1.
a) Give the definition of a contraction mapping. (2)
b) Prove that any contraction mapping is continuous. (2)
c) Is it true that any continuous mapping of a Banach space to itself is a contraction?
If "yes", prove it, if "no", give an example. (2)

2.
a) Prove that the algebraic equation $x = (1+x)^{\frac{1}{5}}$ has a root inside the segment $[1, 2]$. (3)
b) Find any three successive approximations to this root (only algebraic expressions, without evaluation). (2)

3. A function $f(x) : [0, 1] \to \mathbb{R}$ is measurable with respect to the usual σ - additive measure defined on the σ - algebra of all measurable subsets of $[0, 1]$. Prove that $e^{f(x)}$ is also measurable. (3)

4. Evaluate the Lebesgue integral $\int\limits_{\mathbb{R}_+} f(x)\, d\mu$, where

$$f(x) = \begin{cases} e^{-x}, & x \in \mathbb{R}_+ \setminus \mathbb{N}; \\ e^{x}, & x \in \mathbb{N}. \end{cases}$$

Explain your answer. (3)

5. Given

$$f(x) = \begin{cases} 1, & x \in [0, 1] \cap \mathbb{Q}; \\ -2, & \text{otherwise}, \end{cases}$$

evaluate
a) $\|f(x)\|_{L_\infty(0,1)}$, (2)
b) $\|f(x)\|_{L_2(0,1)}$. (2)
Explain your answer.

6. Consider the operator

$$T(x(t)) = \int_0^t x(\tau)\, d\tau$$

from $C[0, 2]$ to $C[0, 2]$, $\|x(t)\|_{C[0,2]} = \sup\limits_{0 \le t \le 2} |x(t)|$.

Prove that the operator is
a) linear (2)
b) bounded (2)
c) continuous. (1)
d) Evaluate its norm. (2)

7. Let $A : X \to Y$ be a linear bounded invertible operator, where both X and Y are Banach spaces.
a) Prove that the inverse operator A^{-1} is linear. (3)
b) Is A^{-1} necessarily bounded? Explain your answer. (3)

8.
a) Give the definition of the dual space of a normed linear space X. (2)
b) Describe the space l_3. (2)
c) What space is dual to l_3? (Only answer, without proof). (2)