Compensated Integrability and Applications to Mathematical Physics

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Lesson#2 - Motivations from Functional Analysis

This lesson is dedicated to the analysis of the structure characterized by the row-wise divergence over symmetric tensors, in the light of the Calculus of Variations.
Our operator Div replaces the row-wise rotational operator Curl, whose kernel was made of Jacobian matrices. The natural question of weak lower- (or upper-) semi-continuity remains the same.

It leads us to the notion of Div-Quasiconvexity, in the spirit of Dacorogna [1] and of I. Fonseca & S. Müller [2].

1. *Weak continuity and weak lower semicontinuity for nonlinear functionals*, LNM **922**, Springer-Verlag, NY, 1982.
2. $\mathcal{A}$-Quasiconvexity, lower semicontinuity, and Young measures. *SIAM J. Math. Anal.* **30** (1999) pp 1355–1390.

Recall the Jensen Inequality :

Let $K \subset \mathbb{R}^N$ be a convex subset and $\phi : K \to \mathbb{R}$ be a convex function. If μ is a probability over a domain Ω and $u : \Omega \to K$ is μ-measurable, then

$$\phi\left(\int_\Omega u\, d\mu\right) \leq \int_\Omega \phi(u)\, d\mu. \tag{1}$$

Here are two examples, where the barred integral denotes the mean value over a domain $\Omega \subset \mathbb{R}^m$ of finite Lebesgue measure :

- With $K = \mathbb{R}$,
$$\left| ⨍_\Omega u(x)\,dx \right|^2 \le ⨍_\Omega |u(x)|^2\,dx.$$
- If S takes values in $\mathbf{Sym}_n^+$ (hence $N = \binom{n}{2}$), then
$$⨍_\Omega (\det S)^{\frac{1}{n}}\,dx \le \left(\det ⨍_\Omega S(x)\,dx \right)^{\frac{1}{n}}. \qquad (2)$$

  This is a consequence of the concavity of $S \mapsto (\det S)^{\frac{1}{n}}$, see L#1, Corollary 2 .

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Weak semi-continuity

Let $\Omega \subset \mathbb{R}^m$ be an open subset. Let $u_k : \Omega \to K$ be a sequence, bounded in L^∞ (we avoid L^p only for the sake of simplicity). Up to the extraction of a sub-sequence, we may assume that $u_k \overset{*}{\rightharpoonup} u$ in the weak-* topology of L^∞.

What can be said of $\phi(u_k)$?

Mind that another extraction allows us to assume $\phi(u_k) \overset{*}{\rightharpoonup} \ell$ for some $\ell \in L^\infty(\Omega)$. The question is therefore :

Is there a relation between ℓ and $\phi(u)$?

For an arbitrary continuous function $\phi : K \to \mathbb{R}$, the answer is No. Weak convergence does not commute with nonlinear operations.

For convex functions instead, one may use (use a sub-differential if needed)

$$\phi(u_k) \geq \phi(u) + d\phi(u) \cdot (u_k - u)$$

and pass to the weak limit.
One obtains

$$\ell \geq \phi(u). \tag{3}$$

This means that convex functions are weakly-$*$ lower semi-continuous.

The converse happens to be true :
let $a, b \in K$ and $\theta \in (0,1)$ be given. Let $\chi : \mathbb{R} \to \{0,1\}$ be the characteristic function of $(0,\theta)$ modulo 1. Define

$$u_k(x) = \chi(kx_1)a + (1 - \chi(kx_1))b,$$

so that $\phi(u_k)(x) = \chi(kx_1)\phi(a) + (1 - \chi(kx_1))\phi(b)$. We have

$$u_k \stackrel{*}{\rightharpoonup} \theta a + (1-\theta)b, \qquad \phi(u_k) \stackrel{*}{\rightharpoonup} \theta\phi(a) + (1-\theta)\phi(b).$$

If ϕ is weakly-$*$ lower semi-continuous, then (3) means

$$\phi(\theta a + (1-\theta)b) \leq \theta\phi(a) + (1-\theta)\phi(b).$$

This is the convexity of ϕ.

A fashionable topic in Functional Analysis is to investigate what happens to weak-$*$ semi-continuity when one has some extra information about u_k, in terms of derivatives.

