Compensated Integrability and Applications to Mathematical Physics

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Starting from Theorems 5 and 6 of Lesson #3, we establish statements that can be applied directly to various models of Mathematical Physics. They are needed in the following situation :

- The domain is bounded, but we lack a control of the normal trace $S\vec{N}$.
- There is a time variable and one is interested in Cauchy problems in $(0,\tau)\times\mathbb{R}^d$. Here $n = 1 + d$ and the generic variable x equals (t, y), where y is the space variable.
- In continuum mechanics, we wish to estimate the velocity.

Each of these involves a different version of Compensated Integrability.

This lesson remains at the level of Mathematical Analysis ; the applications will come in the subsequent lessons.

Estimate without normal trace

Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with smooth boundary, and let S be a DPT over Ω. We know that S admits a normal trace $S\vec{N}$, in a rather bad subspace of $\mathcal{D}'(\partial\Omega)$ (L#3).

Here we don't assume that this trace be a (vector-valued) measure. Therefore Theorem 6 of L#3 does not apply. We overcome this obstacle, by truncating S.

Choose a smooth function $\phi \in \mathcal{D}_+(\Omega)$ and apply Theorem 6 to the Div-controlled tensor ϕS (thanks to $(\phi S)\vec{N} \equiv 0$). We use

$$\sum_j \partial_j(\phi s_{ij}) = \phi \sum_j \partial_j s_{ij} + \sum_j s_{ij}\partial_j \phi,$$

$$\mathrm{Div}\,(\phi S) = \phi \mathrm{Div}\, S + S\nabla\phi = S\nabla\phi.$$

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On the one hand, we learn that $(\det \phi S)^{\frac{1}{n}} \in L^{\frac{n}{n-1}}(\Omega)$. Since ϕ is arbitrary, this means that $(\det S)^{\frac{1}{n}} \in L^{\frac{n}{n-1}}_{\rm loc}(\Omega)$. On the other hand, we get the estimate

$$\int_\Omega \det(\phi S)^{\frac{1}{n-1}}\,dx \le c_n \|S\nabla\phi\|_{\mathcal{M}}^{\frac{n}{n-1}}.$$

If in addition ϕ is 1-Lipschitz, then the right-hand side is bounded by $c_n\|S\|_{\mathcal{M}}^{\frac{n}{n-1}}$.

Approaching uniformly the function $\mathrm{dist}(\cdot,\partial\Omega)$ by such functions ϕ, and passing to the limit, we obtain

Theorem 1

If S is a DPT over a bounded open domain Ω of $\mathbb{R}^n$, then $(\det S)^{\frac{1}{n}} \in L^{\frac{n}{n-1}}_{\rm loc}$ and

$$\int_\Omega {\rm dist}(x,\partial\Omega)^{\frac{n}{n-1}}(\det S)^{\frac{1}{n-1}}\,dx \le c_n\|S\|_{\mathcal M}^{\frac{n}{n-1}}. \tag{1}$$

The above result is more in the spirit of Theorem 3 (periodic DPTs), as the right-hand side involves S instead of the normal trace.

Notice that $\|S\|_{\mathcal M}$ actually stands for the total mass of the finite measure $|S|_{\rm op}$, where $|\cdot|_{\rm op}$ is the operator norm [1] over $\mathbf{M}_n(\mathbb{R})$.

1. For symmetric matrices, this norm coincides with the spectral radius.

Evolution problems

We shall apply Compensated Integrability to PDE systems for which the independent variables split into $x = (t, y)$ where $t \in (0, \tau)$ is a time variable.

The space variable y evolves in a physical domain $\omega \subset \mathbb{R}^d$. The overall domain is thus $\Omega = (0, \tau) \times \omega$ and we have $n = 1 + d$.

We shall often label the coordinates x_j from $j = 0$ to d. Thus

$$x_0 = t, \qquad x_j = y_j \text{ otherwise.}$$

Say that physical rules provide a positive semi-definite tensor (often a DPT) S. Its entries s_{ij} are labelled with $0 \le i, j \le d$.

We write blockwise

$$S = \begin{pmatrix} \rho & m^T \\ m & A \end{pmatrix}.$$

where $s_{00} = \rho \ge 0$ is a scalar measure, which can be interpreted as a mass density. The vector-valued measure m plays the role of a flux ; think to $\partial_t \rho + \mathrm{div}_y m = 0$.

