RELAXATION LIMIT FROM THE QUANTUM NAVIER–STOKES EQUATIONS TO THE QUANTUM DRIFT–DIFFUSION EQUATION

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Abstract. The relaxation-time limit from the Quantum-Navier-Stokes-Poisson system to the quantum drift-diffusion equation is performed in the framework of finite energy weak solutions. No assumptions on the limiting solution are made. The proof exploits the suitably scaled a priori bounds inferred by the energy and BD entropy estimates. Moreover, it is shown how from those estimates the Fisher entropy and free energy estimates associated to the diffusive evolution are recovered in the limit. As a byproduct, our main result also provides an alternative proof for the existence of finite energy weak solutions to the quantum drift-diffusion equation.

1. Introduction

This paper studies the relaxation-time limit for the Quantum Navier-Stokes-Poisson (QNSP) system with linear damping, towards the quantum drift-diffusion equation. More precisely, in the three dimensional torus $\mathbb{T}^3$, we consider a compressible, viscous fluid, whose dynamics is prescribed by

$$\begin{aligned}
&\partial_t\rho + \operatorname{div}(\rho u) = 0\\
&\partial_t(\rho u) + \operatorname{div}(\rho u\otimes u) - \operatorname{div}(\rho Du) + \nabla\rho^\gamma + \rho\nabla V = 2\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) - \xi\rho u\\
&-\Delta V = \rho - g.
\end{aligned} \tag{1.1}$$

Here the unknowns ρ, u, and V denote the particle density, the velocity field, and the electrostatic potential respectively. The function g is given and represents the doping profile.

The system arises in the macroscopic description of electron transport in nanoscale semiconductor devices [23], where quantum-mechanical effects must be taken into account. In this context the dissipative term $-\xi\rho u$ describes collisions between electrons and the semiconductor crystal lattice (see, for instance, [7]), and $\tau = 1/\xi$ is the relaxation time. The advantage of using macroscopic models for quantum fluids, with respect to kinetic models, is their reduced complexity, especially from a computational point of view [36]. Moreover, hydrodynamic models correctly describe high field phenomena or submicronic devices. However, in certain regimes, as in particular for low carrier densities and small electric fields, these models can be further reduced to some simpler ones. In the context of semiconductor devices for instance, quantum transport of electrons can be effectively described by the quantum drift-diffusion (QDD) equation [37], given by

$$\begin{aligned}
&\partial_t\rho + \operatorname{div}\left(2\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) - \nabla\rho^\gamma - \rho\nabla V\right) = 0\\
&-\Delta V = \rho - g.
\end{aligned} \tag{1.2}$$

The (QDD) equation can be formally recovered from system (1.1) as a relaxation limit. Precisely, by rescaling the time as follows

$$t' = \epsilon t, \quad (\rho^\epsilon, u^\epsilon)(t', x) = (\rho, u)\left(\frac{t'}{\epsilon}, x\right), \tag{1.3}$$

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where $\epsilon := 1/\xi$, the scaled system reads

$$\begin{aligned}
&\partial_t\rho_\epsilon + \frac{1}{\epsilon}\operatorname{div}(\rho_\epsilon u_\epsilon) = 0\\
&\partial_t(\rho_\epsilon u_\epsilon) + \frac{1}{\epsilon}\operatorname{div}(\rho_\epsilon u_\epsilon\otimes u_\epsilon) - \frac{1}{\epsilon}\operatorname{div}(\rho_\epsilon Du_\epsilon) + \frac{1}{\epsilon}\nabla\rho_\epsilon^\gamma + \frac{1}{\epsilon}\rho_\epsilon\nabla V_\epsilon = \frac{1}{\epsilon}2\rho_\epsilon\nabla\left(\frac{\Delta\sqrt{\rho_\epsilon}}{\sqrt{\rho_\epsilon}}\right) - \frac{1}{\epsilon^2}\rho_\epsilon u_\epsilon \qquad (1.4)\\
&-\Delta V_\epsilon = \rho_\epsilon - g.
\end{aligned}$$

Thus in the limit $\epsilon\to 0$, we formally obtain that

$$\lim_{\epsilon\to 0}\rho_\epsilon\frac{u_\epsilon}{\epsilon} = 2\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) - \nabla\rho^\gamma - \rho\nabla V \qquad (1.5)$$

and therefore the (QDD) equation.

The main purpose of our paper is to rigorously prove the above limit, that is to prove that scaled finite energy weak solutions to (1.1) converge to finite energy weak solutions to (1.2). To this aim, in the following we shall refer to (1.4) with initial datum (ρ^0, u^0) and doping profile g possibly depending in a suitable way on the relaxation parameter ϵ as well.

Theorem 1.1. *Let $(\rho_\epsilon, u_\epsilon, V_\epsilon)$ be a weak solution of* (1.1) *in the sense of Definition 2.1 with data $(\rho^0_\epsilon, u^0_\epsilon, g_\epsilon)$ satisfying*

$$\begin{aligned}
&\{\rho^0_\epsilon\}_\epsilon \text{ is bounded in } L^1\cap L^\gamma(\mathbb{T}^3) \text{ such that } \rho^0_\epsilon\to\rho^0 \text{ in } L^q(\mathbb{T}^3),\ q<3\\
&\{\nabla\sqrt{\rho^0_\epsilon}\}_\epsilon \text{ is bounded in } L^2(\mathbb{T}^3),\\
&\{\sqrt{\rho^0_\epsilon}u^0_\epsilon\}_\epsilon \text{ is bounded in } L^2(\mathbb{T}^3),\\
&\{g_\epsilon\}_\epsilon \text{ is bounded in } L^2(\mathbb{T}^3) \text{ such that } g_\epsilon\rightharpoonup g \text{ in } L^2(\mathbb{T}^3).
\end{aligned}$$

Then, up to subsequences, there exists $\rho\geq 0$ and V such that

$$\begin{aligned}
&\sqrt{\rho_\epsilon}\to\sqrt{\rho} \text{ strongly in } L^2((0,T);H^1(\mathbb{T}^3))\\
&\nabla V_\epsilon\to\nabla V \text{ strongly in } C([0,T);L^2(\mathbb{T}^3)),
\end{aligned}$$

and (ρ, V) is a finite energy weak solution of ρ of (1.2) *with initial datum $\rho(0)=\rho^0$, in the sense of Definition 2.8. Namely there exist Λ, $\mathcal{S}\in L^2((0,T)\times\mathbb{T}^3)$ such that*

$$\sqrt{\rho}\Lambda = 2\operatorname{div}(\sqrt{\rho}\nabla^2\sqrt{\rho} - \nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho} - \rho^\gamma\mathbb{I}) - \rho\nabla V \text{ in } \mathcal{D}'((0,T)\times\mathbb{T}^3) \qquad (1.6)$$

$$\sqrt{\rho}\mathcal{S} = 2\sqrt{\rho}\nabla^2\sqrt{\rho} - 2\nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho} \text{ a.e. in } (0,T)\times\mathbb{T}^3 \qquad (1.7)$$

and $C>0$ such that for a.e. $t\in(0,T)$

$$\int_{\mathbb{T}^3}\left(|\nabla\sqrt{\rho}|^2 + \frac{\rho^\gamma}{\gamma-1} + \frac{1}{2}|\nabla V|^2\right)(t)dx + \int_0^t\int_{\mathbb{T}^3}|\Lambda|^2 dsdx \leq C \qquad (1.8)$$

$$\begin{aligned}
\int_{\mathbb{T}^3}(\rho(\log\rho-1)+1)(t)dx + \int_0^t\int_{\mathbb{T}^3}|\mathcal{S}|^2dsdx + \frac{4}{\gamma}\int_0^t\int_{\mathbb{T}^3}|\nabla\rho^{\frac{\gamma}{2}}|^2dsdx&\\
+\int_0^t\int_{\mathbb{T}^3}\rho(\rho-g)\,dsdx \leq \int_{\mathbb{T}^3}(\rho^0(\log\rho^0-1)+1)dx.&
\end{aligned} \qquad (1.9)$$

Moreover, if in addition, the initial data also satisfy

$$\begin{aligned}
&\sqrt{\rho^0_\epsilon}u^0_\epsilon\to 0 \text{ strongly in } L^2(\mathbb{T}^3)\\
&\nabla\sqrt{\rho^0_\epsilon}\to\nabla\sqrt{\rho^0} \text{ strongly in } L^2(\mathbb{T}^3)\\
&\rho^0_\epsilon\to\rho^0 \text{ strongly in } L^\gamma(\mathbb{T}^3),
\end{aligned}$$

then ρ is an energy dissipating weak solution, meaning that in addition to be a finite energy weak solution for a.e. $t \in (0,T)$ it holds

$$\int_{\mathbb{T}^3}\left(|\nabla\sqrt{\rho}|^2+\frac{\rho^\gamma}{\gamma-1}+\frac{1}{2}|\nabla V|^2\right)(t)dx+\int_0^t\int_{\mathbb{T}^3}|\Lambda|^2 dsdx \leq \int_{\mathbb{T}^3}|\nabla\sqrt{\rho^0}|^2dx+\int_{\mathbb{T}^3}\frac{(\rho^0)^\gamma}{\gamma-1}dx+\int_{\mathbb{T}^3}\frac{1}{2}|\nabla V(0)|^2dx.$$

Let us notice that the estimates (1.8) and (1.9) yield the boundedness of the Fisher entropy and the free energy, respectively. On the other hand, the quantities Λ and $\mathcal{S}$ characterized in (1.6) and (1.7) provide the associated entropy dissipations, in a weaker sense than the estimates derived in [18] and [27]. Indeed, formally

$$\Lambda = 2\sqrt{\rho}\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right)-\frac{\nabla\rho^\gamma}{\sqrt{\rho}}, \qquad \mathcal{S}=\sqrt{\rho}\nabla^2\log\rho, \tag{1.10}$$

but, due to the low regularity setting and the possible presence of vacuum regions it seems not possible to obtain the relations (1.10) in the limit, so the only available information we have is given by formulas (1.6) and (1.7). We refer to Remark 2.4 and Proposition 2.10 below for more details on the tensor Λ and $\mathcal{S}$.

An exhaustive list of all references concerning diffusive relaxation limits and asymptotic behavior for systems of conservation laws with friction, and in particular for hydrodynamic models for semiconductors, is beyond the interest of our presentation. For the theory of diffusive relaxation, we refer here to [17], concerning in particular the case of multidimensional general semilinear systems, and the reference therein. Moreover, concerning in particular the case of high friction limits with relative entropy techniques in the context of Korteweg theories [19], we refer to [34, 14]; see also [33] for the case of Euler equations with friction. Finally, for the particular case of Euler–Poisson models for semiconductors, we recall that the rigorous analysis of the diffusive relaxation limits in the context of weak, entropic solutions started with the seminal paper [35], where the one dimensional case is treated using compensated compactness; see also [32, 31] for the multi–d case.

Besides the modeling point of view, there are some other mathematical aspects which motivate our result. First of all, the study of this singular limit is related to the asymptotic behavior of solutions to (1.1) for large times. Let $(\rho^\star, u^\star) = (r, 0)$ be the stationary solution to (1.1), where, for g constant, $r = \fint \rho = g$ is the mean value of the particle density. Then it can be shown that solutions to (1.1) exponentially converge towards $(\rho^\star, u^\star)$ as $t \to \infty$, see [20] for the one-dimensional problem (with suitable boundary conditions, see also [29] for some extensions) and [11] for the proof of this result in the framework of finite energy weak solutions in the three dimensional torus.

On the other hand, it is also interesting to determine the asymptotic dynamics which governs the exponential convergence to equilibrium. This is indeed achieved by performing the scaling in (1.3), hence the (QDD) equation (1.2) also gives the asymptotic dynamics we are interested in.

