Compensated Integrability and Applications to Mathematical Physics

Denis SERRE

UMPA, UMR 5669 CNRS
École Normale Supérieure de Lyon
France

University of L'Aquila, spring 2020

iNδAM

This lesson presents the basic theorems of the theory. We shall use them in Lesson #5 to establish the results that are applicable to evolution problems such as gas dynamics.

Several facts deserve to be noticed. On the one hand, our results extend two famous inequalities :

- that of Gagliardo, well known for its role in the proof of the Gagliardo–Nirenberg–Sobolev embedding $W^{1,1}(\mathbb{R}^n) \subset L^{\frac{n}{n-1}}(\mathbb{R}^n)$; actually it provides a new proof of the latter,
- the isoperimetric inequality.

Our functional inequalities are sharp. In the case of Divergence-free tensors[1], the equality case is achieved when the tensor belongs to a nonlinear class which we call "special DPTs".

1. From now on, the acronym DPT means Divergence-free positive semi-definite symmetric tensor.

1st Hint : the diagonal case

The simplest example of a Divergence-free Positive symmetric Tensor ($=:$ DPT) is a diagonal tensor

$$D = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}, \qquad a_j \geq 0.$$

The condition $\mathrm{Div}\, D = 0$ writes $\partial_j a_j \equiv 0$ for every $j \in [1, n]$. In other words,

$$a_j = a_j(\hat{x}_j), \qquad \hat{x}_j := (\dots, x_{j-1}, x_{j+1}, \dots)$$

is a function of $n - 1$ variables only.

Say that D is $\mathbb{Z}^n$-periodic. Denoting $K_n = (0,1)^n$ the unit cube, we have for every function $h : \mathbf{Sym}_n^+ \to \mathbb{R}$

$$\fint_{\mathbb{R}^n/\mathbb{Z}^n} h(D)\, dx = \int_{K_n} h(D)\, dx.$$

iNδAM

The function we are interested in (see L#2) is $h = \det^{\frac{1}{n-1}}$. The integrand is therefore

$$\prod_{j=1}^{n} f_j(\hat{x}_j), \qquad f_j := a_j^{\frac{1}{n-1}}.$$

This situation is reminiscent to the following classical result [2].

Theorem 1 (Gagliardo)

Let $f_1, \dots, f_n : \mathbb{R}^{n-1} \to \mathbb{R}$ be given. Define $f : \mathbb{R}^n \to \mathbb{R}$ by

$$f(x) = f_1(\hat{x}_1) \cdots f_n(\hat{x}_n).$$

If $f_1, \dots, f_n \in L^{n-1}(\mathbb{R}^{n-1})$, then $f \in L^1(\mathbb{R}^n)$ and we have

$$\|f\|_{L^1(\mathbb{R}^n)} \leq \prod_{j=1}^{n} \|f_j\|_{L^{n-1}(\mathbb{R}^{n-1})}. \tag{1}$$

2. E. Gagliardo. Proprietà di alcune di funzioni in più variabili. *Richerche Mat.* 7 (1958), 102–137.

Despite some ressemblance, Gagliardo's inequality is very different from Hölder's.

- On the one hand the number n of factors does not agree with their exponent of integrability $n-1$.
- On the other hand f_j, viewed as a function over $\mathbb{R}^n$, does not belong to an L^p.
- Finally, the simple case $n=2$

$$\int_{\mathbb{R}^2} |f_1(x_2)f_2(x_1)|\,dx \leq \int_{\mathbb{R}} |f_1(x_2)|\,dx_2 \cdot \int_{\mathbb{R}} |f_2(x_1)|\,dx_1$$

is nothing but Fubini's Theorem (where we actually have an equality).

Theorem 1 remains valid when the f_j's are defined over the cube K_{n-1} (just extend them to $\mathbb{R}^{n-1}$ by 0).

