Functional analysis in applied mathematics and engineering — Second part

Test of 12 January 2009

Duration: 60 min.

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Exercise 1

Let $f(x) = \text{sign}(x^3 - x)$.

1. Can this function be considered as a regular distribution (regular generalized function)? Explain your answer.
2. Find the derivative of $f(x)$ in the sense of distributions.
3. Can $f'(x)$ be considered as a regular distribution (regular generalized function)? Explain your answer.

Exercise 2

In the space $L_2([0,2];\mathbb{R})$, consider the closed subspace

$$\mathcal{S} = Span\left\{1, t^2\right\}.$$

1. Using the Projection Theorem, find the projection of the vector t on that subspace.
2. Find the same projection, but applying in advance the Gram–Schmidt orthogonalization to $\{1, t^2\}$ and then the Fourier expansion.

Exercise 3

Let $W(t,\tau) : [0,T] \times [0,T] \to \mathbb{R}$ a continuous function. Let $\mathcal{W} : L_2((0,T);\mathbb{R}) \to L_2((0,T);\mathbb{R})$ the linear and bounded operator defined by

$$(\mathcal{W}u)(t) = \int_0^t W(t,\tau)u(\tau)d\tau,$$

for any $u \in L_2((0,T);\mathbb{R})$. Find the adjont operator $\mathcal{W}^*$.
Hint: you shall use in an appropriate way the Fubini Theorem...

Exercise 4

1. Give the definition of a positive operator and provide an example of a positive operator.
2. Assuming a positive operator has a basis of eigenvectors, describe its eigenvalues. Explain your answer.