Compensated Integrability and Applications to Mathematical Physics

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In this Lesson, we give the proofs of Theorems 4 and 5 of Lesson #3.

A striking phenomenon is that the structure of Divergence-controlled tensors is in duality with a non-linear elliptic PDE, namely the Monge–Ampère equation (Prop. 1 below).

More precisely, we shall use the solvability of the so-called second boundary-value problem. This BVP occurs in optimal transport theory, when the cost is quadratic.

We shall therefore recall some basic concepts about optimal transport theory.

Once we have Brenier's Theorem at our disposal, we can pass to the proofs of both versions of Compensated Integrability.

Because our Functional Inequalities are sharp, the proofs are necessarily sharp as well. In particular, it is possible to trace back what happens in the equality cases.

This[1] is the following 2nd-order PDE

$$\det \mathrm{D}^2 u = f, \tag{MAE}$$

where f is a data. Because the determinant is not monotonous over the whole space $\mathbf{Sym}_n$, the equation is not a priori elliptic.

But since $H \mapsto \det H$ is non-decreasing over $\mathbf{Sym}_n^+$, we may claim that (MAE) is elliptic when $\mathrm{D}^2 u \in \mathbf{Sym}_n^+$, that is when u is convex.

Note that the convexity is a constraint that we put on the solution. Notice also that it requires the following condition over the data :

$$f \geq 0 \qquad (f > 0 \text{ prefered}).$$

1. I heard once that Gaspard Monge came to L'Aquila when Bonaparte was in Italy, and that he boosted the university. However, I didn't find a reference.

Maximum principle

In short, it says

Let Ω be a bounded convex domain. Let $u, v \in C(\overline{\Omega}) \cap C^2(\Omega)$ be uniformly convex over every compact subset. If

$$\det \mathrm{D}^2 u \geq \det \mathrm{D}^2 v \quad \textit{in} \quad \Omega,$$
$$u \leq v \quad \textit{over} \quad \partial\Omega,$$

then $u \leq v$ in Ω.

Somehow, this reveals ellipticity.

The duality argument

It is sometimes suitable to replace the data f by its nth root ϕ in (MAE), so that the equation becomes homogeneous in u and ϕ :

$$\left(\det \mathrm{D}^2 u\right)^{\frac{1}{n}} = \phi. \tag{HMA}$$

We notice that the regularity of ϕ is essentially the same as that of f, except in the borderline situation where f vanishes at the boundary. Thus it is meaningfull to assume instead

$$\inf \phi > 0.$$

Let us now consider a Div-controlled tensor S over a domain Ω. We recall that the expression $(\det S)^{\frac{1}{n}}$ is a well-defined finite measure. Suppose that $\phi \in C^\infty(\overline{\Omega})$, and that u is a C^∞-solution of (HMA). Then we have

$$\phi(\det S)^{\frac{1}{n}} = \left(\det(S\mathrm{D}^2 u)\right)^{\frac{1}{n}}.$$

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Because both S and $\mathrm{D}^2 u$ are positive semi-definite, their product has a real, non-negative spectrum (Proposition 2 of Lesson #0)

> *If $A, B \in \mathbf{Sym}_n^+$, then $\sigma(AB) \subset \mathbb{R}_+$. If in addition A, or B is positive definite, then AB is diagonalisable.*

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and the right-hand side above is bounded by the normalized trace (Corollary 1)

> *If $A, B \in \mathbf{Sym}_n^+$, then*
>
> $$(\det(AB))^{\frac{1}{n}} \leq \frac{1}{n}\, \mathit{Tr}(AB).$$

$$\frac{1}{n}\, \mathrm{Tr}\, \left(S\mathrm{D}^2 u\right).$$

Let us now remark that [2]

$$\begin{aligned} \mathrm{Tr}\, \left(S\mathrm{D}^2 u\right) &= \sum_{i,j} s_{ij}\partial_i\partial_j u = \sum_{i,j} (\partial_i(s_{ij}\partial_j u) - (\partial_i s_{ij})\partial_j u) \\ &= \mathrm{div}\,(S\nabla u) - (\mathrm{Div}\, S)\cdot\nabla u. \end{aligned}$$

2. We emphasize that our divergence symbol has a capital letter D when apply to tensors, but is lowercase when applied to vector fields.