Because of positivity, m is absolutely continuous with respect to ρ. We also have formally

$$A \ge \frac{m \otimes m}{\rho}\,.$$

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By Radon-Nikodym, there exists a density $v \in (L^1 \cap L^2)(\rho)$ such that $m = \rho v$ and $A = \rho v \otimes v + \Sigma$ where the tensor Σ is positive semi-definite. The vector field v plays the role of a *velocity field.*

We recall the formula

$$(\det S)^{\frac{1}{n}} = (\rho \det \Sigma)^{\frac{1}{n}}.$$

Let us consider a **Cauchy problem** : the domain ω is the whole space $\mathbb{R}^d$. The boundary of $\Omega_\tau = (0, \tau) \times \mathbb{R}^d$ consists of a bottom $(t = 0)$ and a top $(t = \tau)$ parts.

The unit normal is $\pm\vec{e}_t$, the first element of the canonical basis. The normal trace of S is $\pm S_{\bullet 0}$, the first column of the tensor,

$$S_{\bullet 0} = \begin{pmatrix} \rho \\ m \end{pmatrix}.$$

Therefore ρ and m have traces at $t = 0, \tau$.

Suppose that S is a Div-controlled tensor in Ω_τ, and that its normal traces on top and bottom are finite measures. In other words, the traces $\rho(0,\cdot)$, $m(0,\cdot)$ and the like at time τ are finite measures. Then the extension of S by 0_n to the complement of Ω_τ yields a Div-controlled tensor $\tilde{S}$ in $\mathbb{R}^{1+d}$.

Applying Theorem 5 (Lesson #3) to $\tilde{S}$, we obtain the following version of Theorem 6.

Theorem 2

Let S be a Div-controlled tensor in $(0,\tau)\times\mathbb{R}^d$. Assume that the traces of ρ and m are finite measures. Then $(\det S)^{1/n}\in L^{n/d}(\Omega_\tau)$ and we have

$$\int_0^\tau dt\int_{\mathbb{R}^d}(\det S)^{\frac{1}{d}}\,dy \le c_n\left(\|S_{\bullet 0}(0)\|_{\mathcal{M}}+\|S_{\bullet 0}(\tau)\|_{\mathcal{M}}+\|\mathrm{Div}\,S\|_{\mathcal{M}}\right)^{1+\frac{1}{d}}. \tag{2}$$

Notice that $\|S_{\bullet 0}(0)\|_{\mathcal{M}}$ is the total mass of the finite measure

$$\sqrt{\rho(0)^2+|m(0)|^2}\,. \tag{3}$$

Homogeneity

Recall that if the equations $\mathrm{Div}\, S = \cdots$ have a physical meaning, then $\det S$ is a meaningful quantity from a physical point of view, in that it has a well-defined physical dimension. We might therefore expect that our estimate (2) be homogeneous, physically speaking.

Amazingly, this is not at all the case, for if ρ has a dimension D (*density*), the first equation $\partial_t \rho + \mathrm{div}\, m = \cdots$ tells us that m has dimension DLT^{-1}. Therefore even the normal trace $S_{\bullet 0}$ is not homogeneous and the formula (3) does not make sense for a physicist.

This paradox can be overcome by applying a scaling argument.

For the sake of simplicity, we assume that S is divergence-free. This implies in particular that the total mass is conserved :

$$\int_{\mathbb{R}^d} \rho(t,y)\,dy \equiv \int_{\mathbb{R}^d} \rho(0,y)\,dy =: M_0.$$

If $\mu > 0$ is a parameter, we build from S another DPT S' by rescaling both the dependent and independent variables :

$$\tau' = \mu\tau, \qquad t' = \mu t, \quad y' = y, \quad \rho' = \mu^2\rho, \quad m' = \mu m, \quad A' = A.$$

Applying (2) to S' over the slab $\Omega_{\tau'}$, we have

$$\begin{aligned}\int_0^{\tau'} dt' \int_{\mathbb{R}^d} (\det S')^{\frac{1}{d}}\,dy' &\leq c_n \left(\|S'_{\bullet 0}(0)\|_{\mathcal{M}} + \|S'_{\bullet 0}(\tau')\|_{\mathcal{M}}\right)^{1+\frac{1}{d}} \\ &\leq c_n \left(2M'_0 + \|m'(0)\|_{\mathcal{M}} + \|m'(\tau')\|_{\mathcal{M}}\right)^{1+\frac{1}{d}}.\end{aligned}$$

Using $\det S' = \mu^2 \det S$, $dt' = \mu dt$, $dy' = dy$, and $M_0' = \mu^2 M_0$, we deduce

$$\int_0^\tau dt \int_{\mathbb{R}^d} (\det S)^{\frac{1}{d}} \, dy \leq c_n \mu^{-\frac{1}{d}} \left(2\mu M_0 + \|m(0)\|_{\mathcal{M}} + \|m(\tau)\|_{\mathcal{M}}\right)^{1+\frac{1}{d}} . \tag{4}$$

