$ are uniformly bounded. Hence, by Helly's theorem, there exists a subsequence, still called $\widetilde{p}_\nu$, converging to a BV function $\widehat{p} = \widehat{p}(\mu, \cdot)$ in $\mathbf{L^1}([-R, 0])$ for all $\mu \in [0, M']$ and therefore, because of (5.16), also in $\mathbf{L}^\infty\big([0, M']; \mathbf{L^1}([-R, 0])\big)$.

To prove the convergence on the whole interval $x < 0$, we use the fact that $\int_{-\infty}^{-R} |p(t, x) - 1|\, dx$ is small for R sufficiently large. More precisely, let $\varepsilon_0 > 0$ and choose $R > 0$ large enough so that

$$\int_{-\infty}^{-R} |\bar{p}(x) - 1|\, dx \quad < \quad \varepsilon_0 \, .$$

Thanks to (3.13), we deduce that

$$\int_{-\infty}^{-R} |p_\nu(t, x) - 1|\, dx \quad < \quad \varepsilon_0$$

for all ν and all $t > 0$. A similar argument is applied to equation (5.12). Therefore we can conclude that the convergence holds in $\mathbf{L}^\infty\big([0, M']; \mathbf{L^1}(\mathbb{R}_-)\big)$.

Step 2. For later use we derive some estimates for the flux function, hp, of the equation for h.

In more detail, for a given $\xi \leq 0$, consider the flux through the interval with endpoints (t', ξ) and (t'', ξ), namely

$$\Phi(\xi) \doteq \int_{t'}^{t''} p(t,\xi)\, h(t,\xi)\, dt\,.$$

Now integrate the first equation in (5.11) on the domain $[t', t''] \times [\xi_1, \xi_2]$, for some $\xi_1 < \xi_2 \leq 0$. One can obtain the following estimates on Φ:

$$\begin{aligned}
\Phi(\xi_2) &\leq \Phi(\xi_1) + \int_{\xi_1}^{\xi_2} \left(\left[-\frac{\bar{p}(x)-1}{\bar{p}(x)} + Cg(x) \right] \cdot \Phi(x) + 2\|h\|_{\mathbf{L}^\infty} \right) dx \\
\Phi(\xi_2) &\geq \Phi(\xi_1) + \int_{\xi_1}^{\xi_2} \left(\left[-\frac{\bar{p}(x)-1}{\bar{p}(x)} - Cg(x) \right] \cdot \Phi(x) - 2\|h\|_{\mathbf{L}^\infty} \right) dx
\end{aligned}$$

where $\bar{p}(x) = p(t', x)$ and g is defined at (5.14). By using a suitable Gronwall-type estimate, one can obtain the estimates

$$\begin{aligned}
\Phi(\xi) &\leq (\Phi(0) + 2M\|h\|_{\mathbf{L}^\infty}) \cdot \exp \int_\xi^0 \left[\frac{\bar{p}(x)-1}{\bar{p}(x)} + Cg(x) \right] dx \\
\Phi(\xi) &\geq (\Phi(0) - 2M\|h\|_{\mathbf{L}^\infty}) \cdot \exp \int_\xi^0 \left[\frac{\bar{p}(x)-1}{\bar{p}(x)} - Cg(x) \right] dx
\end{aligned}$$

where M is independent on ν, for $\xi \in [-R, 0]$. Now introduce the map

$$k(t', \xi) \doteq \exp \int_\xi^0 \frac{\bar{p}(x)-1}{p(x)}\, dx\,,$$

and recall that $\Phi(0) = \int_{t'}^{t''} F(t)\, dt$. Assuming that

$$\Phi(0) = \delta\,, \qquad \|h\|_{\mathbf{L}^\infty} = \mathcal{O}(1) \cdot \delta^2$$

for δ sufficiently small, and using (5.15), we obtain the key estimate

$$\Phi(\xi) - \delta k(t', \xi) = \mathcal{O}(1)\delta^2 \tag{5.17}$$

for $\xi \in [-R, 0]$.

Step 3. Show that the limit solution $\widehat{p}$ obtained in the previous step is a weak solution to the conservation law (5.12).

