Functional analysis in applied mathematics and engineering — First part

Test of 27 January 2009

Duration: 60 min.

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Exercise 1

Using contraction mapping theorem, prove existence and uniqueness of solutions for the following Cauchy problem:

$$\begin{cases} x' = f(x,t) \\ x(0) = x_0, \end{cases}$$

under suitable hypotheses on the function $f(x,t)$.

Exercise 2

Consider the function $f : [0,1] \to \mathbb{R}$ defined by:

$$f(x) = \begin{cases} n + 3n^2 & \text{if } x = \dfrac{n}{n+1} \\ 0 & \text{otherwise,} \end{cases}$$

where $n \in \mathbb{N}$.

1. Prove that f is measurable (explain your answer).
2. Prove that $f = 0$ almost everywhere in $[0,1]$ (explain your answer).

Exercise 3

In the space $C[0,2]$ with the usual sup norm, consider the operator

$$T(x(t)) = t \int_0^t x(\tau)\, d\tau.$$

Prove that the operator is linear, bounded, continuous and evaluate its norm.

Exercise 4

Let $\{x^k\}$ be the sequence (of sequences!) defined by:

$$x^k = (0, \dots, 0, \underbrace{1}_{k\text{-th place}}, 0, 0, \dots).$$

1. Prove that $\{x^k\} \subset \ell_p$ for any $1 \leq p \leq \infty$.
2. Prove that $x_k \rightharpoonup 0$ in ℓ_p for any $1 < p < \infty$.