Going back to the context of the diagonal DPT, the assumption that $f_j \in L^{n-1}$ means that $a_j \in L^1(K_{n-1})$, and the conclusion $f \in L^1$ tells us that $(\det D)^{\frac{1}{n-1}} \in L^1(\mathbb{R}^n)$.

We therefore have

Proposition 1

Let D be a $\mathbb{Z}^d$-periodic diagonal DPT. Assume that $D \in L^1(\mathbb{R}^n/\mathbb{Z}^n)$. Then $(\det D)^{\frac{1}{n-1}} \in L^1(\mathbb{R}^n/\mathbb{Z}^n)$, and we have

$$\fint_{\mathbb{R}^n/\mathbb{Z}^n} (\det D)^{\frac{1}{n-1}}\, dx \le \left(\det \fint_{\mathbb{R}^n/\mathbb{Z}^n} D\, dx\right)^{\frac{1}{n-1}}.$$

Therefore $\det^{\frac{1}{n-1}}$ is Div-quasiconcave, as far as diagonal tensors are concerned.

iNδAM

2nd Hint : “Special” DPTs

The case $n = 2$ is always simpler than higher-dimensional cases. Somehow it is trivial. This was clear in the diagonal case, because of Fubini.

Claim : it is true in full generality.

Suppose S is a 2×2 DPT over a simply connected domain Ω. From

$$\partial_1 a_{11} + \partial_2 a_{21} = 0, \qquad \partial_1 a_{12} + \partial_2 a_{22} = 0,$$

we derive the existence of potentials ϕ_j such that

$$a_{11} = \partial_2\phi_1, \quad a_{21} = -\partial_1\phi_1, \quad a_{12} = \partial_2\phi_2, \quad a_{22} = -\partial_1\phi_2.$$

Now the symmetry tells us $\partial_1\phi_1 + \partial_2\phi_2 = 0$, from which we infer

$$\phi_1 = \partial_2\theta, \qquad \phi_2 = -\partial_1\theta$$

for some potential θ.

iNδAM

We conclude that

$$S = \begin{pmatrix} \partial_2^2\theta & -\partial_1\partial_2\theta \\ -\partial_1\partial_2\theta & \partial_1^2\theta \end{pmatrix}. \tag{2}$$

We notice that the positivity of S amounts to the convexity of θ.

Warning : The fact that every 2×2 DPT can be parametrized by a single potential does not extend to higher dimension,

When $n \geq 3$, a DPT involves $\binom{n+1}{2}$ entries, while there are only n differential constraints, and

$$\binom{n+1}{2} - n > 1.$$

Something remains however.

To see this, let us observe that the formula (2) can be recast as $S = \widehat{\mathrm{D}^2\theta}$, the cofactor matrix of the Hessian of θ.

When $n \geq 3$, not all DPTs can be written that way, but we have

Theorem 2

Let $\Omega \subset \mathbb{R}^n$ be a convex open set, and $\theta \in W^{2,n-1}(\Omega)$ be a convex function. Then the tensor $S = \widehat{\mathrm{D}^2\theta}$ is symmetric, positive and divergence-free (a DPT).

The role of the $W^{2,n-1}$ assumption is to make S integrable.

Such DPT's are called special. In $n = 2$ space dimensions, every DPT is special, but this is not the case if $n \geq 3$. We emphasize that the collection of special DPTs is highly nonlinear.

iNδAM

Proof

$S = \widehat{\mathrm{D}^2\theta}$ inherits the symmetry of the Hessian.

Because θ is convex, $\mathrm{D}^2\theta$ is positive ; then the cofactor $\widehat{H}$ of an $H \in \mathbf{Sym}_n^+$ is positive. To see this, we may assume by density and continuity that H is positive definite, in which case $\widehat{H} = (\det H)H^{-1}$ is positive.

It remains to prove the divergence-freeness. For this, we start with the differential forms $\alpha_j := d(\partial_j\theta)$ of degree 1, that is

$$\alpha_j = \sum_{i=1}^{n} \partial_i\partial_j\theta \, dx_i.$$

Since α_j is exact, it is closed : $d\alpha_j = 0$.