Combining these observations, we obtain

Proposition 1 (Duality Div-c. *vs* M.-A.)

Let S be a Div-controlled tensor. Let (u,ϕ) be a pair of smooth functions, with u convex, tied by the homogeneous Monge-Ampère equation (HMA). Then we have

$$\phi(\det S)^{\frac{1}{n}} \leq \frac{1}{n}\left(\operatorname{div}(S\nabla u) - (\operatorname{Div} S)\cdot\nabla u\right). \tag{1}$$

Warning ! At this stage, we have not yet discusses the boundary conditions. Therefore (1) cannot be exploited immediately to estimate $(\det S)^{\frac{1}{n}}$.

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Why is there a 2nd BVP ?

Because there is a 1st BVP, namely the Dirichlet BVP, with prescribed boundary data

$$u|_{\partial\Omega} = g.$$

The theory of the Dirichlet BVP for (MAE) is extremely complex and requires a full book to be carried out. We recommend A. Figalli's monograph *The Monge-Ampère equation and its Applications*, Zurich lectures in advanced mathematics ; European Math. Soc. (2017).

An obvious necessary condition for a general solvability (remeber that we look for a convex solution) : Ω must be strictly convex.

We shall not explore further the Dirichlet BVP, as we don't need it in the proofs of Compensated Integrability. We content ourselves with mentionning a result concerning smooth data :

Theorem 1

Suppose that Ω is uniformly convex, with a $C^{k+2,\alpha}$-boundary for some integer $k \geq 2$ and $\alpha \in (0,1)$. Let $f \in C^{k,\alpha}(\overline{\Omega})$ and $g \in C^{k+2,\alpha}(\partial\Omega)$ be given, with $f > 0$. Then there exists one and only one solution of the Dirichlet boundary-value problem for (MAE), which turns out to be of class $C^{k+2,\alpha}(\overline{\Omega})$.

This theorem was established in 1983-84 independently by N. M. Ivochkina, by N. V. Krylov, and by L. Caffarelli, L. Nirenberg & J. Spruck.

Optimal transport theory

The optimal transport problem was first posed by G. Monge [3]. In its original form, one has two metric spaces X and Y, equipped with positive finite Radon measures μ and ν of equal masses : $\|\mu\| = \|\nu\|$.

One defines the set $\mathrm{Trans}(\mu, \nu)$ of transport maps $T : X \to Y$ by the constraint

$$T_{\#}\mu = \nu$$

("ν is the push-forward of μ by T"). In other words,

$$\mu(T^{-1}(A)) = \nu(A)$$

for every Borel subset $A \subset Y$.

Then one minimizes a cost

$$\int_X c(x, T(x))\, d\mu(x),$$

over $\mathrm{Trans}(\mu, \nu)$.

3. *Mémoire sur la théorie des déblais et des remblais* (1781).

Obstruction : $\mathrm{Trans}(\mu, \nu)$ might be empty. Think to $\mu = \delta$ and $\nu = dx$.

Therefore L. Kantorovitch reformulated the problem. He symmetrized the roles of X and Y, and relaxed the constraint.

One considers instead the set $\Gamma(\mu, \nu)$ of measures γ over $X \times Y$, whose marginals over X and Y are μ and ν respectively :

$$\gamma(B \times Y) = \mu(B), \qquad \gamma(X \times A) = \nu(A).$$

Then one minimizes the cost

$$\int\int_{X \times Y} c(x, y)\, d\gamma(x, y)$$

over $\Gamma(\mu, \nu)$.

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The link between both formulations :

- If $T \in \mathrm{Trans}(\mu,\nu)$, then Monge's cost equals Kantorovitch's for the measure γ defined by

$$\int\int_{X\times Y} g(x,y)\,d\gamma(x,y) := \int_X g(x,T(x))\,d\mu(x).$$

- In particular, a solution of Monge's problem provides a solution of Kantorovitch's.
- Conversely, if a minimizer γ of the Kantorovitch problem can be associated that way with a function T, then T solves Monge's problem.

