Functional analysis in applied mathematics and engineering — First part

Test of 12 January 2009

Duration: 60 min.

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Exercise 1

1. Using contraction mapping theorem, prove the following Cauchy problem has an unique solution:
$$\begin{cases} x' = 2x \\ x(0) = -1. \end{cases}$$
2. Find this unique solution by means of successive approximations technique.

Exercise 2

Consider the measurable function $f : [0,1] \to \mathbb{R}$ defined by:

$$f(x) = \begin{cases} n & \text{if } x = \dfrac{1}{2n+1} \\ 5 & \text{otherwise,} \end{cases}$$

where $n \in \mathbb{N}$.

1. Is $f \in L_\infty[0,1]$? Evaluate $\|f\|_{L_\infty[0,1]}$.
2. Using theoretical results, prove from point 1. that $f \in L_1[0,1]$ (explain your answer).
3. Evaluate $\|f\|_{L_1[0,1]}$.

Exercise 3

In the space $L_2[0,1]$, consider the operator

$$(Au)(t) = \begin{cases} u(t), & \text{for } 0 \leq t \leq \lambda, \\ 0, & \text{for } \lambda < t \leq 1, \end{cases}$$

for a fixed $\lambda \in (0,1)$. Prove that the operator is linear, bounded, continuous and evaluate its norm.

Exercise 4

1. Give the definition of weak and *-weak convergence in a normed linear space X.
2. Describe these notions in the space ℓ_5 (only answers, without proofs).
3. Is this space reflexive? If yes, recast results of point 2.