For $k \in [1, n]$, let us define the $(n-1)$-form

$$\omega_k = \cdots \wedge \alpha_{k-1} \wedge \alpha_{k+1} \wedge \cdots$$

where only α_k is omitted.

Because of Leibniz formula (in which the factors are 1-forms)

$$d(\alpha \wedge \beta \wedge \cdots) = (d\alpha) \wedge \beta \wedge \cdots - \alpha \wedge (d\beta) \wedge \cdots + \cdots,$$

we find that $d\omega_k = 0$, that is ω_k is a closed form.

We recall that for an $(n-1)$-form, written coordinate-wise

$$\omega = q_1\, dx_2 \wedge \cdots \wedge dx_n - \cdots + (-1)^{n-1} q_n\, dx_1 \wedge \cdots \wedge dx_{n-1},$$

we have $d\omega = (\operatorname{div} \vec{q})\, dx_1 \wedge \cdots \wedge dx_n$ the closedness $d\omega = 0$ is equivalent to the identity $\operatorname{div} \vec{q} = 0$.

When writing $d\omega_k = 0$ for $k \in [1, n]$, we therefore receive a collection of n identities $\operatorname{div} \vec{q}^{\,k} = 0$. It turns out that the coordinates of the vector field $(-1)^{k-1} \vec{q}^{\,k}$ are the entries of the k-th row of $\widehat{\mathrm{D}^2\theta}$.

Therefore this cofactor matrix is row-wise divergence-free.

Remark 1

Consider a potential $\theta \in \mathcal{D}(\mathbb{R}^n)$. Of course θ is not convex, unless being $\equiv 0$. The same construction

$$S = \widehat{\mathrm{D}^2\theta}$$

provides a symmetric divergence-free tensor, which is smooth and compactly supported. Yet $S \not\equiv 0$. This shows that the positivity was an essential assumption in Proposition 3 of Lesson #1.

P3.L#1. *The only DPT over $\mathbb{R}^n$ with finite mass is $S \equiv 0_n$.*

We now turn towards our beloved function $\det^{\frac{1}{n-1}}$.
The formula (see Lesson #0) $\det \widehat{M} = (\det M)^{n-1}$ yields

$$(\det S)^{\frac{1}{n-1}} = \det \mathrm{D}^2\theta.$$

The right-hand side can be viewed as a null-Lagrangian (see Lesson #2), since the Hessian is a particular case of a Jacobian.

iNδAM

Periodic case

Suppose that S, a special DPT, is Γ-periodic according to some lattice. *Warning* : The potential θ is not periodic (it could not be convex !) Only $\mathrm{D}^2\theta$ is periodic.

The potential decomposes as a sum

$$\theta(x)=\underbrace{\frac{1}{2}\,x^T\Sigma x}_{\text{quadratic}}+\underbrace{\vec{v}\cdot x}_{\text{linear}}+\underbrace{\rho(x)}_{\text{periodic}}$$

with constant $\Sigma\in\mathbf{Sym}_n$ and $v\in\mathbb{R}^n$. Mind that the linear part is irrelevant since it does not contribute to the Hessian ; we shall always ignore it.

Integrating $\mathrm{D}^2\theta=\Sigma+\mathrm{D}^2\rho$, we see that

$$\Sigma=⨍_{\mathbb{R}^n/\Gamma}\mathrm{D}^2\theta(x)\,dx\in\mathbf{Sym}_n^+.$$

iNδAM

In the expression

$$S = \widehat{\Sigma + \mathrm{D}^2\rho},$$

every s_{ij} appears to be a $\det(A+B)$ where A and B are $(n-1)$-blocs of Σ and $\mathrm{D}^2\rho$, respectively. Using the expansion formula (2) of Lesson $\#0$, we obtain that s_{ij} is the sum of $\widehat{\Sigma}_{ij}$ and a linear combination of minors of $\mathrm{D}^2\rho$, the latter being null-Lagrangians.