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Kantorovitch's approach yields a dual formulation :

The infimum of the cost equals the supremum of

$$\int_X \phi(x)\,d\mu(x) + \int_Y \psi(y)\,d\nu(y),$$

under the constraint that

$$\phi(x) + \psi(y) \le c(x,y), \qquad \forall (x,y) \in X \times Y.$$

Quadratic cost

We shall be concerned only with a rather special case of the Monge–Kantorovitch problem, when X, $Y \subset \mathbb{R}^n$, and the cost is quadratic

$$c(x,y) = |x-y|^2.$$

In addition, we assume that ν is the Lebesgue measure. The compatibility condition over μ is therefore

$$\int_X d\mu(x) = \mathrm{Vol}(Y). \tag{2}$$

In practice, we shall need only to consider measures μ with smooth, uniformly positive densities :

$$d\mu(x) = f(x)\,dx, \qquad f \in C^\infty(\overline{X}),\ \min f > 0. \tag{3}$$

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We then have the following fundamental result[4].

Theorem 2 (Y. Brenier)

Let X be a bounded open convex subset of $\mathbb{R}^n$. Let $f \in C^\infty(\overline{X})$ be given, such that $\min f > 0$, and let $B(R) \subset \mathbb{R}^n$ be a ball whose volume equals $\int_X f(x)\,dx$.
Then Monge's optimal transport problem from $(X, f(x)dx)$ to $(B(R), dx)$, with quadratic cost, admits a unique solution T.
This transport map turns out to be the gradient ∇u of some C^∞ convex function, which solves the Monge-Ampère equation (MAE), and satisfies

$$\nabla u(X) = B(R). \tag{4}$$

(Brenier's Theorem is far more general ; this restricted version is that needed for our purpose.)

4. See C. Villani. *Topics in optimal transportation*. Graduate Studies in Mathematics **58**, Amer. Math. Society (2003).

Because $\nabla u : X \to B(R)$ is one-to-one, (4) can be read equivalently

$$\nabla u(\partial X) = \partial B(R).$$

It thus plays the role of a boundary condition. The terminology speaks of the second boundary condition.

Since we may always assume that $B(R)$ is centered at the origin, the condition above reads

$$|\nabla u(x)| \equiv R, \qquad x \in \partial X,$$

where the radius R is determined by

$$R^n = \frac{1}{\mathrm{Vol}(B_n)} \int_X f(x)\,dx.$$

Proof of Theorem 5

Recall that Thm 5 of Lesson #3 concerns Div-controlled tensors S whose entries are bounded measures in $\mathbb{R}^n$.

We first consider the case where S is compactly supported.

Let X be an open ball, large enough that it contains $\mathrm{Supp}\, S$. For every $\phi \in C^\infty(\overline{X})$, Brenier's Theorem provides a solution $u \in C^\infty(\overline{X})$ of the homogeneous Monge-Ampère equation (HMA), such that $\nabla u(X) = B(R)$ with

$$R^n = \frac{1}{\mathrm{Vol}(B_n)} \int_X \phi(x)^n \, dx. \tag{5}$$

Recall the duality property (1) :

$$\phi(\det S)^{\frac{1}{n}} \leq \frac{1}{n} \left(\mathrm{div}(S\nabla u) - \mathrm{Div}\, S \cdot \nabla u\right).$$

Integrating, we obtain

$$\int_X \phi d(\det S)^{\frac{1}{n}} \leq -\frac{1}{n} \int_X \nabla u \cdot d(\mathrm{Div}\, S),$$

where we have used the fact that $S\nabla u$ is compactly supported.
Using Cauchy–Schwarz, we infer

$$\int_X \phi d(\det S)^{\frac{1}{n}} \leq \frac{R}{n} \int_X d|\mathrm{Div}\, S| = \frac{R}{n} \|\mathrm{Div}\, S\|_{\mathcal{M}}.$$

Because the value of R is given by (5), this amounts to writing

$$\int_X \phi d(\det S)^{\frac{1}{n}} \leq \frac{1}{n\mathrm{Vol}(B_n)^{\frac{1}{n}}} \|\mathrm{Div}\, S\|_{\mathcal{M}} \|\phi\|_n. \tag{6}$$

This estimate, valid for every smooth and positive ϕ, remains valid by density for positive $\phi \in L^n(X)$, and therefore for every $\phi \in L^n(X)$. This tells us on the one hand that the measure $(\det S)^{\frac{1}{n}}$ is absolutely continuous with respect to the Lebesgue measure (qualitative part).