The best known situation occurs when ∇u_k is a bounded sequence in some L^p space. Then Rellich–Kondrachov Theorem tells us that u_k is relatively compact in L^q whenever

$$\frac{1}{q} > \frac{1}{p} - \frac{1}{m}\,.$$

This implies that $\ell = \phi(u)$ (weak continuity) whenever $\phi(v) = O(|v|^q)$.

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The situation is more complex, and perhaps more interesting, when the information concerns Pu_k where P is some non-elliptic differential operator. Historically, this occured in the context of Calculus of Variations, where $K = \mathbf{M}_{r\times m}(\mathbb{R})$ and P is the row-wise Curl operator.

The fields u_k are Jacobian matrices ∇v_k, and we have $\mathrm{Curl}\, u_k \equiv 0$. Say that Ω is bounded and v_k is a minimizing sequence of some functional

$$I[v] = \int_\Omega F(x, \nabla v)\, dx,$$

under given boundary conditions. One may think of a Dirichlet BC,

$$v = g \qquad \text{over} \quad \partial\Omega.$$

If F satisfies the reasonable property that

$$\frac{1}{C}\,(|u| - 1)^p \le F(x, u) \le C(|u| + 1)^p, \tag{4}$$

then the sequence $\nabla v_k = u_k$ is bounded in L^p and we may assume that $v_k \rightharpoonup \bar{v}$ in $W^{1,p}$. In particular, $\bar{v}$ satisfies the same boundary condition at $\partial\Omega$.

Since the sequence $F(x, \nabla v_k)$ is also bounded in L^1, we can extract a subsequence so that

$$F(x, \nabla v_k) \rightharpoonup \ell$$

in the vague sense. Here ℓ is a finite measure over Ω. In particular, we have

$$\inf I[v] = \lim_{k\to\infty} I[v_k] = \int_\Omega d\ell.$$

Since we are looking for a minimizer of the functional [3], a natural question is whether $\bar{v}$ is the winner. This will certainly be the case if we know that $\ell \geq F(x, \nabla \bar{v})$, because then we shall have

$$\inf I[v] \geq \int_\Omega F(x, \nabla \bar{v})\, dx = I[\bar{v}].$$

3. Mind that we don't know *a priori* whether such a minimizer exists. This is a part of the problem.

We are therefore led to the following question.

> *What are the continuous functions F, satisfying (4), with the property that whenever $v_k \rightharpoonup v$ in $W^{1,p}$ and $F(x, \nabla v_k) \rightharpoonup \ell$, one has*
> $$F(x, \nabla v) \leq \ell. \qquad (5)$$

This property is nothing but weak-$*$ lower semi-continuity over $W^{1,p}(\Omega : \mathbf{M}_{r\times m}(\mathbb{R}))$.

We already know that functions that are convex in their last argument u are weakly-$*$ lsc. But the fact that the argument is a Jacobian makes the theory much richer, and many other functions F have the same property, without being convex.

Example : Null-Lagrangians

These are minors of ∇v.

Say that $p = 2$, and consider the function

$$g(\nabla v) = \partial_i v_\alpha \partial_j v_\beta - \partial_j v_\alpha \partial_i v_\beta$$

for some indices $i \neq j$ and $\alpha \neq \beta$. This can be rewritten as

$$g(\nabla v) = \partial_i (v_\alpha \partial_j v_\beta) - \partial_j (v_\alpha \partial_i v_\beta).$$

When $v^k \rightharpoonup v$ in $W^{1,2}$, then $v^k \to v$ in L^2 strongly (Rellich–K.). Therefore

$$v_\alpha^k \partial_j v_\beta^k \rightharpoonup v_\alpha \partial_j v_\beta$$

in L^1.

Because derivatives are continuous over $\mathcal{D}'$, this tells us that

$$g(\nabla v^k) \rightharpoonup \partial_i (v_\alpha \partial_j v_\beta) - \partial_j (v_\alpha \partial_i v_\beta) = g(\nabla v),$$

in the sense of distributions.

If on the other hand $g(\nabla v^k) \rightharpoonup \ell$ in the vague sense of measure, then $\ell = g(\nabla v)$, because the vague convergence implies that in $\mathcal{D}'$.

In conclusion 2×2 minors of ∇v are weakly-$*$ continuous functions over $W^{1,2}$.

A bootstrap argument shows that $p \times p$ minors are weakly-$*$ continuous functions over $W^{1,p}$.