Likewise $\det \mathrm{D}^2\theta$ is the sum of $\det\Sigma$ and a linear combination of minors of $\mathrm{D}^2\rho$.

Because a null-Lagrangian rewrites as the divergence of some vector field, we obtain that

$$s_{ij} = \widehat{\Sigma}_{ij} + \mathrm{div}(\cdots), \qquad (\det S)^{\frac{1}{n-1}} = \det\Sigma + \mathrm{div}(\cdots),$$

where the dots are expressions in ρ and $\nabla\rho$, hence periodic vector fields.

Applying Green's formula, we observe that

$$\int_{\mathbb{R}^n/\mathbb{Z}^n} \mathrm{div}(\text{periodic})\,dx = 0.$$

We infer

$$\fint_{\mathbb{R}^n/\Gamma} S(x)\,dx = \widehat{\Sigma}, \qquad \fint_{\mathbb{R}^n/\Gamma} (\det S(x))^{\frac{1}{n-1}}\,dx = \det \Sigma.$$

Using again the formula $\det \widehat{\Sigma} = (\det \Sigma)^{n-1}$, we conclude

Theorem 3

For a periodic special DPT, we have

$$\fint_{\mathbb{R}^n/\Gamma} (\det S(x))^{\frac{1}{n-1}}\,dx = \left(\det \fint_{\mathbb{R}^n/\Gamma} S(x)\,dx\right)^{\frac{1}{n-1}}. \qquad (3)$$

As far as special DPTs are concerned, the function $\det^{\frac{1}{n-1}}$ is not only Divergence-quasiconcave, but it is Div-quasiaffine ! We shall see below that the equality case in the Div-quasiconcavity correspond to special DPTs.

iNδAM

Recall that, because our tensors S are distributional and are positive semi-definite, their entries are locally finite measures.

We have seen how to define the measure $(\det S)^{\frac{1}{n}}$, which obeys the same rules of calculation as if S was integrable. For instance the results of Lesson #0 remain valid :

- the map $S \mapsto (\det S)^{\frac{1}{n}}$ is concave,
- if $A : \Omega \to \mathbf{Sym}_n^+$ is continuous, then

$$(\det A)^{\frac{1}{n}} (\det S)^{\frac{1}{n}} \leq \frac{1}{n} \mathsf{Tr}\,(AS).$$

In what follows, we denote $\mathcal{M}(\Omega)$ the space of finite measures over $\Omega \subset \mathbb{R}^n$.

If $\mu \in \mathcal{M}(\Omega)$, then $|\mu|$ is a positive measure, whose total mass is denoted $\|\mu\|_{\mathcal{M}}$. This defines the natural norm over $\mathcal{M}(\Omega)$, whose restriction to $L^1(\Omega)$ coincides with $\|\cdot\|_1$.

If μ is a vector-valued measure, its Euclidian norm $|\mu|$ is well-defined, and its total mass is still denoted $\|\mu\|_{\mathcal{M}}$.

Besides the notion of divergence-free tensors, we consider the following.

Definition 1

A symmetric tensor S over Ω is divergence-controlled if

- *it is positive semi-definite,*
- *its entries are finite measures,*
- $\mathrm{Div}\, S$ *is a (vector-valued) finite measure.*

This means in particular that the distribution $\mathrm{Div}\, S$ is of order 0, instead of the expected order -1.

iNδAM

Periodic case

Our first result solves Question 2 of Lesson #2 : it states that $\det^{\frac{1}{n-1}}$ is Div-quasiconcave.