Inequality (4) speaks of our original tensor, and its right-hand side is parametrized by $\mu > 0$. We choose the parameter which minimizes the rhs

$$\mu = \frac{\|m(0)\|_{\mathcal{M}} + \|m(\tau)\|_{\mathcal{M}}}{2dM_0}$$

and obtain the following result, in which the Functional Inequality is now homogeneous from the physical point of view

$\longrightarrow$

Theorem 3

Let S be a DPT over $(0,\tau)\times\mathbb{R}^d$. If its normal traces at $t=0,\tau$ are finite measures, then we have

$$\int_0^\tau dt\int_{\mathbb{R}^d}(\det S)^{\frac{1}{d}}\,dy\le k_d M_0^{\frac{1}{d}}\left(\|m(0)\|_{\mathcal{M}}+\|m(\tau)\|_{\mathcal{M}}\right),\tag{5}$$

where

$$k_d:=\frac{1}{d}\left(\frac{2n}{|S_d|}\right)^{\frac{1}{d}},\,M_0\equiv\int_{\mathbb{R}^d}\rho(t,y)\,dy.$$

Both sides have the same physical dimension $M^{1+1/d}V$, where M is a mass and V a velocity.

Evolution problems in bounded domains

When the domain ω is bounded and we lack a control of the normal trace over the lateral boundary $(0,\tau)\times\partial\omega$, we must combine the arguments used above in the bounded case and for the Cauchy problem.

To begin with, we carry out the scaling procedure when the tensor S is Div-controlled (instead of divergence-free) over $\Omega_\tau=(0,\tau)\times\mathbb{R}^d$.

We start from the estimate (we use $|S_{\bullet 0}|\le\rho+|m|$)

$$\begin{aligned}\int_0^\tau dt\int_{\mathbb{R}^d}(\det S)^{\frac{1}{d}}\,dy \quad &\le \quad c_n\,(\|\rho(0)\|_{\mathcal{M}}+\|\rho(\tau)\|_{\mathcal{M}}+\|m(0)\|_{\mathcal{M}}+\|m(\tau)\|_{\mathcal{M}}\\ &\qquad +\|\partial_t\rho+\operatorname{div}m\|_{\mathcal{M}}+\|\partial_t m+\operatorname{Div}A\|_{\mathcal{M}})^{1+\frac{1}{d}}\,,\end{aligned}$$

where the masses are taken either on $\mathbb{R}^d$ (first line) or Ω_τ (second one).

We now apply the same scaling as before to create another Div-controlled tensor S' defined in $(0, \tau') \times \mathbb{R}^d$ with $\tau' = \mu\tau$. We apply the above estimate to S' and rewrite it in terms of S. This gives us a parametrized estimate

$$\int_0^\tau dt \int_{\mathbb{R}^d} (\det S)^{\frac{1}{d}} \, dy \leq c_n \mu^{-\frac{1}{d}} (a\mu + b)^{1+\frac{1}{d}}$$

with

$$\begin{aligned} a &:= \|\rho(0)\|_{\mathcal{M}} + \|\rho(\tau)\|_{\mathcal{M}} + \|\partial_t \rho + \operatorname{div} m\|_{\mathcal{M}}, \\ b &:= \|m(0)\|_{\mathcal{M}} + \|m(\tau)\|_{\mathcal{M}} + \|\partial_t m + \operatorname{Div} A\|_{\mathcal{M}}. \end{aligned}$$

Remarks :

- either a or b is homogeneous,
- the total mass of $\rho(t, \cdot)$ is not any more constant ; this is why we don't have a quantity such as M_0.

We next choose the value $\mu = b/da$ of the parameter that minimizes the right-hand side, to obtain the estimate

$$\int_0^\tau dt \int_{\mathbb{R}^d} (\det S)^{\frac{1}{d}}\, dy \leq \frac{1}{d}\, c_n n^{1+\frac{1}{d}} a^{\frac{1}{d}} b.$$

In other words :

Proposition 1

Let S be Div-controlled in $(0,\tau)\times\mathbb{R}^d$. We have

$$\begin{aligned}\int_0^\tau dt \int_{\mathbb{R}^d} (\det S)^{\frac{1}{d}}\, dy & \\ \leq \quad & K_d \left(\|\rho(0)\|_{\mathcal{M}} + \|\rho(\tau)\|_{\mathcal{M}} + \|\partial_t\rho + \operatorname{div} m\|_{\mathcal{M}}\right)^{\frac{1}{d}} \\ & \cdot \left(\|m(0)\|_{\mathcal{M}} + \|m(\tau)\|_{\mathcal{M}} + \|\partial_t m + \operatorname{Div} A\|_{\mathcal{M}}\right)\end{aligned}$$

where

$$K_d := \frac{1}{d}\, c_n n^{1+\frac{1}{d}}.$$

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When a DPT S is defined in the bounded domain $(0,\tau)\times\omega$ instead, we apply Proposition 1 to the extension by 0_d of the Div-controlled tensor $\phi(y)S$ where $\phi\in\mathcal{D}_+(\omega)$ approaches uniformly the function $\mathrm{dist}(\cdot,\partial\omega)$. In particular $|\nabla\phi|\le 1$.