Polyconvex functions

Suppose now that the function $F(x,u)$ is given in the form

$$F(x,u) = \phi(x, \mathrm{Minors}(u))$$

where ϕ is a convex function over[4] $\mathbb{R}^N$. J. M. Ball[5] says that F is polyconvex.

If $v_k \rightharpoonup v$ in $W^{1,p}$ for $p \geq \min(r,m)$, then every minor has the property that

$$\mathrm{Min}(\nabla v_k) \rightharpoonup \mathrm{Min}(\nabla v).$$

Since the convexity of $\phi(x,\cdot)$ implies its weak-$*$ semi-continuity, we obtain that

$$*\lim F(x, \nabla v_k) \geq F(x, \nabla v),$$

that is, F is weakly-$*$ lsc over $W^{1,p}$.

4. This dimension N is rather large !

5. Convexity conditions and existence theorems in non linear elasticity. *Arch. Rat. Mech. Anal.*, **63** (1977), 337–403.

Remarks

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- Although polyconvexity implies w-$*$ lower semi-continuity, the converse is not true.
- Polyconvexity is difficult to characterize, because the range of the algebraic map

$$M \mapsto (\text{Minors}(M))$$

is far from being convex!

Quasi-convexity

C. B. Morrey [6] characterized those functions $F(x, \nabla v)$ that are w-$*$ lsc over $W^{1,p}$. Under a reasonable growth assumption, these are the functions such that for every $z \in \Omega$, the function $g(u) := F(z, u)$ is Quasi-convex.

The definition of quasi-convexity is

For every open $\omega \subset \mathbb{R}^m$, $A \in \mathbf{M}_{r\times m}(\mathbb{R})$ *and* $v \in \mathcal{D}(\omega; \mathbb{R}^r)$,

$$g(A) \leq \fint_\omega g(A + \nabla v)\, dx. \tag{6}$$

Equivalently

For every lattice Γ *of* $\mathbb{R}^m$, $A \in \mathbf{M}_{r\times m}(\mathbb{R})$ *and* Γ*-periodic field* v,

$$g(A) \leq \fint_{\mathbb{R}^n/\Gamma} g(A + \nabla v)\, dx. \tag{7}$$

6. *Multiple integrals in the calculus of variations.* Springer-Verlag, NY, 1966.

The fact that w-$*$ lsc implies quasi-convexity is not too difficult :

If v is periodic, let us define the sequence $v^k : \mathbb{R}^m \to \mathbb{R}^r$ by

$$v^k(x) = Ax + \frac{1}{k}\, v(kx).$$

Then $\nabla v^k(x) = A + (\nabla v)(kx)$ is a bounded sequence in L^∞, while $v^k(x) \to \bar{v}(x) := Ax$ uniformly. Thus $v^k \rightharpoonup \bar{v}$ n $W^{1,\infty}$ weak-$*$. On the other hand, $g(\nabla v^k) \rightharpoonup \bar{g}$, where $\bar{g}$ is the rhs of (7). Thus the lower semi-continuity

$$g(*\lim \nabla v^k) \le *\lim g(\nabla v^k)$$

writes as (7).

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Rank-one convexity

We say that a function $g : \mathbf{M}_{r\times m}(\mathbb{R}) \to \mathbb{R}$ is rank-one convex if its restriction to every segment $[A, B]$ such that $\mathrm{rk}(B - A) = 1$, is convex. In other words, if $s \mapsto g(A + sa \otimes b)$ is convex for every matrix A and vectors a, b.

Proposition 1

Quasi-convexity implies rank-one convexity.

Proof

Let $A, B = A + a \otimes b$ be given.

Let χ be as above the characteristic function of $(0, \theta)$ modulo 1, and f be its primitive. Define

$$w(x) = Ax + f(b \cdot x)a.$$

Let us complete b as a basis of $\mathbb{R}^m$, and denote Γ the corresponding lattice. The field $v(x) = w(x) - \theta Ax - (1-\theta)Bx$ is Γ-periodic.

