Functional analysis in applied mathematics and engineering

Test of 12 October 2009

Duration: approx. 60 min.

Family and first name: ______________________________

Matricola: ______________________________

CFU: _____

Exercise 1 [only for 9 CFU]

1. Using the function $F_1(x) = \sqrt[3]{1+x}$ and the Contraction (Fixed Point) Theorem, prove the existence of a zero of the function $f(x) = x^3 - x - 1$ in $[1,2]$.
2. Find that zero with an error less than $1/100$.
3. Explain why the following choises are not useful to apply the above technique:
   (a) $F_2(x) = x^3 - 1$ in $\mathbb{R}$.
   (b) $F_3(x) = \dfrac{1}{x^2-1}$ in $[1+\delta, +\infty)$, for a $\delta > 0$ sufficiently small.

Exercise 2 [6 and 9 CFU]

Consider the operator $F : L_1(0,1) \to L_1(0,1)$ defined by $(Fu)(t) = tu(t),\ t \in [0,1]$. Prove F is linear, bounded, and evaluate its norm.
Hint: to evaluate the norm, consider the sequence

$$u_n(t) = \begin{cases} 0, & \text{for } 0 \le t \le 1 - \dfrac{1}{n}, \\ n, & \text{for } 1 - \dfrac{1}{n} < t \le 1. \end{cases}$$

Exercise 3 [6 and 9 CFU]

Describe the notions of dual spaces and adjont operators in Hilbert spaces.

Exercise 4 [only for 9 CFU]

Consider the operator $A : L_2((a,b);\mathbb{C}) \to L_2((a,b);\mathbb{C})$ defined by $(Ax)(t) = ix(t)$. Prove that A is a linear, bounded, normal operator.