It tells us on the other hand that its density belongs to $L^{\frac{n}{n-1}}(X)$, the dual space of $L^n(X)$, and satisfies

$$\|(\det S)^{\frac{1}{n}}\|_{\frac{n}{n-1}} \leq \frac{1}{n\mathrm{Vol}(B_n)^{\frac{1}{n}}} \|\mathrm{Div}\, S\|_{\mathcal{M}}.$$

This is exactly the functional inequality (quantitative part) of Theorem 5, because of $n\mathrm{Vol}(B_n) = |S_{n-1}|$.

There remains to treat the general case, where the entries are finite measures.

For this, we choose a cut-off $\rho \in \mathcal{D}(\mathbb{R}^n)$ such that $0 \leq \rho \leq 1$, $\rho \equiv 1$ over a ball $B(L)$ and $|\nabla \rho| \leq 1$ otherwise.

The tensor ρS is Div-controlled, compactly supported, with

$$\mathrm{Div}(\rho S) = \rho \mathrm{Div}\, S + S\nabla\rho.$$

Because the theorem has been proved in the compactly supported case, we have that $\rho(\det S)^{\frac{1}{n}} \in L^{\frac{n}{n-1}}(\mathbb{R}^n)$ and

$$\begin{aligned}
\int_{B(L)} (\det S)^{\frac{1}{n-1}}\, dx &= \int_{B(L)} \rho^{\frac{n}{n-1}} (\det S)^{\frac{1}{n-1}}\, dx \\
&\leq \int_{\mathbb{R}^n} \rho^{\frac{n}{n-1}} (\det S)^{\frac{1}{n-1}}\, dx \\
&\leq c_n (\|\rho \mathrm{Div}\, S\|_{\mathcal{M}} + \|S\nabla\rho\|_{\mathcal{M}})^{\frac{n}{n-1}} \\
&\leq c_n (\|\mathrm{Div}\, S\|_{\mathcal{M}} + \|S\nabla\rho\|_{\mathcal{M}(\mathbb{R}^n \setminus B(L))})^{\frac{n}{n-1}}.
\end{aligned}$$

Letting $L \to +\infty$, and using $\|S\|_{\mathcal{M}(\mathbb{R}^n \setminus B(L))} \to 0$, we obtain the desired conclusion.

Q.E.D.

One can prove Theorem 6 (bounded domain) the same way if Ω is uniformly convex :

Take $\phi \in C^\infty(\overline{\Omega})$ with $\min \phi > 0$, let $T = \nabla u$ be the optimal transport plan from $(\Omega, \phi^n \, dx)$ to $(B(R), dx)$. Integrating (1) over Ω and using Green's formula, we obtain

$$\int_\Omega \phi d(\det S)^{\frac{1}{n}} \leq \frac{1}{n} \left(\int_{\partial\Omega} \vec{N} \cdot d(S\nabla u) - \int_\Omega \nabla u \cdot d\mathrm{Div}\, S \right).$$

Because S is symmetric, the boundary integral is that of $\nabla u \cdot d(S\vec{N})$.

By Cauchy–Schwarz, and using $|\nabla u| \leq R$ everywhere, we infer

$$\int_\Omega \phi d(\det S)^{\frac{1}{n}} \leq \frac{R}{n} (\|S\vec{N}\|_{\mathcal{M}} + \|\mathrm{Div}\, S\|_{\mathcal{M}}).$$

We conclude as in the previous proof, by expressing R in terms of $\|\phi\|_n$ and by using duality. We obtain $(\det S)^{\frac{1}{n}} \in L^{\frac{n}{n-1}}(\Omega)$ and

$$\int_\Omega (\det S)^{\frac{1}{n-1}} \, dx \leq c_n (\|S\vec{N}\|_{\mathcal{M}} + \|\mathrm{Div}\, S\|_{\mathcal{M}})^{\frac{n}{n-1}}. \tag{7}$$

Equality case

The latter calculation can be used to characterize those tensors S for which (7) is an equality, at least when S is smooth and uniformly positive over $\overline{\Omega}$.

Let us choose the test function

$$\phi_0 := (\det S)^{\frac{1}{n(n-1)}}.$$

Remark that we have both

$$\phi_0(\det S)^{\frac{1}{n}} = (\det S)^{\frac{1}{n-1}} \quad \text{and} \quad \|\phi_0\|_n = \left(\int_\Omega (\det S)^{\frac{1}{n-1}}\,dx\right)^{\frac{1}{n}}.$$

Denote ∇u_0 the optimal transport plan and R_0 the radius of the target ball.