We have $\nabla w = A + \chi(b \cdot x) a \otimes b$. Let us consider the sequence $v^k(x) = \frac{1}{k}\, v(kx)$, which tends weakly-$*$ to 0. Equivalently $w^k(x) = Ax + \frac{1}{k}\, f(kb \cdot x) a$ tends to $(\theta A + (1-\theta)B)x$. Then

$$g(\nabla w^k) = g(A + \chi(kb \cdot x) a \otimes b) \rightharpoonup \theta g(A) + (1-\theta) g(B).$$

The lsc $g(* \lim \nabla w^k) \leq * \lim g(\nabla w^k)$ thus gives

$$g(\theta A + (1-\theta)B) \leq \theta g(A) + (1-\theta) g(B).$$

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Remark. In elasticity, the integrand $F(x,\cdot)$ is defined only on $\mathbf{GL}_n^+$ (defined by $\det > 0$). Rank-one convexity makes sense because $\mathbf{GL}_n^+$ itself is a rank-one convex subset ; see L#0.

To summarize :

Theorem 1

For $g \in C(\mathbf{M}_{r\times m}(\mathbb{R});\mathbb{R})$, we have

$$(1) \Longrightarrow (2) \Longleftrightarrow (3) \Longrightarrow (4)$$

where

1. *g is polyconvex,*
2. *$v \mapsto g(\nabla v)$ is weakly-$*$ lower semi-continuous over $W^{1,\infty}$,*
3. *g is quasi-convex,*
4. *g is rank-one convex.*

The converse of the arrows $\Longrightarrow$ are false if $r, m \geq 2$.

The quadratic case

A special case of the Compensated Compactness theory (Tartar & Murat) gives the following.

Theorem 2

For quadratic forms $g : \mathbf{M}_{r\times m}(\mathbb{R}) \to \mathbb{R}$,

$$\textit{(4)} \Longleftrightarrow \textit{(2,3)} \Longleftrightarrow (g(a\otimes b)\geq 0, \quad \forall a, b).$$

The proof of (4) $\Longrightarrow$ (3) involves Fourier transform and the Plancherel formula.

When either r or m equals 2, all the four properties are equivalent to each other in the case of quadratic forms. This is false if $r, m \geq 3$.

Extension to general differential constraint

We wish to mimic, as close as possible, the theory of Calculus of Variations, when the information $\mathrm{Curl}\, u = 0$ (that is $u = \nabla v$) is replaced by another differential constraint

$$P(\nabla)u = 0. \tag{8}$$

Here P is a linear operator with constant coefficients acting over fields $u : \mathbb{R}^n \to \mathbb{R}^N$. It is of homogeneous order ; in practice, it will be of order 1. We are of course interested in the case where $\mathbb{R}^N \sim \mathrm{Sym}_n$ and $P(\nabla) = \mathrm{Div}$.

A rather important extension of the theory is that we don't need that the control $P(\nabla)u$ vanish, but only that it is more regular than ∇u. In Compensated Compactness for instance, one assumes that u is given in a bounded set of $L^2(\Omega)$, though $P(\nabla)u$ belongs to a compact set of $H^{-1}(\Omega)$.

Let us write

$$P(\nabla) = \sum_{i=1}^{n} P^i \partial_i, \qquad P^i \in \mathbf{M}_{\ell \times N}(\mathbb{R}).$$

The symbol of the operator is defined as

$$P(\xi) := \sum_{i=1}^{n} \xi_i P^i \in \mathbf{M}_{\ell \times N}(\mathbb{R}), \qquad \forall \xi \in \mathbb{R}^n.$$

A technical, though important assumption [7], is that the rank of $P(\xi)$ does not depend upon $\xi \neq 0$. This is satisfied in every physical application, because of the invariance of the laws of Physics under a change of observer.

7. F. Murat. Compacité par Compensation : condition nécessaire et suffisante de continuité faible sous une hypothèse de rang constant. *Ann. Scuola Norm. Sup. Pisa, Cl. Sci.* **8** (1981), 68–102.

Consider a functional of the form

$$I[u] = \int_\Omega F(x, u(x))\, dx,$$

where the integrand satisfies

$$F(x, u) \le C(x)(1 + |u|^p), \qquad C \in L^1(\Omega).$$

A general question of Functional Analysis is whether I is lower semi-continuous along sequences u_k that satisfy

$$u_k \overset{*L^p}{\rightharpoonup} u, \qquad P(\nabla)u_k \overset{W^{-1,p}}{\rightarrow} 0, \tag{9}$$

where we warn that the first convergence holds in the weak topology, while the second one is in the strong topology. For instance, one might have $P(\nabla)u_k \equiv 0$.

Fonseca & Müller (*ibid.*) proved the following result, extending that of Morrey.

