Functional analysis in applied mathematics and engineering — Second part

Test of 27 January 2009

Duration: 60 min.

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Exercise 1

Let $f(x)$ be defined by:

$$f(x) = \begin{cases} 0, & \text{if } x \leq -1, \\ x^2, & \text{if } -1 < x \leq 1, \\ 2 - x, & \text{if } 1 < x \leq 2, \\ 0, & \text{if } x > 2; \end{cases}$$

1. Can this function be considered as a regular distribution (regular generalized function)? Explain your answer.
2. Find the derivative of $f(x)$ in the sense of distributions.

Exercise 2

In the space $L_2([0,1];\mathbb{R})$, prove that the set of vectors $\{t, t^3, t^5\}$ is linearly independent and then apply to it the Gram–Schmidt orthogonalization.

Exercise 3

In $H = L_2([a,b];\mathbb{C})$ wiht the usual inner product, consider the operator $T : H \to H$ defined by:

$$(Tu)(t) = \int_a^b k(t,s)u(s)ds,$$

where $k(t,s) \in L_2((a,b)\times(a,b);\mathbb{C})$. Find the adjoint T^* of T.

Exercise 4

1. Define the notion of eigenvalues and eigenvectors for a linear operator $A : D(A) \subset H \to H$, H being an Hilbert space.

2. Find eigenvalues and eigenvectors of the operator $A : D(A) \subset L_2([0,\pi];\mathbb{R}) \to L_2([0,\pi];\mathbb{R})$ defined by:

$$A(u)(t) = u''(t);$$
$$D(A) = \{u \in L_2([0,\pi];\mathbb{R}) : u', u'' \in L_2([0,\pi];\mathbb{R}) \text{ and } u(0) = 0 = u(\pi)\}.$$