On a related subject, let us also comment on the inviscid counterpart of system (1.1), namely the quantum hydrodynamic (QHD) system [1, 2]. Due to the dissipative term $-\xi\rho u$, also in this case it is possible to show both the exponential convergence towards the stationary solution [21, 22] and the relaxation limit [25], again towards the (QDD) equation. However the only available results here deal with small, regular perturbations around stationary solutions. This can be seen as due to the lack of regularizing effect of the viscosity, by means of the BD entropy estimates.

Notice that in Theorem 1.1 the only assumption needed is the initial energy associated to the system (1.1) to be uniformly bounded at the initial time. In particular no assumptions on the limiting solution to (1.2) are given. Consequently, as a byproduct our main Theorem also provides an alternative proof for the existence of finite energy weak solutions to (1.2), see [27] and [18]. Furthermore, in the proof of our main result it is possible to see how the energy and BD entropy estimates, respectively, associated to (1.1) yield, in the limit $\varepsilon \to 0$, the Fisher and free energy, respectively, associated to (1.2). Those facts were already noticed, in a similar context, in the recent preprint [10], see also [9]. More precisely, the Authors in [10] consider the one dimensional shallow water equations with a nonlinear damping term. By using a similar scaling as in (1.3), the authors study the convergence towards a lubrication type model. In particular in [10] the

Authors emphasize how the BD entropy for the hydrodynamical system converge towards the so called Bernis-Friedman [8] entropy, associated to the limiting diffusive equation.

Let us remark that also in the context of semiconductor device modeling it would make sense to consider a nonlinear damping term, as in [10]. Indeed this would correspond to the case when the relaxation time τ is no longer a constant, but a function of the particle density. This is consistent with the derivation of the hydrodynamic system from kinetic theories, as in general the relaxation coefficient may depend on the particle density. Finally, our result can also be seen as related to the derivation of (1.1) and (1.2) from kinetic equations. These macroscopic models for quantum transport are usually derived from collisional Wigner-type equations, with a suitable choice of the collision operator, see [16, 15] and [24] for a more comprehensive discussion about those issues. In particular, the QNS system with a linear damping was derived in [28] by applying the moment method to a Wigner-type equation whose collisional operator is chosen to be the sum of a BGK and a Caldeira-Leggett-type operator, see also [26] where an alternative derivation is given by avoiding the Chapman-Enskog expansion. Actually in [28] and [26] the authors derive the full QNS system, where also the dynamics of the energy density is given, in our paper we only consider the isentropic dynamics given by (1.1). We also mention [13] where the QNS system without damping is derived. On the other hand the QDD equation can also be derived from the same Wigner-type equation by using a diffusive scaling. In this sense our result, obtained by using the scaling (1.3), can be seen as linking the two different scalings used to derive (1.1) and (1.2) directly from kinetic models.

Organization of the paper. The paper is organized as follows. In Section 2 we give the definition of weak solution for the Quantum-Navier-Stokes system and the Quantum-Drift-Diffusion equation, we give a formal proof of the ϵ-independent estimates and we state the main theorem and in Section 3 we prove the main result of the paper.

Notations. We denote with $L^p(\mathbb{T}^n)$ the standard Lebesgue spaces. The Sobolev space of L^p functions with k distributional derivatives in L^p is denoted $W^{k,p}$, in the case $p = 2$ we write $H^k(\mathbb{T}^n)$. The spaces $W^{-k,p}$ and H^{-k} denote the dual spaces of $W^{k,p'}$ and H^k where p' is the usual Hölder conjugate of p. Given a Banach space B, the classical Bochner space of real valued function with values in B is denote by $L^p(0,T;B)$ and sometimes also the abbreviation $L^p_t(B_x)$ will be used. Given a function $f \in L^p(\mathbb{T}^3)$ we denote the average of f alternatively $\fint f$ or $\bar{f}$, and throughout the paper we can assume without loss of generality that $|\mathbb{T}^3| = 1$. We denote with $Du = (\nabla u + \nabla u^t)/2$ the symmetric part of the Jacobian matrix ∇u and with $Au = (\nabla u - \nabla u^t)/2$ the antisymmetric part. Finally, the subscript ϵ used to denote sequences of functions has to be always understood running over a countable set.

2. Definition of Weak Solutions and Main Result

In this section we give the definition of weak solutions for the system (2.1) and (1.2). The existence of such solutions is out of the main aims of this paper; we underline here this result can be proved via a compactness argument as done in [30, 4] with minor modifications; see also [3, 5, 6]. However, it is worth observing that with this technique, thanks to the particular choice of the approximating sequence, the constructed weak solutions verify various appropriate bounds, instrumental for our convergence result. For this reason, we shall assume these bounds are valid in our framework; see Definition 2.1 and Remarks 2.4 and 2.5 below.

2.1. **Weak solutions of the Quantum-Navier-Stokes-Poisson equations.**

Let us consider the following system in $(0,T) \times \mathbb{T}^3$ for a given $g_\epsilon : \mathbb{T}^3 \to \mathbb{R}$,

$$
\begin{aligned}
&\partial_t \rho_\epsilon + \frac{1}{\epsilon}\operatorname{div}(\rho_\epsilon u_\epsilon) = 0\\
&\partial_t(\rho_\epsilon u_\epsilon) + \frac{1}{\epsilon}\operatorname{div}(\rho_\epsilon u_\epsilon \otimes u_\epsilon) - \frac{1}{\epsilon}\operatorname{div}(\rho_\epsilon D u_\epsilon) + \frac{1}{\epsilon}\nabla \rho_\epsilon^\gamma + \frac{1}{\epsilon}\rho_\epsilon \nabla V_\epsilon = \frac{1}{\epsilon} 2\rho_\epsilon \nabla\left(\frac{\Delta\sqrt{\rho_\epsilon}}{\sqrt{\rho_\epsilon}}\right) - \frac{1}{\epsilon^2}\rho_\epsilon u_\epsilon \\
&-\Delta V_\epsilon = \rho_\epsilon - g_\epsilon
\end{aligned}
\tag{2.1}
$$

with initial data

$$\begin{aligned} &\rho_\epsilon(0,x)=\rho_\epsilon^0(x),\\ &\rho_\epsilon(0,x)u_\epsilon(0,x)=\rho_\epsilon^0(x)u_\epsilon^0(x), \end{aligned} \tag{2.2}$$

on $\{t=0\}\times\mathbb{T}^3$ and zero average condition for V_ϵ, namely

$$⨍_{\mathbb{T}^3} V_\epsilon(x,t)dx=0. \tag{2.3}$$

We emphasize that in order to solve $(2.1)_3$, (2.3) we need the following assumption on the doping profile g_ϵ: $⨍_{\mathbb{T}^3} g_\epsilon(x)dx=M_\epsilon$ and $M_\epsilon>0$ where $M_\epsilon:=⨍_{\mathbb{T}^3}\rho_\epsilon^0(x)$; see also Remark 2.6 below.

The definition of weak solution is the following.

Definition 2.1. *Given ρ_ϵ^0 positive and such that $\sqrt{\rho_\epsilon^0}\in H^1(\mathbb{T}^3)$ and $\rho_\epsilon^0\in L^\gamma(\mathbb{T}^3)$, $g_\epsilon\in L^2(\mathbb{T}^3)$ and u_ϵ^0 such that $\sqrt{\rho_\epsilon^0}u_\epsilon^0\in L^2(\mathbb{T}^3)$, then $(\rho_\epsilon,u_\epsilon,V_\epsilon)$ with $\rho_\epsilon\geq 0$ and V_ϵ with zero average is a weak solution of the Cauchy problem (2.1)-(2.2) if the following conditions are satisfied.*

(1) Integrability conditions:

$$\begin{aligned} &\sqrt{\rho_\epsilon}\in L^\infty(0,T;H^1(\mathbb{T}^3))\cap L^2(0,T;H^2(\mathbb{T}^3)), && \sqrt{\rho_\epsilon}u_\epsilon\in L^\infty(0,T;L^2(\mathbb{T}^3)),\\ &\rho_\epsilon^\gamma\in L^\infty(0,T;L^1(\mathbb{T}^3)), && \mathcal{T}_\epsilon\in L^2(0,T;L^2(\mathbb{T}^3)),\\ &\rho_\epsilon\in C(0,T;L^2(\mathbb{T}^3)), && V_\epsilon\in C(0,T;H^2(\mathbb{T}^3)). \end{aligned}$$

(2) Continuity equation:
For any $\phi\in C^\infty([0,T)\times\mathbb{T}^3;\mathbb{R})$

$$\int\rho_\epsilon^0\phi(0)dx+\iint\rho_\epsilon\phi_t+\frac{1}{\epsilon}\sqrt{\rho_\epsilon}\sqrt{\rho_\epsilon}u_\epsilon\nabla\phi\, dsdx=0. \tag{2.4}$$

(3) Momentum equation:
For any $\psi\in C^\infty([0,T)\times\mathbb{T}^3;\mathbb{R}^3)$

$$\begin{aligned} &\int\rho_\epsilon^0u_\epsilon^0\psi(0)dx+\iint\sqrt{\rho_\epsilon}(\sqrt{\rho_\epsilon}u_\epsilon)\psi_t\, dsdx+\frac{1}{\epsilon}\iint\sqrt{\rho_\epsilon}u_\epsilon\sqrt{\rho_\epsilon}u_\epsilon\cdot\nabla\psi\, dsdx\\ &-\frac{1}{\epsilon}\iint\sqrt{\rho_\epsilon}\mathcal{T}_\epsilon^s:\nabla\psi\, dsdx+\frac{1}{\epsilon}\iint\rho_\epsilon^\gamma\operatorname{div}\psi\, dsdx-\frac{1}{\epsilon}\iint\rho_\epsilon\nabla V_\epsilon\psi\, dsdx\\ &-\frac{1}{\epsilon}\iint 2\sqrt{\rho_\epsilon}\,\nabla^2\sqrt{\rho_\epsilon}:\nabla\psi\, dsdx+\frac{1}{\epsilon}\iint 2\nabla\sqrt{\rho_\epsilon}\otimes\nabla\sqrt{\rho_\epsilon}\,:\nabla\psi\, dsdx+\frac{1}{\epsilon^2}\iint\rho_\epsilon u_\epsilon\psi\, dsdx=0. \end{aligned} \tag{2.5}$$

(4) Poisson equation:
For a.e. $(t,x)\in(0,T)\times\mathbb{T}^3$ it holds that

$$-\Delta V_\epsilon=\rho_\epsilon-g_\epsilon. \tag{2.6}$$

(5) Energy dissipation:
For any $\varphi\in C^\infty([0,T)\times\mathbb{T}^3;\mathbb{R})$

$$\iint\sqrt{\rho_\epsilon}\mathcal{T}_{\epsilon,i,j}\varphi\, dsdx=-\iint\rho_\epsilon u_{\epsilon,i}\nabla_j\varphi\, dsdx-\iint 2\sqrt{\rho_\epsilon}u_{\epsilon,i}\otimes\nabla_j\sqrt{\rho_\epsilon}\,\varphi\, dsdx. \tag{2.7}$$

(6) Energy Inequality:
For a.e. $t\in(0,T)$

$$\begin{aligned} &\int_{\mathbb{T}^3}\left(\frac{1}{2}\rho_\epsilon|u_\epsilon|^2+\frac{\rho_\epsilon^\gamma}{\gamma-1}+2|\nabla\sqrt{\rho_\epsilon}|^2+\frac{1}{2}|\nabla V_\epsilon|^2\right)(t)dx+\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}|\mathcal{T}_\epsilon^s|^2dsxd+\frac{1}{\epsilon^2}\int_0^t\int_{\mathbb{T}^3}\rho_\epsilon|u_\epsilon|^2dsdx\\ &\leq\int_{\mathbb{T}^3}\left(\frac{1}{2}\rho_\epsilon^0|u_\epsilon^0|^2+\frac{(\rho_\epsilon^0)^\gamma}{\gamma-1}+2|\nabla\sqrt{\rho_\epsilon^0}|^2+\frac{1}{2}|\nabla V_\epsilon(0)|^2\right)dx. \end{aligned} \tag{2.8}$$