Theorem 3

Assume the constant rank condition for the operator $P(\nabla)$. Assume also the growth $F(x,u) \leq C(x)(1+|u|^p)$. Then (9) implies

$$I[u] \leq \liminf I[u_k]$$

if, and only if every $g = F(\bar{x}, \cdot)$ is P-quasiconvex, that is

$$g\left(\fint_{\mathbb{R}^n/\Gamma} U(x)\,dx \right) \leq \fint_{\mathbb{R}^n/\Gamma} g(U(x))\,dx \tag{10}$$

for every periodic field U satisfying $P(\nabla)U = 0$.

Of course, convex functions are P-quasiconvex, by Jensen.

Changing g into $-g$ yields the notion of P-quasiconcavity, which is equivalent to the upper semi-continuity of I.

The proof that lower semi-continuity implies (10) is essentially the same as in Morrey : just consider sequences

$$u_k(x) = V(x, kx)$$

where each $U := V(\bar{x}, \cdot)$ satisfies $P(\nabla)U = 0$.

As before, P-quasiconvexity implies an algebraic condition of directional convexity :

Proposition 2

Let Λ be the characteristic cone of P :

$$\Lambda = \left\{w \in \mathbb{R}^N \,|\, \exists \xi \neq 0,\ P(\xi)w = 0\right\} = \bigcup_{\xi \neq 0} \ker P(\xi).$$

If g is P-quasiconvex, then it is Λ-convex, that is $s \mapsto g(\bar{u} + sw)$ is convex for every $\bar{u} \in \mathbb{R}^N$ and $w \in \Lambda$.

Proof

Consider as before a field

$$U(x) = \bar{u} + \chi(k\xi \cdot x)w$$

where χ is periodic. It is periodic and satisfies $P(\nabla)U = 0$. We have

$$\fint U(x)\,dx = (1-\theta)(\bar{u}+w)+\theta\bar{u}, \qquad \fint g(U(c))\,dx = \theta g(\bar{u}+w)+(1-\theta)g(\bar{u})$$

where θ is the mean value of χ.

■

A necessary condition for the lower semi-continuity property (under the control by $P(\nabla)$) is therefore the Λ-convexity.

Compensated Compactness : when g is quadratic, the P-quasiconvexity is equivalent to the Λ-convexity, which reduces to $g(w) \geq 0$ for every $w \in \Lambda$.

We apply the previous ideas to symmetric tensors ($\mathbb{R}^N \sim \mathbf{Sym}_n$) that are controlled through their row-wise divergence : $P(\nabla) = \mathrm{Div}$.

The symbol $P(\xi)$ acts by $P(\xi)S = S\xi$. It has full range when $\xi \neq 0$, hence the constant rank condition is satisfied.

The characteristic cone is obviously

$$\Lambda = \{S \in \mathbf{Sym}_n \mid \det S = 0\}.$$

Recall that we are interested in functions of the determinant, especially in powers, because :

- on the one hand, they have a well-defined physical dimension,
- on the other hand $\det^{\frac{1}{n}}$ is concave over $\mathbf{Sym}_n^+$.

The latter property suggests however to restrict to tensors that are positive semi-definite. Recall that such tensors have entries in the space $\mathcal{M}(\Omega)$ of finite (or merely locally finite) measures. This will be our framework throughout the theory.

Defining $\det^{\frac{1}{n}}$

Let $T := \mathrm{Tr}\, S$, which is a non-negative measure. Because of

$$|s_{ij}| \leq \frac{1}{2}(s_{ii} + s_{jj}),$$

we have $s_{ij} = f_{ij}\, T$ where f_{ij} is bounded, T-measurable, and takes values in $\mathbf{Sym}_n^+$.

There is a natural way to define $\det^{\frac{1}{n}}$, using the fact that this is a positively homogeneous function of degree one :

$$(\det S)^{\frac{1}{n}} := (\det f)^{\frac{1}{n}}\, T.$$

The Jensen inequality applies with this definition (mind that $\det^{\frac{1}{n}}$ is concave) :

$$\fint_\Omega (\det S)^{\frac{1}{n}} \leq \left(\det \fint_\Omega S\right)^{\frac{1}{n}}.$$

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We are thus interested in the following questions

Q1. When is $\det^\alpha$ upper semi-continuous over positive semi-definite symmetric tensors, under the control of their row-wise divergence ?