We use the fact that

$$|\partial_t(\phi\rho)+\mathrm{div}(\phi m)| = |m\cdot\nabla\phi| \le |m|$$

and likewise

$$|\partial_t(\phi m)+\mathrm{div}(\phi A)| = |A\cdot\nabla\phi| \le |A|_{\mathrm{op}}.$$

Denoting $R = \sup \operatorname{dist}(\cdot, \partial\omega)$ the radius of ω, we infer

Theorem 4

Let $\omega \subset \mathbb{R}^d$ be a bounded domain. If S is a DPT over a domain $(0,\tau) \times \omega$, we have

$$\begin{aligned} & \int_0^\tau dt \int_\omega \operatorname{dist}(y, \partial\omega)^{1+\frac{1}{d}} (\det S)^{\frac{1}{d}} \, dy \\ \leq \quad & K_d \left(R(\|\rho(0)\|_{\mathcal{M}} + \|\rho(\tau)\|_{\mathcal{M}}) + \|m\|_{\mathcal{M}} \right)^{\frac{1}{d}} \\ & \cdot \left(R(\|m(0)\|_{\mathcal{M}} + \|m(\tau)\|_{\mathcal{M}}) + \|A\|_{\mathcal{M}} \right). \end{aligned}$$

Homework

Let S be Div-controlled over $\mathbb{R}^n$.

Ex. #1. Prove that

$$\int_{\mathbb{R}^n} (\det S)^{\frac{1}{n-1}}\,dx \le K_n \left(\prod_{j=1}^{n} \|(\mathrm{Div}\, S)_j\|_{\mathcal{M}} \right)^{\frac{1}{n-1}}.$$

Ex. #2. If $(\mathrm{Div}\, S)_1 \equiv 0$, prove that

$$S\vec{e}_1 \equiv 0.$$

Integrating in time first

Let S be a DPT in $(0,\tau)\times\mathbb{R}^d$, whose normal traces at $t=0,\tau$ are finite measures. Theorem 2 allows us to estimate the integral of

$$(\rho\det\Sigma)^{\frac{1}{d}},\qquad \Sigma := A-\rho v\otimes v.$$

The Cauchy stress tensor Σ is the part of A which is invariant under Galilean changes of variables ; it does not depend upon the velocity field v. This means that a direct application of (2), or even (5), will provide an estimate of $\rho^{1/d}\Sigma$, but will provide no information about v.

To resolve this flaw, we define the *marginal* of the tensor A :

$$\mathcal{A}(y) := \int_0^\tau A\,dt\ :\mathbb{R}^d\to \mathrm{Sym}_d^+,$$

whose entries are finite measures over $\mathbb{R}^d$.

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Thanks to the conservation law $\partial_t m + \mathrm{Div}\, A = 0$, we have

$$\mathrm{Div}\,\mathcal{A} = m(0,\cdot) - m(\tau,\cdot),$$

which shows that $\mathcal{A}$ is Div-controlled.

Applying Theorem 5 of L#3, we have on the one hand $(\det\mathcal{A})^{\frac{1}{d}} \in L^{\frac{d}{d-1}}(\mathbb{R}^d)$ and on the other hand

$$\|(\det\mathcal{A})^{\frac{1}{d}}\|_{\frac{d}{d-1}} \le c_d\left(\|m(0,\cdot)\|_{\mathcal{M}} + \|m(\tau,\cdot)\|_{\mathcal{M}}\right). \tag{6}$$

Using the concavity of $\det^{\frac{1}{d}}$ over $\mathbf{Sym}_d^+$, and its homogeneity of degree one, we see that

$$(\Delta(y) :=) \quad \int_0^\tau (\det A)^{\frac{1}{d}}\,dt \le (\det\mathcal{A})^{\frac{1}{d}}.$$

This, together with (6), imply

$$\|\Delta\|_{\frac{d}{d-1}} \le k_d\left(\|m(0,\cdot)\|_{\mathcal{M}} + \|m(\tau,\cdot)\|_{\mathcal{M}}\right). \tag{7}$$

Conclusion

The results of this lesson will allow us to get estimates of $\rho \det \Sigma$ and of $\det A$ (or $\det \mathcal{A}$) in evolution problems.

We shall apply them in Lesson 6 to various models of gas dynamics. We shall be able to estimate the pressure (as announced at the beginning of the course), and also the velocity field.

Because these estimates involve a time integral, they are reminiscent to Strichartz estimates that are ubiquitous in the theory of dispersive equations.