1.
a) Give the definition of the Hilbert space. (1)
b) How the inner product induces a norm in an inner product space? (1)
c) Is it true that any norm in a linear normed space is generated by an inner product? Explain your answer. (2)

2. $H = L_2(0, \pi)$.
a) Give an example of the orthogonal basis in H. (1)
b) Is any orthonormal infinite sequence of elements of H a basis of this space? Explain your answer. (1)
c) Find the projection on element $x(t) = t$ to the linear subspace S generated by element $\sin 2t$. (1)
d) Find the projection of this element on $S^{\perp}$. (2)

3. Let H be a Hilbert space, $\{\phi_n\}$, $n \in \mathbb{N}$ be an orthonormal basis on H. Does it exist an element $x \in H$ such that the generalized Fourier coefficients $\alpha_n = (x, \phi_n) = \frac{1}{\sqrt{n}}$, $n \in \mathbb{N}$? (2)

4. Let $H = l_2$. Proof that if the sequence $x_n = (x_{n1}, x_{n2}, ..., x_{nk}, ...)$ converges to an element $x_0 = (x_{01}, x_{02}, ..., x_{0k}, ...)$ weakly as $n \to \infty$, then $x_{nk} \to x_{0k}$ as $n \to \infty$ for any $k \in \mathbb{N}$. (2)

5.
a) Let $f(x) = \text{sign}(\text{x}^2 - 1)$. Can this function be considered as a regular distribution (regular generalized function) on $\mathbb{R}$? Explain your answer (1)
b) Find the derivative of $f(x)$ in the sense of distributions. (2)
c) Give the definition of $\delta(x)$. (1)
d) Give an example of sequence of regular generalized functions converging to $\delta(x)$ in the sense of distributions (without proof). (2)

6.
a) Give a definition of space $H^1(\Omega)$. (1)
b) Give a definition of space $H_0^1(\Omega)$. (1)
c) Give an example of function from the space $H_0^1([-1, 1]$. Explain your answer. (2)
c) Let Ω be a bounded domain of $\mathbb{R}^n$. For what dimension n a function $f(x) \in H_0^1(\Omega)$ is necessarily continuous? Explain your answer. (2)

7.
a) Give the definition of weak solution of the homogeneous Dirichlet problem for the Poisson equation. (1)
b) List the main step in the proof of existence and uniqueness of this solution. (3)

8.
a) Give the definition of compact operator on a Hilbert space. (1)
b) Give any example of compact operator (without proof). (1)
c) Describe the spectrum and eigenvector space of self-adjoint compact operator on a Hilbert space. (2)
d) Let A be compact inversible operator on a finite or infinite dimensional Hilbert space. Can A^{-1} be bounded? Explain your answer. (2)

9.
a) Give a definition of unitary operator. (1)
b) Give an example of unitary operator (without proof). (1)
c) Proof that the eigenvectors of unitary operator corresponding to distinct eigenvalues are orthogonal. (3)