(7) *BD Entropy: Let $w_\epsilon = u_\epsilon + \nabla \log \rho_\epsilon$, then there exists a tensor $\mathcal{S}_\epsilon \in L^2(0,T;L^2(\mathbb{T}^3))$ such that*

$$\sqrt{\rho_\epsilon}\mathcal{S}_\epsilon = 2\sqrt{\rho_\epsilon}\nabla^2\sqrt{\rho_\epsilon} - 2\nabla\sqrt{\rho_\epsilon}\otimes\nabla\sqrt{\rho_\epsilon} \;\; a.e. \; in \; (0,T)\times\mathbb{T}^3 \tag{2.9}$$

we have for a.e. $t \in (0,T)$

$$\begin{aligned}
&\int_{\mathbb{T}^3}\left(\frac{1}{2}\epsilon\rho_\epsilon|w_\epsilon|^2 + \epsilon\frac{\rho_\epsilon^\gamma}{\gamma-1} + \epsilon 2|\nabla\sqrt{\rho_\epsilon}|^2 + \frac{1}{2}\epsilon|\nabla V_\epsilon|^2 + (\rho_\epsilon(\log\rho_\epsilon - 1)+1)\right)(t)dx \\
&+\int_0^t\int_{\mathbb{T}^3}|\mathcal{T}_\epsilon^a|^2\,dsdx + \int_0^t\int_{\mathbb{T}^3}|\mathcal{S}_\epsilon|^2\,dsdx + \frac{4}{\gamma}\int_0^t\int_{\mathbb{T}^3}|\nabla\rho_\epsilon^{\frac{\gamma}{2}}|^2\,dsdx + \frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\rho_\epsilon|u_\epsilon|^2\,dsdx \\
&+\int_0^t\int_{\mathbb{T}^3}\rho_\epsilon(\rho_\epsilon - g_\epsilon)\,dsdx \le \\
&\int_{\mathbb{T}^3}\left(\frac{1}{2}\epsilon\rho_\epsilon^0|w_\epsilon^0|^2 + \epsilon\frac{(\rho_\epsilon^0)^\gamma}{\gamma-1} + \epsilon 2|\nabla\sqrt{\rho_\epsilon^0}|^2 + +\frac{1}{2}\epsilon|\nabla V_\epsilon(0)|^2 + (\rho_\epsilon^0(\log\rho_\epsilon^0 - 1)+1)\right)dx.
\end{aligned} \tag{2.10}$$

(8) *There exists an absolute constant C such that*

$$\begin{aligned}
\int_0^t\int_{\mathbb{T}^3}|\nabla^2\sqrt{\rho_\epsilon}|^2 + |\nabla\rho_\epsilon^{\frac{1}{4}}|^4 dsdx &\le C\int_{\mathbb{T}^3}\left(\frac{1}{2}\epsilon\rho_\epsilon^0|w_\epsilon^0|^2 + \epsilon\frac{(\rho_\epsilon^0)^\gamma}{\gamma-1} + \epsilon 2|\nabla\sqrt{\rho_\epsilon^0}|^2\right)dx \\
&+ C\int_{\mathbb{T}^3}\left(\frac{1}{2}\epsilon|\nabla V_\epsilon(0)|^2 + (\rho_\epsilon^0(\log\rho_\epsilon^0 - 1)+1)\right)dx
\end{aligned} \tag{2.11}$$

The following remarks aim at explaining some peculiar points of the definition of weak solutions. In particular, we explain the presence of the tensors $\mathcal{T}_\epsilon$ and $\mathcal{S}_\epsilon$ in the following remarks, and the reason why we must further assume the bound (2.11).

Remark 2.2 (*Weak formulation of the quantum term*). We emphasize that in the weak formulation introduced above, the third-order term in the momentum equation can be written in different ways:

$$2\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) = \operatorname{div}(\rho\nabla^2\log\rho) = \operatorname{div}(2\sqrt{\rho}\nabla^2\sqrt{\rho} - 2\nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho}), \tag{2.12}$$

and we are using the last expression to give a distributional meaning to the third-order tensor term.

Remark 2.3 (*The velocity field and the vacuum set*). We stress that in the definition of weak solutions the vacuum region can be of positive measure. As a consequence, the velocity it is not uniquely defined in the vacuum set, namely if we change its value on $\{\rho = 0\}$ we would still have the same weak solution of (2.1)-(2.2). Moreover, even if we choose $u = 0$ on $\{\rho = 0\}$ we can not deduce any a priori bound in any Lebesgue space.

Remark 2.4 (*About the tensors $\mathcal{T}_\epsilon$ and $\mathcal{S}_\epsilon$*). The presence of the tensor $\mathcal{T}_\epsilon$ is due to the possible presence of vacuum regions. Indeed, if the density is bounded away from zero, (2.7) implies that $\mathcal{T}_\epsilon = \sqrt{\rho_\epsilon}\nabla\, u_\epsilon$. On the other hand, even in the case when the vacuum has zero Lebesgue measure, ∇u_ϵ also can not be defined in a distributional sense. Therefore the tensor $\mathcal{T}_\epsilon$ arises as a weak L^2-limit of the sequence $\{\sqrt{\rho^n}\nabla u_\epsilon^n)\}_n$, where $\{\rho_\epsilon^n, u_\epsilon^n\}_n$ is a suitable sequence of approximations, see [30, 4]. Analogously, $\mathcal{S}_\epsilon$ is again motivated by the presence of vacuum regions. Indeed, also in this case, if the density is bounded away from zero, by using the identity (2.12), we have that $\mathcal{S}_\epsilon = \sqrt{\rho_\epsilon}\nabla^2\log\rho_\epsilon$. As for $\mathcal{T}_\epsilon$, the tensor $\mathcal{S}_\epsilon$ arises as an L^2 weak limit of the sequence $\{\sqrt{\rho^n}\nabla^2\log\rho_\epsilon^n\}_n$, where $\{\rho_\epsilon^n, u_\epsilon^n\}_n$ is a suitable sequence of approximations. Notice that the fact that $\{\sqrt{\rho^n}\nabla^2\log\rho_\epsilon^n\}_n$ is bounded in $L^2((0,T)\times\mathbb{T}^3)$ because of the BD Entropy, see Proposition 2.7.

Remark 2.5 (*About the inequality* (2.11)). Regarding (2.11) we recall the following inequality proved in [27] : There exists $C > 0$ depending only on the dimension, such that for any function $\rho \ge 0$ with $\sqrt{\rho} \in H^2(\mathbb{T}^3)$

$$\iint|\nabla\rho^{\frac{1}{4}}|^4\,dsdx + \iint|\nabla^2\sqrt{\rho}|^2\,dsdx \le C\iint\rho|\nabla^2\log\rho|^2\,dsdx. \tag{2.13}$$

Therefore, (2.11) follows by applying (2.13) to some approximation ρ_ϵ^n, for which the quantity $\sqrt{\rho^n}\nabla^2\log\rho_\epsilon^n$ is well-defined, using the bound on $\sqrt{\rho^n}\nabla^2\log\rho_\epsilon^n$ inferred from the BD Entropy and using the weak lower-semicontinuity of the norms. For completeness we give the proof of (2.11) in the appendix.

Remark 2.6 (*About the integrability conditions and the Poisson equation*). We notice that by using the integrability hypothesis and (2.4) the average of the density is conserved, namely, $\fint_{\mathbb{T}^3}\rho_\epsilon(t,x)\,dx = M_\epsilon$ for any $t\in(0,T)$. Therefore, $\rho_\epsilon - g$ has zero average for any $t\in(0,T)$ and then the compatibility for the Poisson equation $(2.1)_3$ is always satisfied.
Next, from the mimimun regularity required for ρ_ϵ to be a weak, finite energy solution, we readily obtain $\rho_\epsilon\in C([0,T];L^2(\mathbb{T}^3))$; this (redundant) condition is thus listed explicitly in Definition 2.1. Hence $\rho_\epsilon|_{t=0}=\rho_\epsilon^0$ in $L^2(\mathbb{T}^3)$ and we have that $V_\epsilon(0)$ is well-defined and coincides with the solution of the corresponding elliptic equation at $t=0$, that is:

$$-\Delta\, V_\epsilon(0) = \rho_\epsilon^0 - g_\epsilon.$$

Finally, since ρ_ϵ^0 and g_ϵ are both bounded in $L^2(\mathbb{T}^3)$ we have that $V_\epsilon|_{t=0} = V_\epsilon(0)$ is bounded in $H^2(\mathbb{T}^3)$.

Weak solutions of (2.1)-(2.2) can be obtained by using the same approximation argument of [30] with minor changes. In particular, being the proof of the existence based on a compactness argument, (2.8) and (2.10) should be proved for suitable approximate solutions and then would be obtained trough a limiting argument. On the other hand, as usual in PDE they can be motivated by formal estimate. This is exactly the content of the next proposition.

Proposition 2.7. *Let $\epsilon>0$ and $(\rho_\epsilon,u_\epsilon,V_\epsilon)$ be a sequence of smooth solutions of* (2.1) *with initial data* (2.2). *Define $w_\epsilon = u_\epsilon + \nabla\log\rho_\epsilon$. Then, for any $t\in[0,T)$ the pair $(\rho_\epsilon,u_\epsilon,V_\epsilon)$ satisfies the following estimates:*

(1) (Energy Estimate)

$$\begin{aligned}&\int_{\mathbb{T}^3}\left(\frac12\rho_\epsilon|u_\epsilon|^2+\frac{\rho_\epsilon^\gamma}{\gamma-1}+2|\nabla\sqrt{\rho_\epsilon}|^2+\frac12|\nabla V_\epsilon|^2\right)(t)dx+\frac1\epsilon\int_0^t\int_{\mathbb{T}^3}\rho_\epsilon|Du_\epsilon|^2\,dsdx+\frac{1}{\epsilon^2}\int_0^t\int_{\mathbb{T}^3}\rho_\epsilon|u_\epsilon|^2\,dsdx\\&=\int_{\mathbb{T}^3}\left(\frac12\rho_\epsilon^0|u_\epsilon^0|^2+\frac{(\rho_\epsilon^0)^\gamma}{\gamma-1}+2|\nabla\sqrt{\rho_\epsilon^0}|^2+\frac12|\nabla V_\epsilon(0)|^2\right)dx.\end{aligned}\tag{2.14}$$

(2) (BD Entropy)

$$\begin{aligned}&\int_{\mathbb{T}^3}\left(\frac12\epsilon\rho_\epsilon|w_\epsilon|^2+\epsilon\frac{\rho_\epsilon^\gamma}{\gamma-1}+\epsilon2|\nabla\sqrt{\rho_\epsilon}|^2+\epsilon\frac12|\nabla V_\epsilon|^2+(\rho_\epsilon(\log\rho_\epsilon-1)+1)\right)(t)dx+\int_0^t\int_{\mathbb{T}^3}\rho_\epsilon|Au_\epsilon|^2\,dsdx\\&+\int_0^t\int_{\mathbb{T}^3}\rho_\epsilon|\nabla^2\log\rho_\epsilon|^2\,dsdx+\frac4\gamma\int_0^t\int_{\mathbb{T}^3}|\nabla\rho_\epsilon^{\frac\gamma2}|^2\,dsdx+\int_0^t\int_{\mathbb{T}^3}\rho_\epsilon(\rho_\epsilon-g_\epsilon)\,dsdx+\frac1\epsilon\int_0^t\int_{\mathbb{T}^3}\rho_\epsilon|u_\epsilon|^2\,dsdx\\&=\int_{\mathbb{T}^3}\left(\frac12\epsilon\rho_\epsilon^0|w_\epsilon^0|^2+\epsilon\frac{(\rho_\epsilon^0)^\gamma}{\gamma-1}+\epsilon2|\nabla\sqrt{\rho_\epsilon^0}|^2+\frac12\epsilon|\nabla V_\epsilon(0)|^2+(\rho_\epsilon^0(\log\rho_\epsilon^0-1)+1)\right)dx.\end{aligned}\tag{2.15}$$