The equality in (7) gives us

$$\begin{aligned} R_0(\|S\vec{N}\|_{\mathcal{M}} + \|\mathrm{Div}\, S\|_{\mathcal{M}}) &= n\int_\Omega \phi_0(\det S)^{\frac{1}{n}} \\ &\le \int_{\partial\Omega} \nabla u_0 \cdot (S\vec{N})\, dx - \int_\Omega \nabla u_0 \cdot \mathrm{Div}\, S\, dx, \end{aligned}$$

Because $|\nabla u_0| = R_0$ on the boundary, but $|\nabla u_0| < R_0$ in the interior, we infer

$$S\vec{N} \parallel_+ \nabla u_0 \quad \text{over} \quad \partial\Omega, \tag{8}$$

$$\mathrm{Div}\, S \equiv 0 \quad \text{in} \quad \Omega. \tag{9}$$

This is not the end of the story, because every inequality used in the proof must be an equality. In particular, we have

$$\left(\det(S\mathrm{D}^2 u_0)\right)^{\frac{1}{n}} = \frac{1}{n}\,\mathsf{Tr}\,(S\mathrm{D}^2 u_0).$$

This is the equality case in the Arithmetic-Geometric Inequality, which means that the (real ≥ 0) spectrum of $S\mathrm{D}^2 u_0$ is made of equal eigenvalues $\lambda(x)$. Since this matrix is diagonalizable (Proposition 2 of Lesson #0), we deduce $S\mathrm{D}^2 u_0 = \lambda(x) I_n$ for some scalar function $\lambda > 0$. In other words, $S = \mu(x)\widehat{\mathrm{D}^2 u_0}$ for some $\mu > 0$.

Coming back to the definition of ϕ_0, we have

$$(\det \mathrm{D}^2 u_0)^{\frac{1}{n}} = \phi_0 = (\det S)^{\frac{1}{n(n-1)}} = \mu^{\frac{1}{n-1}} (\det \mathrm{D}^2 u_0)^{\frac{1}{n}},$$

whence $\mu \equiv 1$. Hence $S = \widehat{\mathrm{D}^2 u_0}$.

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Eventually, we come back to the condition (8), which writes $\vec{N} \parallel_+ S^{-1}\nabla u_0$, that is $\vec{N} \parallel_+ (\mathrm{D}^2 u_0)\nabla u_0$. This is equivalent to

$$\nabla \frac{1}{2} |\nabla u_0|^2 \parallel_+ \vec{N},$$

which turns out to be trivial since $|\nabla u_0| \equiv R_0$ is constant on the boundary and is less than R_0 in the interior. In conclusion we have

Proposition 2

Let Ω be a uniformly convex domain. Let S be a smooth, uniformly positive Div-controlled tensor. The Functional Inequality (7) is an equality if, and only if $S = \widehat{\mathrm{D}^2 u}$ is a special DPT for a convex potential such that $\nabla u(\Omega)$ is a ball.

Periodic DPTs

We turn towards the proof of Theorem 4. To this end, we need another type of "boundary-value" problem for the Monge-Ampère equation, the one treated by Yanyan Li[5].

Given a lattice Γ of $\mathbb{R}^n$, we have seen that a convex function cannot be periodic (unless being constant), but its Hessian can be. We therefore consider convex functions $u : \mathbb{R}^n \to \mathbb{R}$ of the form

$$u(x) = \frac{1}{2}\, x^T A x + \rho(x)$$

where ρ is periodic and $A \in \mathbf{SPD}_n$. Then (MAE) is recast as

$$\det(A + \mathrm{D}^2\rho) = f. \qquad (10)$$

5. Some existence results of fully nonlinear elliptic equations of Monge–Ampère type. *Comm. Pure & Appl. Math.*, **43** (1990), pp 233–271.

Remark that, when integrating (10) on a fundamental domain, and using the fact that $\det(A + \mathrm{D}^2\rho) - \det A$ is a sum of null-Lagrangians, we obtain the constraint

$$\det A = \fint_{\mathbb{R}^n/\Gamma} f(x)\,dx. \tag{11}$$

The result we use is

Theorem 3 (Yanyan Li)

Let Γ be a lattice of $\mathbb{R}^n$, and f be a smooth, Γ-periodic, positive function. Let $A \in \mathbf{SPD}_n$ satisfy the compatibility condition (11). Then there exists a unique (up to an additive constant) periodic smooth solution ρ of the Monge-Ampère equation (10), such that $\frac{1}{2}\,x^T Ax + \rho(x)$ is convex.