Theorem 4

Let Γ be a lattice of $\mathbb{R}^n$ and $S \in \mathcal{M}(\mathbb{R}^n/\Gamma; \mathbf{Sym}_n^+)$ be divergence-free. Then

1. $$(\det S)^{\frac{1}{n}} \in L^{\frac{n}{n-1}}(\mathbb{R}^n/\Gamma),$$

2. $$\fint_{\mathbb{R}^n/\Gamma} (\det S)^{\frac{1}{n-1}}\,dx \le \left(\det \fint_{\mathbb{R}^n/\Gamma} S\right)^{\frac{1}{n-1}}.$$

Comments

- Point 1 is qualitative. It says that some finite measure is actually absolutely continuous, and its density has an integrability property.
- We call this property Compensated Integrability, because the higher integrability of $(\det S)^{\frac{1}{n}}$ is not shared by its constituants [3] such as $(s_{11}\cdots s_{nn})^{\frac{1}{n}}$.
- Point 2 is quantitative. We wrote the left-hand side in a sloppy way. It stands for

$$\fint_{\mathbb{R}^n/\Gamma}\left((\det S)^{\frac{1}{n}}\right)^{\frac{n}{n-1}}dx.$$

- A complement, which we shall not use, is that the equality case in the functional inequality occurs only for special DPTs, as far as smooth positive definite tensors are concerned. We dont't know whether this is true if smoothness or definiteness is dropped.

3. D. Serre. Compensated integrability. Applications to the Vlasov–Poisson equation and other models in mathematical physics. *J. Math. Pures & Appl.*, **127** (2019), 67–88.

- A more general result holds true when the tensor is only Div-controlled (instead of being a DPT). Then the right-hand side of the functional inequality incorporates a term $\|\mathrm{Div}\, S\|_{\mathcal{M}}$, in the spirit of Theorem 6 below. Since the periodic case is rarely used in the applications, we don't explore further.
- Following Radon–Nikodym Theorem, we can decompose $S = S_a + S_s$ where S_a is absolutely continuous and S_s is singular, that is supported by a Lebesgue-null set. By orthogonality

$$(\det S)^{1/n} = (\det S_a)^{1/n} + (\det S_s)^{1/n}.$$

Theorem 4 tells us that $(\det S_s)^{1/n} = 0$, Tr (S_s)-almost everywhere. This is a manifestation of a generalization [4] of Alberti's rank-one Theorem for BV vector fields.

4. G. De Philippis, Filip Rindler. On the structure of $\mathcal{A}$-free measures and applications. *Annals of Maths.* **184** (2016), 1017–1039

iNδAM

Integrable tensors over $\mathbb{R}^n$

Let us denote B_n the unit ball and S_{n-1} the unit sphere of $\mathbb{R}^n$, and $|B_n|$ or $|S_{n-1}|$ their volume/area.

Theorem 5

Let $S \in \mathcal{M}(\mathbb{R}^n; \mathbf{Sym}_n^+)$ be divergence-controlled. Then

1. $$(\det S)^{\frac{1}{n}} \in L^{\frac{n}{n-1}}(\mathbb{R}^n),$$

2. $$\int_{\mathbb{R}^n} (\det S)^{\frac{1}{n-1}}\,dx \le c_n \|\mathrm{Div}\, S\|_{\mathcal{M}}^{\frac{n}{n-1}}, \qquad c_n := \frac{1}{n|S_{n-1}|^{\frac{1}{n-1}}}.$$

iNδAM

Comments

.

- Once again, the theorem splits into a qualitative part (Compensated Integrability) and a quantitative part (Functional Inequality).
- Even if the total mass of S does not appear in the right-hand side of the latter, the assumption that the entries are finite measures over $\mathbb{R}^n$ is essential. The conclusions are obviously false when $S \equiv I_n$, despite the fact that $\operatorname{Div} S \equiv 0$ is a finite measure.
- The constant c_n is sharp, as we shall see in a moment.
- The functional inequality is consistent with Proposition 3 of Lesson #1.

iNδAM

First applications - I

Gagliardo–Nirenberg–Sobolev Embedding. Let $f:\mathbb{R}^n\to\mathbb{R}$ be of bounded variations : f and its derivatives $\partial_j f$ are bounded measures. Then $|f|$ is BV too. Let us form the tensor $S=|f|I_n$, which is divergence-controlled since $\mathrm{Div}\,S=\nabla|f|$. Compensated Integrability tells us that $f\in L^{\frac{n}{n-1}}(\mathbb{R}^n)$, while the Functional Inequality reads

$$\|f\|_{\frac{n}{n-1}}\le c_n^{\frac{n-1}{n}}\|\nabla f\|_{\mathcal{M}}=c_n^{\frac{n-1}{n}}\,TV(f),$$

a sharp estimate (see below).