Proof. Let us start by recalling the following alternative ways to write the dispersive term:

$$2\rho\nabla\left(\frac{\Delta\sqrt\rho}{\sqrt\rho}\right)=\operatorname{div}(\rho\nabla^2\log\rho)=\operatorname{div}(2\sqrt\rho\nabla^2\sqrt\rho-2\nabla\sqrt\rho\otimes\nabla\sqrt\rho).$$

Then, for smooth solutions $(\rho_\epsilon,u_\epsilon)$ of (2.1), as usual we multiply the continuity equation $(2.1)_1$ by $\gamma\rho_\epsilon^{\gamma-1}/(\gamma-1)$ and the momentum equation $(2.1)_2$ by u_ϵ and integrate in space to get:

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{T}^3}\left(\frac12\rho|u_\epsilon|^2\right)dx+\frac1\epsilon\int_{\mathbb{T}^3}\rho|Du_\epsilon|^2dx+\frac1\epsilon\int_{\mathbb{T}^3}\nabla\rho^\gamma\cdot u_\epsilon dx+\frac1\epsilon\int_{\mathbb{T}^3}\rho_\epsilon\nabla^2\log\rho_\epsilon:\nabla u_\epsilon dx\\&+\frac{1}{\epsilon^2}\int_{\mathbb{T}^3}\rho|u_\epsilon|^2dx+\frac1\epsilon\int_{\mathbb{T}^3}\rho_\epsilon\nabla V_\epsilon u_\epsilon dx=0,\end{aligned}\tag{2.16}$$

$$\frac{d}{dt}\int_{\mathbb{T}^3}\frac{1}{2}|\nabla V_\epsilon|^2\,dx = -\int_{\mathbb{T}^3}(\Delta V_\epsilon)_t V_\epsilon\,dx = \int_{\mathbb{T}^3}\partial_t\rho_\epsilon V_\epsilon\,dx = -\frac{1}{\epsilon}\int_{\mathbb{T}^3}\operatorname{div}(\rho_\epsilon u_\epsilon)V_\epsilon\,dx = \frac{1}{\epsilon}\int_{\mathbb{T}^3}\rho_\epsilon\nabla V_\epsilon u_\epsilon\,dx, \tag{2.17}$$

$$\frac{d}{dt}\int_{\mathbb{T}^3}\frac{\rho_\epsilon^\gamma}{\gamma-1}\,dx - \frac{1}{\epsilon}\int_{\mathbb{T}^3}\nabla\rho_\epsilon^\gamma u_\epsilon dx = 0. \tag{2.18}$$

Moreover, using the equation satisfied by the so called *effective velocity* $\nabla\log\rho_\epsilon$, (see [12] for its formal derivation)

$$(\rho_\epsilon\nabla\log\rho_\epsilon)_t + \frac{1}{\epsilon}\operatorname{div}(\rho_\epsilon\nabla\log\rho_\epsilon\otimes u_\epsilon) + \frac{1}{\epsilon}\operatorname{div}(\rho_\epsilon^t\nabla u_\epsilon) = 0,$$

after multiplication by $\nabla\log\rho_\epsilon$ itself and space integration we obtain

$$\frac{d}{dt}\int_{\mathbb{T}^3}2|\nabla\sqrt{\rho}_\epsilon|^2\,dx - \frac{1}{\epsilon}\int_{\mathbb{T}^3}\rho_\epsilon\nabla^2\log\rho_\epsilon : \nabla u_\epsilon\,dx = 0. \tag{2.19}$$

Hence summing (2.16), (2.18) and (2.19) and integrating time we end up to (2.14). For the BD Entropy relation, we take once again advantage of the effective velocity, and in particular introducing the quantity $w_\epsilon = u_\epsilon + \nabla\log\rho_\epsilon$. Then, using the relations

$$\begin{aligned}
\frac{1}{\epsilon}\operatorname{div}(\rho_\epsilon\nabla\log\rho_\epsilon) &= \frac{1}{\epsilon}\Delta\rho_\epsilon,\\
(\rho_\epsilon\nabla\log\rho_\epsilon)_t &= -\frac{1}{\epsilon}\nabla\operatorname{div}(\rho_\epsilon u_\epsilon),\\
\frac{1}{\epsilon}\operatorname{div}(\rho_\epsilon u_\epsilon\otimes\nabla\log\rho_\epsilon + \rho_\epsilon\nabla\log\rho_\epsilon\otimes u_\epsilon) &= \frac{1}{\epsilon}\Delta(\rho_\epsilon u_\epsilon) - \frac{2}{\epsilon}\operatorname{div}(\rho_\epsilon Du_\epsilon) + \frac{1}{\epsilon}\nabla\operatorname{div}(\rho_\epsilon u_\epsilon),\\
\frac{1}{\epsilon}\operatorname{div}(\rho_\epsilon\nabla\log\rho_\epsilon\otimes\nabla\log\rho_\epsilon) &= \frac{1}{\epsilon}\Delta(\rho_\epsilon\nabla\log\rho_\epsilon) - \frac{1}{\epsilon}\operatorname{div}(\rho_\epsilon\nabla^2\log\rho_\epsilon).
\end{aligned}$$

we obtain the following alternative version of (2.1):

$$\begin{aligned}
&\partial_t\rho_\epsilon + \frac{1}{\epsilon}\operatorname{div}(\rho_\epsilon w_\epsilon) = \frac{1}{\epsilon}\Delta\rho_\epsilon\\
&\partial_t(\rho_\epsilon w_\epsilon) + \frac{1}{\epsilon}\operatorname{div}(\rho_\epsilon w_\epsilon\otimes w_\epsilon) + \frac{1}{\epsilon}\nabla\rho_\epsilon^\gamma + \frac{1}{\epsilon}\operatorname{div}(\rho_\epsilon Dw_\epsilon) - \frac{1}{\epsilon}\Delta(\rho_\epsilon w_\epsilon)\\
&\quad -\frac{1}{\epsilon}\operatorname{div}(\rho_\epsilon\nabla^2\log\rho_\epsilon) + \frac{1}{\epsilon}\rho_\epsilon\nabla V_\epsilon + \frac{1}{\epsilon^2}\rho_\epsilon u_\epsilon = 0.
\end{aligned} \tag{2.20}$$

Now we pass to compute the energy associated the the system (2.20). To this end, as already done for (2.1), we use the continuity equation $(2.20)_1$ and multiply the equation $(2.20)_2$ by w_ϵ to conclude

$$\begin{aligned}
&\partial_t\left(\frac{1}{2}\rho|w_\epsilon|^2\right) + \frac{1}{\epsilon}\operatorname{div}\left(\rho_\epsilon w_\epsilon\frac{1}{2}|w_\epsilon|^2\right) + \frac{1}{\epsilon}\left(\frac{1}{2}\Delta\rho_\epsilon|w_\epsilon|^2 - \Delta(\rho_\epsilon w_\epsilon)\cdot w_\epsilon\right) +\\
&\frac{1}{\epsilon}\nabla\rho_\epsilon^\gamma\cdot w_\epsilon + \frac{1}{\epsilon}\operatorname{div}(\rho_\epsilon Dw_\epsilon)\cdot w_\epsilon - \frac{1}{\epsilon}\operatorname{div}(\rho_\epsilon\nabla^2\log\rho_\epsilon)\cdot w_\epsilon + \frac{1}{\epsilon^2}\rho_\epsilon u_\epsilon w_\epsilon + \frac{1}{\epsilon}\rho_\epsilon\nabla V_\epsilon w_\epsilon = 0.
\end{aligned}$$

Since

$$\frac{1}{\epsilon}\left(\frac{1}{2}\Delta\rho_\epsilon|w_\epsilon|^2 - \Delta(\rho_\epsilon w_\epsilon)\cdot w_\epsilon\right) = \frac{1}{\epsilon}\left(\operatorname{div}\left(\nabla\rho_\epsilon\frac{1}{2}|w_\epsilon|^2\right) - \operatorname{div}(\rho_\epsilon\cdot\nabla w_\epsilon)\cdot w_\epsilon\right),$$

when we integrate the equality above over $\mathbb{T}^3$ we get:

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{T}^3}\frac{1}{2}\rho_\epsilon|w_\epsilon|^2dx + \frac{1}{\epsilon}\int_{\mathbb{T}^3}\rho_\epsilon|\nabla w_\epsilon|^2dx + \frac{1}{\epsilon}\int_{\mathbb{T}^3}\nabla\rho_\epsilon^\gamma\cdot w_\epsilon dx + \frac{1}{\epsilon}\int_{\mathbb{T}^3}\rho_\epsilon\nabla V_\epsilon w_\epsilon\,dx\\
&-\frac{1}{\epsilon}\int_{\mathbb{T}^3}\rho_\epsilon|D(w_\epsilon)|^2dx + \frac{1}{\epsilon^2}\int_{\mathbb{T}^3}\rho_\epsilon u_\epsilon w_\epsilon dx + \frac{1}{\epsilon}\int_{\mathbb{T}^3}\rho_\epsilon\nabla^2\log\rho_\epsilon : \nabla w_\epsilon\,dx = 0.
\end{aligned} \tag{2.21}$$

We observe that $|\nabla w_\epsilon|^2 - |D(w_\epsilon)|^2 = |A(w_\epsilon)|^2 = |A(u_\epsilon)|^2$, and, from the definition of w_ϵ, we infer

$$\nabla\rho_\epsilon^\gamma\cdot w_\epsilon = \nabla\rho_\epsilon^\gamma\cdot u_\epsilon + \frac{4}{\gamma}|\nabla\rho_\epsilon^{\gamma/2}|^2;$$

$$\rho_\epsilon u_\epsilon w_\epsilon = \rho_\epsilon u_\epsilon^2 + \rho_\epsilon u_\epsilon\nabla\log\rho_\epsilon;$$

$$\rho_\epsilon \nabla^2 \log \rho_\epsilon : \nabla w_\epsilon = \rho_\epsilon \nabla^2 \log \rho_\epsilon : \nabla u_\epsilon + \rho_\epsilon |\nabla^2 \log \rho_\epsilon|^2$$
$$\rho_\epsilon \nabla V_\epsilon w_\epsilon = \rho_\epsilon \nabla V_\epsilon u_\epsilon + \rho_\epsilon \nabla V_\epsilon \nabla \log \rho_\epsilon$$

Moreover, we recall (2.18), (2.19) and (2.17) to compute:

$$\frac{d}{dt}\int_{\mathbb{T}^3} \big(\rho_\epsilon(\log\rho_\epsilon - 1) + 1\big)dx = \frac{1}{\epsilon}\int_{\mathbb{T}^3} \rho_\epsilon u_\epsilon \nabla \log \rho_\epsilon \, dx,$$

$$\frac{1}{\epsilon}\int_{\mathbb{T}^3} \rho_\epsilon \nabla V_\epsilon w_\epsilon \, dx = \frac{d}{dt}\int_{\mathbb{T}^3} \frac{1}{2}|\nabla V_\epsilon|^2 \, dx + \frac{1}{\epsilon}\int_{\mathbb{T}^3} \rho_\epsilon \nabla V_\epsilon \nabla \log \rho_\epsilon \, dx,$$

the last term can be rewrite:

$$\frac{1}{\epsilon}\int_{\mathbb{T}^3} \rho_\epsilon \nabla V_\epsilon \nabla \log \rho_\epsilon \, dx = \frac{1}{\epsilon}\int_{\mathbb{T}^3} \nabla V_\epsilon \nabla \rho_\epsilon \, dx = -\frac{1}{\epsilon}\int_{\mathbb{T}^3} \Delta V_\epsilon \rho_\epsilon \, dx = \frac{1}{\epsilon}\int_{\mathbb{T}^3} \rho_\epsilon(\rho_\epsilon - g_\epsilon) \, dx$$

