Compensated Integrability and Applications to Mathematical Physics

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1. Thermodynamical models

Now that useful forms of Compensated Integrability and the corresponding Functional Inequalities have been established (Theorem 1 to 5 of Lesson #5), we are in good position to develop applications.

The first domain is that of compressible gas dynamics, where we have at our disposal various models :

- Euler system,
- its variant with heat diffusion (Euler-Fourier),
- its relativistic variant.

The second part of the lesson (next lecture) will be devoted to kinetic models for dilute gases, especially to the Boltzmann equation.

The situation is significantly different for Vlasov-type models where particles don't collide, but interact through a self-induced field. We shall postpone their study to Lesson #9.

Euler system for a compressible gas

Internal variables are the mass density $\rho \geq 0$, the pressure $p \geq 0$, the temperature $\vartheta \geq 0$, the specific internal energy $e \geq 0$ and the entropy s. Of these five quantities, only two are functionally independent, as the three others can be determined through one equation of state.

For instance, one can start from a function $F(\rho, \vartheta)$ called Helmholtz free energy, and define

$$p = -\rho^2 \frac{\partial F}{\partial \rho} \qquad s = -\frac{\partial F}{\partial \vartheta}\,, \qquad e = \vartheta s + F.$$

The specific energy can be viewed as the partial Legendre transform of $-F$ with respect to the temperature.

The velocity field is denoted v. We ignore the diffusion processes ; in particular the gas is inviscid.

The gas obeys the Euler system which expresses the conservation of mass, momentum and energy :

$$\partial_t \rho + \mathrm{div}(\rho v) = 0, \qquad (1)$$

$$\partial_t(\rho v) + \mathrm{Div}(\rho v \otimes v) + \nabla p = 0, \qquad (2)$$

$$\partial_t \left(\frac{1}{2}\rho|v|^2 + \rho e \right) + \mathrm{Div}\left((\frac{1}{2}\rho|v|^2 + \rho e + p)v \right) = 0. \qquad (3)$$

The mass-momentum tensor

As noted in Lesson #1, the two first equations can be recast as $\mathrm{Div}_{t,y} S = 0$ where

$$S = \begin{pmatrix} \rho & \rho v^T \\ \rho v & \rho v \otimes v + p I_d \end{pmatrix}$$

is positive semi-definite.

Question : *Is this a DPT ?* $\Longleftrightarrow$ *Are* $\rho, \rho v, \rho|v|^2 + p$ *integrable ?*

The answer is positive from the conservation of total mass M_0 and the conservation/decay of total energy $E(t) \leq E_0$.

To summarize, we shall work in the following context

- Ms. The total mass M_0 is a finite constant.
- En. The total energy at time $t > 0$ is bounded above by E_0, where the energy E_0 at time $t = 0$ is finite.
- St. There exists a finite constant C such that $p \leq C\rho(1 + e)$.

Flows satisfying (**Ms**, **En**) will be called below admissible.

Structure assumption (**St**) is satisfied by most reasonable equations of state. For instance perfect gas obey to

$$p = \underbrace{(\gamma - 1)}_{\text{constant}} \rho e.$$

These assumptions ensure that S is a DPT over $(0, \tau) \times \omega$.

Observe that the control of energy

$$\int_\omega \left(\frac{1}{2}\,\rho|v|^2 + \rho e\right)\,dy \le E_0, \qquad \forall\, t \in (0,\tau)$$

serves also to estimate the right-hand side in the functional inequalities that involve $\|m(0,\cdot)\|_{\mathcal M} + \|m(\tau,\cdot)\|_{\mathcal M}$.

Because of $m = \rho v$, Cauchy–Schwarz yields

$$\begin{aligned}\|m(t,\cdot)\|_{\mathcal M} &\le \left(\int_\omega \rho(t,y)\,dy \cdot \int_\omega \rho(t,y)|v|^2\,dy\right)^{1/2} \\ &\le \sqrt{2M_0E_0}\,.\end{aligned}$$

Recall that $\det S = \rho p^d$. Applying Theorem 3 of Lesson #5, we obtain

Proposition 1

*Assume the structure condition (**St**). Then every admissible flow in $(0,\tau) \times \mathbb{R}^d$ with finite mass and energy satisfies*

$$\int_0^\tau dt \int_{\mathbb{R}^d} \rho^{\frac{1}{d}} p \, dy \leq k_d M_0^{\frac{1}{d}} \sqrt{8 M_0 E_0} \, . \tag{4}$$

We emphasize that the estimate (4) does not depend at all upon the length τ of the time interval. Therefore, if the flow is globally defined, the result above is **valid even with** $\tau = +\infty$.