Q2. When is $\det^\alpha$ Div-quasiconcave, that is

$$⨍ (\det S)^\alpha \, dx \leq \left(\det ⨍ S \, dx \right)^\alpha \tag{11}$$

for every smooth periodic $S : \mathbb{R}^n/\Gamma \to \mathbf{Sym}_n^+$ satisfying $\mathrm{Div}\, S = 0$?

Q3. When is $\det^\alpha$ concave in the singular directions (*i.e.* $\det = 0$) over $\mathbf{Sym}_n^+$?

At this stage, the relation between these three properties is unclear, apart from the fact that each implies the next one :

- We cannot involve Fonseca & Müller to say that the Div-quasiconcavity implies the upper semi-continuity, because (11) is valid only for positive tensors.
- The concavity in the singular directions does not immediately imply (11), because $\det^\alpha$ is not quadratic.

Remark finally that we are only interested in exponents $\alpha > \frac{1}{n}$. Lower exponents ($\alpha \le \frac{1}{n}$) satisfy all the properties because then $\det^\alpha$ is concave over $\mathbf{Sym}_n^+$; just compose $\det^{\frac{1}{n}}$ with $s \mapsto s^{n\alpha}$, which is increasing and concave.

Question **Q1** and **Q2** will be answered later on. In this chapter, we content ourselves with the following.

Proposition 3

Consider positive exponents α. The map

$$\begin{array}{rcl} \mathbf{Sym}_n^+ & \to & \mathbb{R}_+ \\ S & \mapsto & (\det S)^\alpha \end{array}$$

is concave in the directions of singular matrices if, and only if

$$\alpha \leq \frac{1}{n-1}\,.$$

Proof

Again, by composition with $s \mapsto s^\beta$ ($\beta \in (0,1)$) it suffices to prove that $S \mapsto (\det S)^\alpha$ has this concavity property for $\alpha = \frac{1}{n-1}$, and that it has not if $\alpha > \frac{1}{n-1}$.

To begin with, consider the matrices

$$A = \begin{pmatrix} 0 & 0 \\ 0 & I_{n-1} \end{pmatrix}, \qquad S = \begin{pmatrix} 1 & 0 \\ 0 & 0_{n-1} \end{pmatrix}.$$

We have $\det(S + tA) = t^{n-1}$, and $A \in \Lambda$. If $\det^\alpha$ is Λ-concave over $\mathbf{Sym}_n^+$, we thus have $(n-1)\alpha \leq 1$.

There remains the case $\alpha = \frac{1}{n-1}$. Suppose that $S, S + A \in \mathbf{Sym}_n^+$, with $A \in \Lambda$, that is $\det A = 0$. By density and continuity, we may assume that both $S, S + A \in \mathbf{SPD}_n$. Then

$$\begin{aligned} \det(S + tA) &= \det S \cdot \det(I_n + S^{-1}A) \\ &= \det S \cdot \det(I_n + S^{-1/2}AS^{-1/2}) =: c \det(I_n + tB) \end{aligned}$$

where $c > 0$, $B \in \Lambda$ and $I_n + B \in \mathbf{SPD}_n$. We have to prove that $t \mapsto (\det(I_n + tB))^{\frac{1}{n-1}}$ is concave over $[0, 1]$.

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For this, we use an orthogonal diagonalisation

$$B = U^T \mathrm{diag}(c_1, \ldots, c_{n-1}, 0)\, U = U^T \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix} U.$$

We have

$$(\det(I_n + tB))^{\frac{1}{n-1}} = (\det(I_{n-1} + tC))^{\frac{1}{n-1}},$$

which is know to be concave, because the exponent $\frac{1}{n-1}$ is the inverse of the size of these positive symmetric matrices (L#0, Corollary 2).

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Conclusion

Because of Proposition 3, the map

$$S \longmapsto (\det S)^{\frac{1}{n-1}}$$

is a good candidate for being Divergence-Quasiconcave.

If it turns out to be Div-quasiconcave, it will be a good candidate for being upper semi-continuous over $\mathbf{Sym}_n^+$, under a control by the row-wise divergence.

This suggests that $(\det S)^{\frac{1}{n}}$, *a priori* a finite measure, is actually a function in the Lebesgue space $L^{\frac{n}{n-1}}(\Omega)$. This gain of integrability will be called in the sequel

Compensated Integrability.

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