Therefore we multiply (2.21) by ϵ and, using the previous identities, we obtain the final estimate

$$\frac{d}{dt}\int_{\mathbb{T}^3} \left[\epsilon\left(\frac{1}{2}\rho|w_\epsilon|^2 + \frac{\rho_\epsilon^\gamma}{\gamma-1} + 2|\nabla\sqrt{\rho_\epsilon}|^2 + \frac{1}{2}|\nabla V_\epsilon|^2\right) + \rho_\epsilon(\log\rho_\epsilon - 1) + 1\right] dx$$
$$+ \int_{\mathbb{T}^3} \rho|A(u_\epsilon)|^2 \, dx + \frac{4}{\gamma}\int_{\mathbb{T}^3} |\nabla \rho_\epsilon^{\frac{\gamma}{2}}|^2 \, dx + \int_{\mathbb{T}^3} \rho_\epsilon |\nabla^2 \log\rho_\epsilon|^2 \, dx + \frac{1}{\epsilon}\int_{\mathbb{T}^3} \rho_\epsilon u_\epsilon^2 \, dx + \int_{\mathbb{T}^3} \rho_\epsilon(\rho_\epsilon - g_\epsilon) \, dx = 0$$

which gives (2.15) upon time integration. $\square$

2.2. **Weak solution of the Quantum-Drift-Diffusion equation.** Next, we consider the Quantum Drift–Diffusion–Poisson equation in $(0,T)\times\mathbb{T}^3$ with $g : \mathbb{T}^3 \to \mathbb{R}$

$$\begin{aligned} &\partial_t \rho + \operatorname{div}\left(2\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) - \nabla\rho^\gamma - \rho\nabla V\right) = 0 \\ &-\Delta V = \rho - g, \end{aligned} \tag{2.22}$$

with initial datum

$$\rho|_{t=0} = \rho^0 \geq 0, \tag{2.23}$$

and V such that

$$⨍_{\mathbb{T}^3} V dx = 0. \tag{2.24}$$

As before, to solve $(2.22)_2$, (2.24) we need the following assumption for g: defining $M := ⨍_{\mathbb{T}^3}\rho^0$, we assume that $⨍_{\mathbb{T}^3} g = M$.

The following definition specifies the framework of weak solutions to (2.22) we shall obtain at the limit.

Definition 2.8. *Given $g \in L^2(\mathbb{T}^3)$, we say that (ρ, V) with $\rho \geq 0$ is a finite energy weak solution of (2.22) if*

(1) *Integrability condition:*

$$\sqrt{\rho} \in L^\infty(0,T;H^1(\mathbb{T}^3)) \cap L^2(0,T;H^2(\mathbb{T}^3)); \qquad \rho \in L^\infty(0,T;L^\gamma(\mathbb{T}^3))$$
$$V \in C((0,T);H^2(\mathbb{T}^3))$$

(2) *Continuity equation:*

$$\int_{\mathbb{T}^3} \rho^0\phi(0) \, dx + \iint \big(\rho\phi_t - \rho^\gamma \operatorname{div}\nabla\phi - 2\nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho} : \nabla^2\phi + 2\sqrt{\rho}\nabla^2\sqrt{\rho} : \nabla^2\phi + \rho\nabla V\cdot\phi\big) \, dsdx = 0,$$

for any $\phi \in C_c^\infty([0,T);C^\infty(\mathbb{T}^3))$,

(3) *Poisson equation:*
For a.e. $(t,x) \in (0,T)\times\mathbb{T}^3$ it holds that

$$-\Delta V = \rho - g \tag{2.25}$$

(4) *Entropy inequalities:*
there exist Λ, $\mathcal{S} \in L^2((0,T)\times\mathbb{T}^3)$ *such that*

$$\sqrt{\rho}\Lambda = \operatorname{div}(2\sqrt{\rho}\nabla^2\sqrt{\rho} - 2\nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho} - \rho^\gamma\mathbb{I}) - \rho\nabla V \ \textit{in}\ \mathcal{D}'((0,T)\times\mathbb{T}^3) \tag{2.26}$$

$$\sqrt{\rho}\mathcal{S} = 2\sqrt{\rho}\nabla^2\sqrt{\rho} - 2\nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho} \ \textit{a.e. in}\ (0,T)\times\mathbb{T}^3 \tag{2.27}$$

and constant $C>0$ *such that for a.e.* $t\in(0,T)$

$$\int_{\mathbb{T}^3}\left(|\nabla\sqrt{\rho}|^2 + \frac{\rho^\gamma}{\gamma-1} + \frac{1}{2}|\nabla V|^2\right)(t)\,dx + \int_0^t\int_{\mathbb{T}^3}|\Lambda|^2\,dsdx \le C, \tag{2.28}$$

$$\begin{aligned}&\int_{\mathbb{T}^3}(\rho(\log\rho-1)+1)(t)\,dx + \int_0^t\int_{\mathbb{T}^3}|\mathcal{S}|^2\,dsdx + \frac{4}{\gamma}\int_0^t\int_{\mathbb{T}^3}|\nabla\rho^{\frac{\gamma}{2}}|^2\,dsdx + \int_0^t\int_{\mathbb{T}^3}\rho(\rho-g)\,dsdx\\ &\le \int_{\mathbb{T}^3}(\rho^0(\log\rho^0-1)+1)\,dx.\end{aligned} \tag{2.29}$$

In addition, (ρ,V) *is called* energy dissipating weak solution *if in particular it holds that for a.e.* $t\in(0,T)$*:*

$$\begin{aligned}\int_{\mathbb{T}^3}\left(|\nabla\sqrt{\rho}|^2 + \frac{\rho^\gamma}{\gamma-1} + \frac{1}{2}|\nabla V|^2\right)(t)\,dx + \int_0^t\int_{\mathbb{T}^3}|\Lambda|^2\,dsdx \le \int_{\mathbb{T}^3}|\nabla\sqrt{\rho^0}|^2\,dx + \int_{\mathbb{T}^3}\frac{\rho^{0\gamma}}{\gamma-1}\,dx\\ + \int_{\mathbb{T}^3}\frac{1}{2}|\nabla V(0,x)|^2.\end{aligned} \tag{2.30}$$

Remark 2.9. The terminology finite energy and energy dissipating weak solutions is motivated by the fact that both (2.28) and (2.30) arises as a weak limit of the energy inequality (2.8). As in Proposition 2.7 we derive the formal estimates in the framework of smooth solutions.

Proposition 2.10. *Let* (ρ,V) *be a smooth solution of* (2.22) *with data satisfying* (2.23),(2.24). *Then, for any* $t\in[0,T)$ *the pair* (ρ,V) *satisfies the following estimates:*

$$\frac{d}{dt}\int_{\mathbb{T}^3}\left(2|\nabla\sqrt{\rho}|^2 + \frac{\rho^\gamma}{\gamma-1} + \frac{1}{2}|\nabla V|^2\right)dx + \int_{\mathbb{T}^3}\rho\left|2\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) - \gamma\rho^{\gamma-2}\nabla\rho - \nabla V\right|^2 dx = 0, \tag{2.31}$$

$$\frac{d}{dt}\int_{\mathbb{T}^3}\rho(\log\rho-1)+1+\ dx\int_{\mathbb{T}^3}\rho|\nabla^2\log\rho|^2\,dx + \frac{4}{\gamma}\int_{\mathbb{T}^3}|\nabla\rho^{\frac{\gamma}{2}}|^2\,dx + \int_{\mathbb{T}^3}\rho(\rho-g)\,dx = 0. \tag{2.32}$$

Proof. The energy (2.32) is achieved by multiplying (2.22) by $\log\rho$ and integrating by parts:

$$\begin{aligned}&\int_{\mathbb{T}^3}\rho_t\log\rho\,dx = \frac{d}{dt}\int_{\mathbb{T}^3}\big((\rho(\log\rho-1))+1\big)\,dx,\\ &\int_{\mathbb{T}^3}\operatorname{div}\left(2\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) - \nabla\rho^\gamma\right)\log\rho\,dx = \int_{\mathbb{T}^3}\operatorname{div}\big(\operatorname{div}\big(\rho\nabla^2\log\rho\big) - \nabla\rho^\gamma\big)\log\rho\,dx\\ &= \int_{\mathbb{T}^3}\rho|\nabla^2\log\rho|^2\,dx + \frac{4}{\gamma}\int_{\mathbb{T}^3}|\nabla\rho^{\frac{\gamma}{2}}|^2\,dx,\\ &-\int_{\mathbb{T}^3}\operatorname{div}(\rho\nabla V)\log\rho\,dx = \int_{\mathbb{T}^3}\rho\nabla V\nabla\log\rho\,dx = -\int_{\mathbb{T}^3}\Delta V\rho\,dx = \int_{\mathbb{T}^3}\rho(\rho-g)\,dx.\end{aligned}$$

Moreover, if we multiply (2.22) by $-2\Delta\sqrt{\rho}/\sqrt{\rho} + \gamma\rho^{\gamma-1}/(\gamma-1)$, after integrating by parts we get

$$\begin{aligned}&\int_{\mathbb{T}^3}\rho_t\left(\frac{-2\Delta\sqrt{\rho}}{\sqrt{\rho}}\right)dx = -\int_{\mathbb{T}^3}\operatorname{div}\left(2\frac{\nabla\sqrt{\rho}}{\sqrt{\rho}}\rho_t\right)dx + \int_{\mathbb{T}^3}\nabla\sqrt{\rho}\nabla\left(2\frac{\rho_t}{\sqrt{\rho}}\right)dx = \frac{d}{dt}\int_{\mathbb{T}^3}2|\nabla\sqrt{\rho}|^2\,dx,\\ &\int_{\mathbb{T}^3}\rho_t\left(\frac{\gamma}{\gamma-1}\rho^{\gamma-1}\right)dx = \frac{d}{dt}\int_{\mathbb{T}^3}\frac{\rho^\gamma}{\gamma-1}\,dx,\\ &\int_{\mathbb{T}^3}\operatorname{div}\left(2\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right)\right)\left(-2\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} + \frac{\gamma}{\gamma-1}\rho^{\gamma-1}\right)dx =\end{aligned}$$

$$= \int_{\mathbb{T}^3} \left(4\rho \left| \nabla \left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} \right) \right|^2 - 2\gamma\rho\nabla \left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} \right) \rho^{\gamma-2}\nabla\rho \right) dx,$$

$$\int_{\mathbb{T}^3} \operatorname{div}(-\nabla\rho^\gamma) \left(-2\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} + \frac{\gamma}{\gamma-1}\rho^{\gamma-1} \right) dx = \int_{\mathbb{T}^3} -\nabla\rho^\gamma \nabla \left(2\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} \right) + \rho|\gamma\rho^{\gamma-2}\nabla\rho|^2 \, dx$$

$$= \int_{\mathbb{T}^3} -2\gamma\rho\rho^{\gamma-2}\nabla\rho\nabla \left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} \right) + \rho|\gamma\rho^{\gamma-2}\nabla\rho|^2 \, dx,$$

$$\int_{\mathbb{T}^3} -\operatorname{div}(\rho\nabla V)\frac{\gamma}{\gamma-1}\rho^{\gamma-1} \, dx = \int_{\mathbb{T}^3} \rho\nabla V \gamma\rho^{\gamma-2}\nabla\rho \, dx = \int_{\mathbb{T}^3} \nabla V \nabla\rho^\gamma \, dx = -\int_{\mathbb{T}^3} \Delta V \rho^\gamma \, dx$$

$$\int_{\mathbb{T}^3} \operatorname{div}(\rho\nabla V) 2\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} \, dx = -\int_{\mathbb{T}^3} \rho\nabla V 2\nabla \left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} \right) dx = -\int_{\mathbb{T}^3} \nabla V \operatorname{div}(\rho\nabla^2 \log\rho) \, dx$$