Somehow, the boundary condition has thus been replaced by the prescription that $\fint_{\mathbb{R}^n/\Gamma} \mathrm{D}^2 u\,dx$ equals A.

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Proof of Theorem 4

We now consider a Γ-periodic DPT S.

We choose an arbitrary matrix $A \in \mathbf{SPD}_n$ and a smooth, Γ-periodic test function $\phi > 0$ such that $f := \phi^n$ satisfies (11). We denote ρ the solution given by Li's Theorem. Equivalently $u = \frac{1}{2}x^T Ax + \rho(x)$ solves

$$(\det \mathrm{D}^2 u)^{\frac{1}{n}} = \phi.$$

Duality gives

$$\begin{aligned} \phi(\det S)^{\frac{1}{n}} \quad &\leq \quad \frac{1}{n}\,\mathrm{div}\,(S\nabla u) - \frac{1}{n}\,\mathrm{Div}\,S\cdot\nabla u \\ &= \frac{1}{n}\,\mathrm{div}\,(S(Ax+\nabla\rho)) = \frac{1}{n}\,(\mathsf{Tr}\,(SA) + \mathrm{div}\,(S\nabla\rho)). \end{aligned}$$

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Integrating on a fundamental domain, we get

$$\fint_{\mathbb{R}^n/\Gamma} \phi \, d(\det S)^{\frac{1}{n}} \leq \frac{1}{n} \fint_{\mathbb{R}^n/\Gamma} \mathsf{Tr}\,(SA) = \frac{1}{n}\,\mathsf{Tr}\,(A\bar{S}),$$

where

$$\bar{S} := \fint_{\mathbb{R}^n/\Gamma} S$$

denotes the mean value of the tensor.

We may assume $\bar{S} \in \mathbf{SPD}_n$, otherwise $\det S \equiv 0$ and the results (CI and FI) are obvious.

The next step is to minimize the right-hand side $\mathsf{Tr}\,(A\bar{S})$ with respect to A, keeping ϕ fixed. Recall that A is arbitrary in $\mathbf{SPD}_n$, apart for the constraint (11), here

$$\det A = \fint_{\mathbb{R}^n/\Gamma} \phi(x)^n \, dx.$$

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The minimum of $\mathrm{Tr}\,(A\bar{S})$ is achieved for $A = \lambda \bar{S}^{-1}$ where

$$\lambda = \|\phi\|_n (\det \bar{S})^{\frac{1}{n}}.$$

We obtain therefore

$$\int_{\mathbb{R}^n/\Gamma} \phi \, d(\det S)^{\frac{1}{n}} \leq \lambda = \|\phi\|_n (\det \bar{S})^{\frac{1}{n}},$$

from which we conclude as usual by duality.

Q.E.D.

The equality case works as well. If S is smooth and uniformly positive, we choose the same

$$\phi_0 = (\det S)^{\frac{1}{n(n-1)}}$$

and

$$A_0 = \|\phi_0\|_n (\det \bar{S})^{\frac{1}{n}} \bar{S}^{-1}.$$

The same arguments as in the case of a bounded domain yield the fact that $S \equiv \widehat{\mathrm{D}^2 u_0}$, that is, S is a special DPT.

And conversely, we have seen (Thm 3 of Lesson #3) that every special DPT satisfies the equality in the Functional Inequality

$$\fint_{\mathbb{R}^n/\Gamma} (\det S)^{\frac{1}{n-1}} \, dx \leq \left(\det \fint_{\mathbb{R}^n/\Gamma} S \right)^{\frac{1}{n-1}}.$$

Conclusion

The important points of this lesson are

- The structure of Div-controlled tensors is in duality with a boundary-value problem for the Monge-Ampère equation,
- The proofs are sharp ; one is able to characterize those tensors for which the Functional Inequality is an equality. These are special DPTs.

The fundamental results are not yet in a form that can be used in applications, but they contain everything we need. We shall develop convenient statements in the next Lesson.