First applications - II

Isoperimetric Inequality. Let Ω be a bounded Caccioppoli set of $\mathbb{R}^n$. Let us choose the tensor S so as $S(x) = I_n$ if $x \in \Omega$, and $S(x) = 0_n$ otherwise. This is a particular case of the situation above, where f is the characteristic function of Ω. According to De Giorgi's definition, $TV(f)$ is the perimeter of Ω. Since $\det S$ is the characteristic function of Ω, the Functional Inequality writes

$$\mathrm{Vol}(\Omega) \leq c_n \mathrm{Per}(\Omega)^{\frac{n}{n-1}}.$$

With $c_n = |B_n|/|S_{n-1}|^{\frac{n}{n-1}}$, this is nothing but the Isoperimetric Inequality, with the sharp constant,

$$\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(B_n)} \leq \left(\frac{\mathrm{Per}(\Omega)}{\mathrm{Per}(B_n)}\right)^{\frac{n}{n-1}}.$$

Tensors in bounded domains

Let Ω be a bounded domain. For the sake of simplicity, we assume that $\partial\Omega$ is a smooth hypersurface.

Let S be Div-controlled tensor over Ω.
Each row $\vec{q}^{\,i}$ belongs to

$$\mathcal{M}_{\rm div}(\Omega) = \{\vec{q} \in \mathcal{M}(\Omega; \mathbb{R}^n) \,|\, {\rm div}\, \vec{q} \in \mathcal{M}(\Omega)\}.$$

This space is similar to the space $H_{\rm div}(\Omega)$ used in the mathematical analysis of Navier-Stokes equation, but with $\mathcal{M}$ replacing L^2 in its definition.

Normal trace

Recall that the normal trace operator

$$\vec{q} \mapsto (\vec{q})|_{\partial\Omega} \cdot \vec{N}$$

admits a unique continuous extension

$$\gamma_{\vec{N}} : H_{\rm div}(\Omega) \to H^{-1/2}(\partial\Omega),$$

which obeys the identity

$$\int_\Omega (\vec{q} \cdot \nabla\phi + \phi \,{\rm div}\, \vec{q}) \, dx = \langle \gamma_{\vec{N}} \vec{q}, \gamma_0 \phi \rangle_{H^{-1/2}, H^{1/2}}$$

for every $\phi \in H^1(\Omega)$. Actually this identity is the source of the definition of the normal trace operator.

iNδAM

The same strategy can be used to define in a unique way the normal trace operator over $\mathcal{M}_{\rm div}(\Omega)$. It takes values in the dual of the space of traces of functions in $C^1_b(\Omega)$. At least, this is a space of distributions over $\partial\Omega$, which contains $({\rm Lip}(\partial\Omega))'$.

Extension by 0.
Let $\vec q\in\mathcal{M}_{\rm div}(\Omega)$ be given. Its extension by zero to the complement $\mathbb{R}^n\setminus\Omega$ makes it still a vector-valued finite measure $\widetilde q$ over $\mathbb{R}^n$. However there is no reason why $\widetilde q$ would be divergence-controlled. Thanks to the formula

$$\langle\vec q,\nabla\phi\rangle+\langle\,{\rm div}\,\vec q,\phi\rangle=\langle\gamma_{\vec N}\,\vec q,\phi|_{\partial\Omega}\rangle,$$

we have

$$\langle{\rm div}\,\widetilde q,\phi\rangle=-\langle\widetilde q,\nabla\phi\rangle=\,-\,\langle\vec q,\nabla\phi|_\Omega\rangle=\langle\,{\rm div}\,\vec q,\phi|_\Omega\rangle-\langle\gamma_{\vec N}\,\vec q,\phi|_{\partial\Omega}\rangle.$$