Finally we multiply (2.22) by V, integrating by parts we get:

$$\int_{\mathbb{T}^3} \rho_t V \, dx = -\int_{\mathbb{T}^3} \Delta V_t \, V \, dx = \int_{\mathbb{T}^3} \nabla V_t \nabla V \, dx = \frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^3} |\nabla V|^2 \, dx,$$

$$\int_{\mathbb{T}^3} \operatorname{div}\left(2\rho\nabla \left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} \right) \right) V \, dx = -\int_{\mathbb{T}^3} 2\rho\nabla \left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} \right) \nabla V \, dx = -\int_{\mathbb{T}^3} \nabla V \operatorname{div}(\rho\nabla^2 \log\rho) \, dx,$$

$$\int_{\mathbb{T}^3} -\operatorname{div}(\nabla\rho^\gamma)V = \int_{\mathbb{T}^3} \nabla\rho^\gamma\nabla V = -\int_{\mathbb{T}^3} \rho^\gamma \Delta V$$

and (2.31) follows by summing up all terms. □

Remark 2.11. It is worth to observe that if we perform the Hilbert expansion of (2.1) the limit solution $(\bar\rho, \bar u)$ formally satisfies at the first non trivial order $O(1/\epsilon)$ (the order $O(1/\epsilon^2)$ tells us the momentum expansion starts form the power one in ϵ) the following identities:

$$\bar\rho\bar u = \epsilon\bar\rho \left(2\nabla \left(\frac{\Delta\sqrt{\bar\rho}}{\sqrt{\bar\rho}} \right) - \gamma\bar\rho^{\gamma-2}\nabla\bar\rho - \nabla\bar V \right),$$

$$\partial_t\bar\rho + \operatorname{div}\left(2\bar\rho\nabla \left(\frac{\Delta\sqrt{\bar\rho}}{\sqrt{\bar\rho}} \right) - \nabla\bar\rho^\gamma - \bar\rho\nabla\bar V \right) = 0,$$

which is exactly the Quantum Drift–Diffusion equation (2.22). We underline also that, with the above definition for $\bar u$, namely

$$(\bar\rho, \bar u) = \left(\bar\rho, \epsilon \left(2\nabla \left(\frac{\Delta\sqrt{\bar\rho}}{\sqrt{\bar\rho}} \right) - \gamma\bar\rho^{\gamma-2}\nabla\bar\rho - \nabla\bar V \right) \right),$$

in the formulation for the mechanical energy (2.14) for (2.1), at the limit the latter reduces to (2.31) after a time integration, that is the corresponding conservation of the mechanical energy for the Quantum Drift–Diffusion equation. The same happens for the BD entropy, namely (2.10) for $\epsilon \to 0$ reduces to (the time integrated version of) (2.32). This fact is coherent with our analysis and it will be validated by Theorem 3.1 below.

Remark 2.12. We remark that if ρ is a weak solution in the sense of the previous definition the initial datum ρ^0 is attained for example in the strong topology of $L^2(\mathbb{T}^3)$.

3. Main result

In this section we prove Theorem 1.1, which we rewrite for reader's convenience.

Theorem 3.1. *Let $(\rho_\epsilon, u_\epsilon, V_\epsilon)$ be a weak solution of* (2.1) *in the sense Definition 2.1 with data* (2.2) *satisfying*

$$\begin{aligned} &\{\rho_\epsilon^0\}_\epsilon \textit{ is bounded in } L^1 \cap L^\gamma(\mathbb{T}^3) \textit{ such that } \rho_\epsilon^0 \to \rho^0 \textit{ in } L^q(\mathbb{T}^3),\ q < 3 \\ &\{\nabla\sqrt{\rho_\epsilon^0}\}_\epsilon \textit{ is bounded in } L^2(\mathbb{T}^3), \\ &\{\sqrt{\rho_\epsilon^0}u_\epsilon^0\}_\epsilon \textit{ is bounded in } L^2(\mathbb{T}^3), \\ &\{g_\epsilon\}_\epsilon \in L^2(\mathbb{T}^3) \textit{ and } g_\epsilon \rightharpoonup g \textit{ weakly in } L^2(\mathbb{T}^3). \end{aligned} \tag{3.1}$$

Then, up to subsequences, there exist $\rho \geq 0$ and V such that

$$\sqrt{\rho_\epsilon} \to \sqrt{\rho} \text{ strongly in } L^2((0,T);H^1(\mathbb{T}^3))$$
$$\nabla V_\epsilon \to \nabla V \text{ strongly in } C([0,T);L^2(\mathbb{T}^3))$$

and (ρ, V) is a finite energy weak solution of (2.22) – (2.23) in the sense of Definition 2.8. If in addition to (3.1), it also hold that

$$\begin{aligned}&\sqrt{\rho_\epsilon^0}u_\epsilon^0 \to 0 \text{ strongly in } L^2(\mathbb{T}^3)\\ &\nabla\sqrt{\rho_\epsilon^0} \to \nabla\sqrt{\rho^0} \text{ strongly in } L^2(\mathbb{T}^3)\\ &\rho_\epsilon^0 \to \rho^0 \text{ strongly in } L^\gamma(\mathbb{T}^3)\end{aligned} \tag{3.2}$$

then (ρ, V) is an energy dissipating weak solution.

The proof of Theorem 3.1 requires several preliminaries, collected in the following section.

3.1. **Preliminary results.** We start by proving the uniform bounds obtained by the Energy estimate and BD Entropy.

Lemma 3.2. *There exists a constant $C > 0$ such that the following uniform bounds on the data*

$$\begin{aligned}&\sup_t \int_{\mathbb{T}^3} \rho_\epsilon \, dx \leq C; \quad \sup_t \int_{\mathbb{T}^3} \rho_\epsilon u_\epsilon^2 \, dx \leq C; \quad \int_0^T \int_{\mathbb{T}^3} |\nabla \rho_\epsilon^{\frac{\gamma}{2}}|^2 \, dsdx \leq C; \quad \sup_t \int |\nabla\sqrt{\rho_\epsilon}|^2 \, dsdx \leq C;\\ &\frac{1}{\epsilon}\int_0^T \int_{\mathbb{T}^3} |\mathcal{T}_\epsilon^s|^2 \, dsdx \leq C; \quad \sup_t \int \rho_\epsilon^\gamma \, dx \leq C; \quad \frac{1}{\epsilon^2}\int_0^T \int \rho_\epsilon u_\epsilon^2 \, dsdx \leq C;\\ &\int_0^T \int_{\mathbb{T}^3} |\nabla^2\sqrt{\rho_\epsilon}|^2 \, dsdx \leq C; \quad \int_0^T \int_{\mathbb{T}^3} |\mathcal{S}_\epsilon|^2 \, dsdx \leq C; \quad \sup_t \int_{\mathbb{T}^3} |\nabla^2 V_\epsilon|^2 \, dx \leq C\end{aligned} \tag{3.3}$$

Proof. We first notice that under the hypothesis (3.1) we have that

$$\{\rho_\epsilon^0 \log \rho_\epsilon^0\}_\epsilon \text{ is bounded in } L^1(\mathbb{T}^3). \tag{3.4}$$

This is obtained very easily under the hypothesis (3.1). Indeed, one can easily show that for $q > p \geq 1$ there exists $C > 0$ such that for $s \geq 0$:

$$|s \log s|^q \leq C(1 + s^p)$$

and therefore, by taking $q = \gamma$ and $1 \leq p < \gamma$ we have that

$$\{\rho_\epsilon^0 \log \rho_\epsilon^0\}_\epsilon \text{ is bounded in } L^q(\mathbb{T}^3). \tag{3.5}$$

which implies (3.4).

Next, as shown in Remark 2.6, $\{\nabla V_\epsilon|_{t=0}\}_\epsilon$ is bounded in $L^2(\mathbb{T}^3)$ and we have proved that the right-hand sides of (2.8) and (2.10) are bounded uniformly in ϵ. Then, to obtain (3.3) is enough to use the bound

$$\int_0^t \int_{\mathbb{T}^3} \rho_\epsilon(\rho_\epsilon - g_\epsilon) \, dsdx \geq \frac{1}{2}\int_0^t \int_{\mathbb{T}^3} |\rho_\epsilon|^2 \, dsdx - \frac{1}{2}\int_0^t \int_{\mathbb{T}^3} |g_\epsilon|^2 \, dsdx$$

in the energy inequalities (2.8) and (2.10). Finally, the bound on the potential follows from the Poisson equation. ☐

The following lemma we prove the convergence needed to pass to the limit as $\epsilon \to 0$.

Lemma 3.3. *There exist $\rho \geq 0$ such that $\sqrt{\rho} \in L^\infty(0,T);H^1(\mathbb{T}^3) \cap L^2(0,T;H^2(\mathbb{T}^3))$ and the following hold:*

$$\rho_\epsilon \to \rho \text{ in } C([0,T];L^q(\mathbb{T}^3)) \text{ , } q < 3, \tag{3.6}$$
$$\sqrt{\rho}_\epsilon \rightharpoonup \sqrt{\rho} \text{ weakly in } L^2(0,T;H^2(\mathbb{T}^3)), \tag{3.7}$$
$$\sqrt{\rho_\epsilon} \overset{*}{\rightharpoonup} \sqrt{\rho} \text{ weakly* in } L^\infty(0,T;H^1(\mathbb{T}^3)), \tag{3.8}$$
$$\sqrt{\rho_\epsilon} \to \sqrt{\rho} \text{ strongly in } L^2(0,T;H^1(\mathbb{T}^n)). \tag{3.9}$$
$$\rho_\epsilon(\log\rho_\epsilon + 1) + 1 \to \rho(\log\rho + 1) + 1 \text{ strongly in } L^1(0,T;L^1(\mathbb{T}^3)), \tag{3.10}$$

$$\rho_\epsilon^\gamma \to \rho^\gamma \ \textit{in } L^1(0,T;L^1(\mathbb{T}^n)) \tag{3.11}$$

$$\tag{3.12}$$

Proof. From (3.3) and Sobolev embedding have that:

$$\{\sqrt{\rho_\epsilon}\}_\epsilon \text{ is bounded in } L^2(0,T;L^q(\mathbb{T}^3)) \quad q\in[1,6] \tag{3.13}$$

Therefore since $\nabla\rho_\epsilon = 2\nabla\sqrt{\rho_\epsilon}\sqrt{\rho_\epsilon}$ by using (3.13) and (3.3) we can infer that

$$\{\rho_\epsilon\}_\epsilon \text{ is bounded in } L^\infty(0,T;W^{1,\frac{3}{2}}(\mathbb{T}^3)).$$

Next, by using the weak formulation of the continuity equation and the bounds (3.3) we get that

$$\{\partial_t\rho_\epsilon\}_\epsilon \text{ is bounded in } L^2(0,T;W^{-1,\frac{3}{2}}(\mathbb{T}^3)).$$

Indeed, it is enough to note that we have

$$\int_0^T \|\partial_t\rho_\epsilon\|^2_{W^{-1,\frac{3}{2}}}\,ds \le \frac{1}{\epsilon^2}\int_0^T \|\rho_\epsilon u_\epsilon\|^2_{L_x^{\frac{3}{2}}}\,ds \le \int_0^T \|\sqrt{\rho_\epsilon}\|^2_{L_x^6}\left\|\sqrt{\rho_\epsilon}\frac{u_\epsilon}{\epsilon}\right\|^2_{L_x^2}\,ds.$$