This is achieved by showing that it satisfies the weak formulation of (5.12). In detail, we show that for any fixed $0 \leq \mu_1 < \mu_2$ and any test function $\psi \in \mathcal{C}_c^1(\mathbb{R}_-)$, we have

$$\int_{-R}^0 \psi(x)\, [\widehat{p}(\mu_2, x) - \widehat{p}(\mu_1, x)]\, dx = \int_{\mu_1}^{\mu_2} \int_{-R}^0 \psi_x(x) \cdot \frac{\widehat{p}(\mu, x) - 1}{\widehat{p}(\mu, x)} \cdot \widehat{k}(\mu, x)\, dx\, d\mu\,, \tag{5.18}$$

where

$$\widehat{k}(\mu,x) \doteq \exp\int_x^0 \frac{\widehat{p}(\mu,\zeta)-1}{\widehat{p}(\mu,\zeta)}\,d\zeta\,.$$

We start from the weak formulation of the second equation in (5.11)

$$\int_{-R}^0 \psi(x)\Big[p(t(\mu_2),x)-p(t(\mu_1),x)\Big]\,dx = \int_{t(\mu_1)}^{t(\mu_2)}\int_{-R}^0 \psi_x(x)\cdot[p(t,x)-1]\cdot h(t,x)\,dx\,dt \tag{5.19}$$

and show that it converges to (5.18), as $\|\bar h\|_{\mathbf{L}^\infty}, \|F\|_{\mathbf{L}^\infty}\to 0$.

Step 3.1: convergence of the left hand sides. Since the sequence $p_\nu(t_\nu(\mu),\cdot) = \widetilde{p}_\nu(\mu,\cdot)$ converges to $\widehat{p}(\mu,\cdot)$ in $\mathbf{L^1}([-R,0])$, then the left hand side of (5.19) converges to the corresponding left hand side of (5.18).

Step 3.2: convergence of the right hand sides. Rewrite the flux of the second equation in (5.11) as follows

$$(p-1)h \;=\; \underbrace{\frac{p-1}{p}}_{\textbf{(I)}}\cdot\underbrace{(ph)}_{\textbf{(II)}}\,.$$

Arguing as before, we know that the factor **(I)** converges strongly in $\mathbf{L^1}\left([0,M']\times[-R,0]\right)$ to the corresponding limit:

$$\frac{\widetilde{p}(\mu,x)-1}{\widetilde{p}(\mu,x)} \quad\to\quad \frac{\widehat{p}(\mu,x)-1}{\widehat{p}(\mu,x)}\,. \tag{5.20}$$

The key point is to prove the convergence of **(II)**. One can prove that the flux ph, rescaled in terms of μ, converges weakly to $\widehat{k}$:

$$\widetilde{p}(\mu,x)\,\widetilde{h}(\mu,x) \;\rightharpoonup\; \widehat{k}(\mu,x) \tag{5.21}$$

on the domain $[0,M']\times[-R,0]$. More precisely, by means of (5.17), we prove that for every x, as $\nu\to\infty$ one has

$$\int_{t_\nu(\mu_1)}^{t_\nu(\mu_2)} p_\nu(t,x)\,h_\nu(t,x)\,dt \quad\to\quad \int_{\mu_1}^{\mu_2}\widehat{k}(\mu,x)\,d\mu\,. \tag{5.22}$$

Together, the estimates (5.20), (5.21) allow us to establish the convergence of the right hand side.

For the reader's convenience, we check it directly in the special case of $F_\nu(t) = \nu^{-1}\cdot\chi_{(0,\nu)}(t)\,$, $\nu\in\mathbb{N}$. Then one has $t_\nu(\mu) = \nu\mu \quad$ for $0\le\mu\le 1$, and

$$\int_{t_\nu(\mu_1)}^{t_\nu(\mu_2)} p_\nu(t,x)\,h_\nu(t,x)\,dt \;=\; \nu\int_{\mu_1}^{\mu_2}(\widetilde{p}_\nu\widetilde{h}_\nu)(\mu,x)\,d\mu\,. \tag{5.23}$$

After changing the time variable, the r.h.s. of (5.19) can be rewritten as

$$\int_{\mu_1}^{\mu_2}\int_{-R}^0 \psi_x(x)\cdot\frac{\widetilde{p}_\nu(\mu,x)-1}{\widetilde{p}_\nu(\mu,x)}\cdot(\widetilde{p}_\nu\widetilde{h}_\nu)(\mu,x)\cdot\nu\,dx\,d\mu\,.$$

By using (5.20), (5.23) and (5.22) we conclude that it converges to the r.h.s. of (5.18).

Step 4. Check the entropy admissibility of the limit function. One needs to check that all shocks jump in the correct direction. Since the flux $kf(q)$ is concave w.r.t. q, a discontinuity is entropic if q jumps up. Recalling Figure 2, all the shocks of the second family jump up in the p variable; this property is still satisfied in the limit.

For a detailed proof we refer to [3].

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