We deduce

Proposition 2

Let $\vec{q} \in \mathcal{M}_{\rm div}(\Omega)$ be given, and $\widetilde{q}$ be its extension by 0. Then $\widetilde{q}$ is divergence-controlled if, and only if the normal trace $\gamma_{\vec{N}}\vec{q}$ is a finite measure over $\partial\Omega$.

Application to tensors.
Since a Div-controlled tensor S is nothing but a symmetric collection of divergence-controlled vector fields, it admits a (now vector-valued) normal trace. If S is continuous over $\overline{\Omega}$, this trace is nothing but $S\vec{N}$; we shall keep this as a notation for the normal trace in the general case. Then $\widetilde{S}$ is Div-controlled iff $S\vec{N} \in \mathcal{M}(\partial\Omega)$.

Applying Theorem 5 to $\widetilde{S}$, we obtain our fundamental result for bounded domains.

Theorem 6

Let Ω be an open bounded domain with smooth boundary. Let S be a divergence-controlled tensor over Ω. Let us assume that its normal trace is a (vector-valued) finite measure over $\partial\Omega$. Then

1. $$(\det S)^{\frac{1}{n}} \in L^{\frac{n}{n-1}}(\Omega),$$

2. $$\int_\Omega (\det S)^{\frac{1}{n-1}}\,dx \leq c_n \left(\|\mathrm{Div}\, S\|_{\mathcal{M}(\Omega)} + \|S\vec{N}\|_{\mathcal{M}(\partial\Omega)} \right)^{\frac{n}{n-1}}.$$

We emphasize that the constant c_n is the same as in Theorem 5.

iNδAM

Weak upper semi-continuity

Div-quasiconcavity (Thm 4) suggests that $\det^{\frac{1}{n-1}}$ enjoys wusc. This turns out to be true :

Theorem 7 (L. De Rosa, R. Tione & D. S., 2019)

Let Ω be an open subset in $\mathbb{R}^n$, and $p > \frac{n}{n-1}$. Let $S_k : \Omega \to \mathrm{Sym}_n^+$ be a given sequence. Assume that S_k is bounded in $L^p(\Omega)$ and that $\mathrm{Div}\, S_k$ is a bounded sequence in $\mathcal{M}(\Omega)$. Up to extraction, we may assume that $S_k \rightharpoonup S$ in $L^p(\Omega)$, and $(\det S_k)^{\frac{1}{n-1}} \rightharpoonup \ell$ in $L^{\frac{p(n-1)}{n}}(\Omega)$. Then

$$\ell \leq (\det S)^{\frac{1}{n-1}}.$$

Notice that the statement is wrong with the limit case $p = \frac{n}{n-1}$.

iNδAM

Conclusion

Three hypotheses are at stake :

- symmetry,
- positivity,
- control of the row-wise divergence.

Altogether, they yield an enhanced integrability of a single quantity, namely that of the measure $(\det S)^{\frac{1}{n}}$.

This integrability has the same exponent $\frac{n}{n-1}$ as S would have if the operator Div was replaced by the gradient or by another elliptic operator of order one.

Because Div is not elliptic, and is actually very far from being so, $(\det S)^{\frac{1}{n}}$ is the only quantity to enjoy enhanced integrability.

iNδAM

Conclusion

The qualitative result is completed with a quantitative one, a Functional Inequality.
FI is established with sharp constants and is found to extend well-known inequalities in functional analysis and geometry.

One of its forms says that $\det^{\frac{1}{n-1}}$ is Div-quasiconcave. It is actually weakly-$*$ upper semi-continuous over divergence-controlled tensors.

The next Lesson is devoted to the proofs of the fundamental results.

iNδAM