Since for $q<3$ we have $W^{1,3/2}(\mathbb{T}^3)\subset L^q(\mathbb{T}^3)$ with compact embedding and $L^q(\mathbb{T}^3)\subset W^{-1,3/2}(\mathbb{T}^3)$ we can apply Aubin-Lion lemma to deduce that there exists a subsequence not relabelled and $\rho\ge 0$ such that (3.6) holds. Moreover, by passing to a further subsequence if necessary, (3.6) implies

$$\rho_\epsilon \to \rho \quad \text{a.e in } (0,T)\times\mathbb{T}^3. \tag{3.14}$$

Then, the convergence (3.7) and (3.11) follow from (3.14), the uniform bounds (3.3) and standard weak compactness considerations. Next, to prove (3.9) we start by proving that

$$\sqrt{\rho_\epsilon}\to\sqrt{\rho} \text{ strongly in } L^2(0,T;L^2(\mathbb{T}^3)). \tag{3.15}$$

First, notice that (3.14) implies that

$$\sqrt{\rho_\epsilon}\to\sqrt{\rho} \quad \text{a.e in (0,T) }\times\mathbb{T}^3.$$

Let $M>0$, then

$$\begin{aligned}\int_0^T\int_{\mathbb{T}^3}|\sqrt{\rho_\epsilon}-\sqrt{\rho}|^2\,dsdx \le &\int_0^T\int_{\{\rho_\epsilon>M\}}|\sqrt{\rho_\epsilon}|^2\,dsdx+\\ &\int_0^T\int_{\mathbb{T}^3}|\sqrt{\rho_\epsilon}\chi_{\{\rho_\epsilon\le M\}}-\sqrt{\rho}\chi_{\{\rho\le M\}}|^2\,dsdx+\\ &\int_0^T\int_{\{\rho>M\}}|\sqrt{\rho}|^2\,dsdx.\end{aligned}$$

Then, by using (3.3), (3.13), the fact that $\sqrt{\rho}\in L^\infty(0,T;L^6(\mathbb{T}^3))$, we have that

$$\begin{aligned}\int_0^T\int_{\mathbb{T}^3}|\sqrt{\rho_\epsilon}-\sqrt{\rho}|^2\,dsdx \le &\frac{1}{M^2}\int_0^T\int|\sqrt{\rho_\epsilon}|^6\,dsdx\\ &+\int_0^T\int_{\mathbb{T}^3}|\sqrt{\rho_\epsilon}\chi_{\{\rho_\epsilon\le M\}}-\sqrt{\rho}\chi_{\{\rho\le M\}}|^2\,dsdx\\ &+\frac{1}{M^2}\int_0^T\int|\sqrt{\rho}|^6\,dsdx\\ &\le\int_0^T\int_{\mathbb{T}^3}|\sqrt{\rho_\epsilon}\chi_{\{\sqrt{\rho_\epsilon}\le M\}}-\sqrt{\rho}\chi_{\{\sqrt{\rho}\le M\}}|^2\,dsdx+\frac{2C}{M^2}\end{aligned}$$

Then, we conclude by first sending $\epsilon\to 0$ in the second term, where we use Dominated Convergence Theorem, and then by choosing suitably $M\to\infty$.

The strong convergence (3.9) of $\sqrt{\rho_\epsilon}$ in $L^2(0,T;H^1(\mathbb{T}^3))$ is a consequence of the following simple interpolation inequality:

$$\|\sqrt{\rho_\epsilon}(t)-\sqrt{\rho}(t)\|_{H^1}\le C\|\sqrt{\rho_\epsilon}(t)-\sqrt{\rho}(t)\|^{\frac{1}{2}}_{L^2}\|\sqrt{\rho_\epsilon}(t)-\sqrt{\rho}(t)\|^{\frac{1}{2}}_{H^2}. \tag{3.16}$$

Next, we prove (3.10). We first notice that by using (3.3) and by the very same argument used to deduce (3.5), we easily have that for some $p > 1$

$$\begin{aligned}&\rho \log \rho \in L^p((0,T)\times\mathbb{T}^3),\\&\{\rho_\epsilon \log \rho_\epsilon\}_\epsilon \text{ is bounded in } L^p((0,T)\times\mathbb{T}^3).\end{aligned}$$

Moreover, since the function $s \to s\log s$ is continuous on $[0,\infty)$ we have that

$$\rho_\epsilon \log\rho_\epsilon \to \rho\log\rho \text{ a.e. in } (0,T)\times\mathbb{T}^3.$$

Let $M > 0$, then

$$\begin{aligned}\int_0^T\int_{\mathbb{T}^3}|\rho_\epsilon\log\rho_\epsilon - \rho\log\rho|\,dsdx \le &\int_0^T\int_{\{\rho_\epsilon>M\}}|\rho_\epsilon\log\rho_\epsilon|\,dsdx\\&+\int_0^T\int_{\mathbb{T}^3}|\rho_\epsilon\log\rho_\epsilon\chi_{\{\rho_\epsilon\le M\}} - \rho\log\rho\chi_{\{\rho\le M\}}|\,dsdx\\&+\int_0^T\int_{\{\rho>M\}}|\rho\log\rho|\,dsdx.\\&\le\int_0^T\int_{\mathbb{T}^3}|\rho_\epsilon\log\rho_\epsilon\chi_{\{\rho_\epsilon\le M\}} - \rho\log\rho\chi_{\{\rho\le M\}}|\,dsdx\\&+\frac{2C}{(M\log M)^{p-1}}\end{aligned}$$

and we conclude as in before. Finally, we prove the convergence of the pressure term. We first note that from (3.3) we have that

$$\{\rho_\epsilon^{\frac{\gamma}{2}}\}_\epsilon \text{ is bounded in } L^\infty(0,T;L^2(\mathbb{T}^3))\cap L^2(0,T;H^1(\mathbb{T}^3)).$$

Then, by Sobolev embedding

$$\{\rho_\epsilon^{\frac{\gamma}{2}}\}_\epsilon \text{ is bounded in } L^2(0,T;L^6(\mathbb{T}^3))$$

and therefore for a.e t:

$$\|\rho_\epsilon^{\frac{\gamma}{2}}(t)\|_{\frac{10}{3}} \le \|\rho_\epsilon^{\frac{\gamma}{2}}(t)\|_2^{\frac{2}{5}}\|\rho_\epsilon^{\frac{\gamma}{2}}(t)\|_6^{\frac{3}{5}}.$$

Therefore, by integrating in time and using (3.3) we have that

$$\{\rho_\epsilon^{\frac{\gamma}{2}}\}_\epsilon \text{ is bounded in } L^{\frac{10}{3}}((0,T)\times\mathbb{T}^3),$$

which is equivalent to say that

$$\{\rho_\epsilon^{\gamma}\}_\epsilon \text{ is bounded in } L^{\frac{5}{3}}((0,T)\times\mathbb{T}^3).$$

Moreover, by (3.14) we have that also $\rho_\epsilon^\gamma \to \rho^\gamma$ a.e. in $(0,T)\times\mathbb{T}^3$ and by Fatou Lemma we have that

$$\rho^\gamma \in L^{\frac{5}{3}}((0,T)\times\mathbb{T}^3).$$

Then, if $M > 0$ we have

$$\begin{aligned}\int_0^T\int_{\mathbb{T}^3}|\rho_\epsilon^\gamma - \rho^\gamma|\,dsdx \le &\int_0^T\int_{\{\rho_\epsilon>M\}}\rho_\epsilon^\gamma\,dsdx\\&+\int_0^T\int_{\mathbb{T}^3}|\rho_\epsilon^\gamma\chi_{\{\rho_\epsilon\le M\}} - \rho^\gamma\chi_{\{\rho\le M\}}|\,dsdx\\&+\int_0^T\int_{\{\rho>M\}}\rho^\gamma\,dsdx\\&\le\int_0^T\int_{\mathbb{T}^3}|\rho_\epsilon^\gamma\chi_{\{\rho_\epsilon\le M\}} - \rho^\gamma\chi_{\{\rho\le M\}}|\,dsdx\\&+\frac{2C}{M^{\frac{2}{3}\gamma}}\end{aligned}$$

and we conclude as before.

□

3.2. **Proof of Theorem 3.1.** First, we notice that by (3.3), (3.6) and the compact embedding of $L^2(\mathbb{T}^3)$ in $H^{-1}(\mathbb{T}^3)$ for $\{g_\epsilon\}_\epsilon$, we have that there exists $V \in C([0,T); H^2(\mathbb{T}^3))$ such that

$$\nabla V_\epsilon \to \nabla V \text{ strongly in } C([0,T); L^2(\mathbb{T}^3)), \tag{3.17}$$

and the Poisson equation is satisfied pointwise.

Regarding the momentum equation, let $\psi \in C^\infty([0,T); C^\infty(\mathbb{T}^3))$ and consider the weak formulation of the momentum equations in Definition 2.1, multiplied by ϵ,

$$\begin{aligned} &\epsilon \int \rho_\epsilon^0 u_\epsilon^0 \psi(0)\, dx + \epsilon^2 \iint \sqrt{\rho_\epsilon}\sqrt{\rho_\epsilon}\frac{u_\epsilon}{\epsilon}\psi_t \, dsdx + \epsilon^2 \iint \sqrt{\rho_\epsilon}\frac{u_\epsilon}{\epsilon} \otimes \sqrt{\rho_\epsilon}\frac{u_\epsilon}{\epsilon} : \nabla\psi \, dsdx \\ &- \sqrt{\epsilon}\iint \sqrt{\rho_\epsilon}\frac{\mathcal{T}_\epsilon^s}{\sqrt{\epsilon}} : \nabla\psi \, dsdx + \iint \rho_\epsilon^\gamma \operatorname{div}\psi \, dsdx - \iint 2\sqrt{\rho_\epsilon}\,\nabla^2\sqrt{\rho_\epsilon} : \nabla\psi \, dsdx \\ &- \iint \rho_\epsilon \nabla V_\epsilon \psi \, dsdx + \iint 2\nabla\sqrt{\rho_\epsilon}\otimes\nabla\sqrt{\rho_\epsilon} \, : \nabla\psi \, dsdx + \iint \rho_\epsilon \frac{u_\epsilon}{\epsilon}\psi \, dsdx = 0. \end{aligned} \tag{3.18}$$

We study the convergence in the limit of $\epsilon \to 0$ of all the terms separately. By using (3.1) and Hölder inequality we conclude

$$\epsilon \left| \int \rho_\epsilon^0 u_\epsilon^0 \psi(0)\, dx \right| \le \epsilon\, \|\psi\|_{L^\infty_{t,x}} \|\sqrt{\rho_\epsilon^0}\|_{L^2} \|\sqrt{\rho_\epsilon^0} u_\epsilon^0\|_{L^2} \le \epsilon\, C \to 0 \text{ as } \epsilon \to 0.$$

Analogously, from (3.3) and Hölder inequality, we get for $\epsilon \to 0$:

$$\begin{aligned} \epsilon^2 \left| \iint \sqrt{\rho_\epsilon}\sqrt{\rho_\epsilon}\frac{u_\epsilon}{\epsilon}\psi_t \, dsdx \right| &\le \epsilon^2 \|\psi_t\|_{L^\infty_{t,x}} \|\sqrt{\rho_\epsilon}\|_{L^2_{t,x}} \left\| \sqrt{\rho_\epsilon}\,\frac{u_\epsilon}{\epsilon} \right\|_{L^2_{t,x}} \le \epsilon^2 C \to 0, \\ \epsilon^2 \left| \iint \sqrt{\rho_\epsilon}\,\frac{u_\epsilon}{\epsilon} \otimes \sqrt{\rho_\epsilon}\,\frac{u_\epsilon}{\epsilon} : \nabla\psi \, dsdx \right| &\le \epsilon^2 \|\nabla\psi\|_{L^\infty_{t,x}} \left\| \sqrt{\rho_\epsilon}\,\frac{u_\epsilon}{\epsilon} \right\|^2_{L^2_{t,x}} \le \epsilon^2 C \to 0, \\ \sqrt{\epsilon} \left| \iint \sqrt{\rho_\epsilon}\frac{\mathcal{T}_\epsilon^s}{\sqrt{\epsilon}} : \nabla\psi \, dsdx \right| &\le \sqrt{\epsilon}\|\nabla\psi\|_{L^\infty_{t,x}} \|\sqrt{\rho_\epsilon}\|_{L^2_{t,x}} \left\| \frac{\mathcal{T}_\epsilon^s}{\sqrt{\epsilon}} \right\|_{L^2_{t,x}} \le \sqrt{\epsilon} C \to 0. \end{aligned}$$

Next, by using (3.7) and (3.9) it follows that for $\epsilon \to 0$

$$\begin{aligned} &\iint 2\sqrt{\rho_\epsilon}\nabla^2\sqrt{\rho_\epsilon} : \nabla\psi \, dsdx \to \iint 2\sqrt{\rho}\nabla^2\sqrt{\rho} : \nabla\psi \, dsdx, \\ &\iint 2\nabla\sqrt{\rho_\epsilon}\otimes\nabla\sqrt{\rho_\epsilon} : \nabla\psi \, dsdx \to \iint 2\nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho} : \nabla\psi \, dsdx. \end{aligned}$$

Moreover, the convergence of ρ_ϵ^γ in (3.11) implies the continuity of the pressure term

$$\iint \rho_\epsilon^\gamma \operatorname{div}\psi \, dsdx \to \iint \rho^\gamma \operatorname{div}\psi \, dsdx$$

as $\epsilon \to 0$. Next, we consider the potential term, by using the (3.6) and (3.17) one gets:

$$\iint \rho_\epsilon \nabla V_\epsilon \psi \, dsdx \to \iint \rho \nabla V \psi \, dsdx \text{ as } \epsilon \to 0.$$

For the damping term, we first note that, that from (3.3), we can infer that there exists Λ such that

$$\sqrt{\rho_\epsilon}\,\frac{u_\epsilon}{\epsilon} \rightharpoonup \Lambda \text{ weakly in } L^2((0,T)\times\mathbb{T}^3), \tag{3.19}$$

by using also (3.9) we get that

$$\iint \sqrt{\rho_\epsilon}\sqrt{\rho_\epsilon}\,\frac{u_\epsilon}{\epsilon}\psi \, dsdx \to \iint \sqrt{\rho}\Lambda\psi \, dsdx \;\text{ as } \epsilon \to 0$$

Therefore from (3.18) we conclude

$$\begin{aligned}\iint \sqrt{\rho}\Lambda\psi \, dsdx = &-2\iint \nabla\sqrt{\rho}\otimes\sqrt{\rho} : \nabla\psi \, dsdx + 2\iint \sqrt{\rho}\nabla^2\sqrt{\rho} : \nabla\psi \, dsdx \\ &- \iint \rho^\gamma \operatorname{div}\psi \, dsdx + \iint \rho\nabla V\psi \, dsdx,\end{aligned}$$

that is

$$\sqrt{\rho}\Lambda = \operatorname{div}(-2\nabla\sqrt{\rho}\otimes\sqrt{\rho} + 2\sqrt{\rho}\nabla^2\sqrt{\rho} - \rho^\gamma\mathbb{I}) + \rho\nabla V \quad \text{in } \mathcal{D}'((0,T)\times\mathbb{T}^3). \tag{3.20}$$

Finally, for the continuity equation we similarly have for $\epsilon \to 0$:

$$\int \rho_\epsilon^0\phi(0)\,dx + \iint \rho_\epsilon\phi_t + \sqrt{\rho_\epsilon}\sqrt{\rho_\epsilon}u_\epsilon\nabla\phi \, dsdx \to \int \rho^0\phi(0)\,dx + \iint \rho\phi_t + \sqrt{\rho}\Lambda\nabla\phi \, dsdx,$$

and therefore taking into account (3.20) we get that ρ satisfies

$$\iint \rho\phi_t + \sqrt{\rho}\Lambda\nabla\phi \, dsdx = 0, \tag{3.21}$$

or any $\phi \in C_c^\infty([0,T);C^\infty(\mathbb{T}^3))$. Next, we prove (2.27). Again, from (3.3) we have that there exists $\mathcal{S}$ such that

$$\mathcal{S}_\epsilon \rightharpoonup \mathcal{S} \text{ weakly in } L^2((0,T)\times\mathbb{T}^3), \tag{3.22}$$

Therefore, for any $\phi \in C^\infty(\mathbb{T}^3)$, we have that

$$\begin{aligned}&\iint \sqrt{\rho_\epsilon}\mathcal{S}_\epsilon\phi \, dsdx \to \iint \sqrt{\rho}\mathcal{S}\phi \, dsdx \\ &\iint \sqrt{\rho_\epsilon}\nabla^2\sqrt{\rho_\epsilon}\phi - \nabla\sqrt{\rho_\epsilon}\otimes\nabla\sqrt{\rho_\epsilon} \, dsdx \to \iint \sqrt{\rho}\nabla^2\sqrt{\rho}\phi - \nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho} \, dsdx\end{aligned}$$

where we have used (3.9), (3.7) and (3.22). Finally, by using (2.9) we get (2.27). Next, we prove the entropy inequalities. By lower semicontinuity we have that for a.e. $t \in (0,T)$

$$\begin{aligned}&\int_{\mathbb{T}^3}\left(\frac{\rho(t,x)^\gamma}{\gamma-1} + 2|\nabla\sqrt{\rho}(t,x)|^2 + \frac{1}{2}|\nabla V(t,x)|^2\right)dx + \int_0^t\int_{\mathbb{T}^3}|\Lambda(t,x)|^2\,dsdx \\ &\le \liminf_{\epsilon\to 0}\int_{\mathbb{T}^3}\left(\frac{1}{2}\rho_\epsilon^0|u_\epsilon^0|^2 + \frac{(\rho_\epsilon^0)^\gamma}{\gamma-1} + 2|\nabla\sqrt{\rho_\epsilon^0}|^2 + \frac{1}{2}|\nabla V_\epsilon^0|^2\right)dx,\end{aligned} \tag{3.23}$$

and then (2.28) and (2.30) follows by using (3.1) and (3.2), respectively, and (3.17).

Finally, regarding (2.29), we recall that we only assume (3.1). We first note that (3.1) implies that, up to a subsequence,

$$\rho_\epsilon^0 \to \rho^0 \text{ a.e. in } (0,T)\times\mathbb{T}^3$$

then by using (3.4) and the very same argument used in Lemma 3.3 to prove (3.10) we get that

$$\rho_\epsilon^0(\log\rho_\epsilon^0 + 1) + 1 \to \rho^0(\log\rho^0 + 1) + 1 \text{ strongly in } L^1(\mathbb{T}^3).$$

Moreover, we have that

$$\lim_{\epsilon\to 0}\int_0^t\int_{\mathbb{T}^3}\rho_\epsilon(\rho_\epsilon - g_\epsilon)\,dsdx = \lim_{\epsilon\to 0}\int_0^t\int_{\mathbb{T}^3}\rho_\epsilon^2\,dsdx - \lim_{\epsilon\to 0}\int_0^t\int_{\mathbb{T}^3}\rho_\epsilon g_\epsilon\,dsdx = \int_0^t\int_{\mathbb{T}^3}\rho(\rho - g)\,dsdx,$$

but this follows directly from (3.6) and the weak convergence of g_ϵ. Therefore considering (2.10) and arguing exactly as done to deduce (3.23) we get (2.29).

Appendix A

For completeness, we give the proof of (2.13).

Lemma A. There exists $C > 0$, depending only on the dimension, such that for any function $\rho \in H^2(\mathbb{T}^3)$, with $\rho > 0$ a.e. on $(0,T)\times\mathbb{T}^3$

$$\iint |\nabla\rho^{\frac{1}{4}}|^4 + \iint |\nabla^2\sqrt{\rho}|^2 \le C\iint \rho|\nabla^2\log\rho|^2.$$

Proof. By a density argument it is enough to prove the lemma for ρ being a smooth function strictly positive everywhere. We first notice that

$$\iint \rho \left|\nabla\left(\frac{\nabla\sqrt{\rho}}{\sqrt{\rho}}\right)\right|^2 = \iint \rho\left|-\frac{1}{2}\nabla\log\rho\otimes\nabla\log\rho+\frac{1}{2\rho}\nabla^2\rho\right|^2 = \frac{1}{4}\iint \rho|\nabla^2\log\rho|^2. \tag{3.24}$$

On the other hand we also have

$$\iint \rho \left|\nabla\left(\frac{\nabla\sqrt{\rho}}{\sqrt{\rho}}\right)\right|^2 = \iint \frac{1}{\rho}|\nabla\sqrt{\rho}|^4+|\nabla^2\sqrt{\rho}|^2-2\frac{1}{\sqrt{\rho}}\nabla^2\sqrt{\rho}:\nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho}. \tag{3.25}$$

We have:

$$\begin{aligned}\iint \frac{1}{\sqrt{\rho}}\partial_{x_i}\partial_{x_j}\sqrt{\rho}\partial_{x_i}\sqrt{\rho}\partial_{x_j}\sqrt{\rho} &= \iint \partial_{x_i}\left(\partial_{x_j}\sqrt{\rho}\frac{\partial_{x_i}\sqrt{\rho}}{\sqrt{\rho}}\partial_{x_j}\sqrt{\rho}\right)\\ -\iint \partial_{x_j}\sqrt{\rho}\partial_{x_i}\left(\frac{\partial_{x_i}\sqrt{\rho}}{\sqrt{\rho}}\right)\partial_{x_j}\sqrt{\rho} &- \iint \frac{1}{\sqrt{\rho}}\partial_{x_i}\partial_{x_j}\sqrt{\rho}\partial_{x_i}\sqrt{\rho}\partial_{x_j}\sqrt{\rho}.\end{aligned}$$

The first term is zero and thus we get

$$2\iint \frac{1}{\sqrt{\rho}}\nabla^2\sqrt{\rho}:\nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho} = -\iint |\nabla\sqrt{\rho}|^2\operatorname{div}\left(\frac{\nabla\sqrt{\rho}}{\sqrt{\rho}}\right).$$

Then, we use Young inequality

$$\begin{aligned}2\left|\iint \frac{1}{\sqrt{\rho}}\nabla^2\sqrt{\rho}:\nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho}\right| &\le \frac{1}{2}\iint \frac{|\nabla\sqrt{\rho}|^4}{\rho}+2\iint \rho\left|\operatorname{div}\left(\frac{\nabla\sqrt{\rho}}{\sqrt{\rho}}\right)\right|^2\\ &\le \frac{1}{2}\iint \frac{|\nabla\sqrt{\rho}|^4}{\rho}+2\iint \rho\left|\nabla\left(\frac{\nabla\sqrt{\rho}}{\sqrt{\rho}}\right)\right|^2,\end{aligned}$$

and finally we get, using (3.24) and (3.25):

$$\iint \frac{|\nabla\sqrt{\rho}|^4}{\rho}+|\nabla^2\sqrt{\rho}|^2 \le C\iint \rho|\nabla^2\log\rho|^2$$

that gives (2.13), being

$$\iint |\nabla\rho^{\frac{1}{4}}|^4 = \iint \frac{|\nabla\sqrt{\rho}|^4}{\rho}.$$

□

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