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A refined estimate

Proposition 1 is not the end of the story : Internal variables such as ρ, p don't depend upon the choice of an inertial frame. Thus the left-hand side of (4) is invariant under a Galilean change of variable. But since the kinetic part of the energy involves the velocity, the right-hand side is not invariant. To keep track of the choice of an inertial frame $\mathcal{F}$, this kinetic part is denoted $E_{k0} = E_{k0}[\mathcal{F}]$.

One can therefore go further, by minimizing $E_{k0}[\mathcal{F}]$ with respect to the frame. Since every inertial frame moves with a constant velocity $\bar{v}$ with respect to the laboratory frame, we calculate

$$\min_{\bar{v}\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{2}\,\rho_0|v_0 - \bar{v}|^2\,dy.$$

This is achieved when $\bar{v}$ is the mean velocity,

$$\bar{v} = \frac{1}{M_0} \int_{\mathbb{R}^d} \rho_0 v_0 \, dy,$$

giving

$$M_0 E_{k0} = \frac{1}{4} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \rho_0(y)\rho_0(y')|v_0(y') - v_0(y)|^4 \, dy \, dy'.$$

This yields

Theorem 1 (Euler/Cauchy pb.)

Assume the structure condition (**St**)*. Then every admissible flow in* $(0, \tau) \times \mathbb{R}^d$ *with finite mass and energy satisfies*

$$\int_0^\tau dt \int_{\mathbb{R}^d} \rho^{\frac{1}{d}} p \, dy \leq k_d M_0^{\frac{1}{d}} \sqrt{8 D_0} \tag{5}$$

where

$$D_0 := \frac{1}{4} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \rho_0(y)\rho_0(y')|v_0(y') - v_0(y)|^4 \, dy \, dy' + M_0 \int_{\mathbb{R}^d} \rho_0 e_0 \, dy.$$

Discussion

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The novelty of (4) is that it involves only the total mass and energy. Yet, it is not contained in the estimates of mass and energy.

The latter tell us that both ρ and ρe (as well as $\rho|v|^2$) belong to $L^\infty(\mathbb{R}_+;L^1(\mathbb{R}^d))$. But (4) tells us that $\rho^{1/d}p\in L^1((0,+\infty)\times\mathbb{R}^d)$.

There are two main differences between these estimates :

- The latter is a space-time integral, instead of being the supremum in time of a space integral.
- The integrand $\rho^{1/d}p$ dominates strictly $p\sim\rho e$.

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For instance, if we are working with an isentropic gas, for which the equation of state is $p = \rho^\gamma$ (the same constant γ as above), then $\rho^{1/d} p = \rho^{\gamma+1/d}$ is a higher power of ρ than the internal energy $\rho e = \frac{1}{\gamma-1}\,\rho^\gamma$.

We have therefore a gain, in that we integrate a "stronger quantity", a higher power of the mass density. This gain has a cost, in that we replace the supremum in time by an integral. Somehow, (4) ressembles the Strichartz estimates that arise in the Cauchy problem for dispersive equations.

Interpolation

Let us consider the polytropic equation of state $p = \rho^\gamma$. The finite mass and energy tell us that $\rho \in L^\infty(\mathbb{R}_+; L^q(\mathbb{R}^d))$ for $q = 1$ and $q = \gamma$, respectively. By interpolation (Hölder inequality), this remains true for every $q \in [1, \gamma]$.

Turning towards (4), we see that

$$\rho \in L^{\gamma+\frac{1}{d}}(\mathbb{R}_+ \times \mathbb{R}^d) = L^{\gamma+\frac{1}{d}}(\mathbb{R}_+; L^{\gamma+\frac{1}{d}}(\mathbb{R}^d)).$$

Interpolating again, we infer that $\rho \in L^r(\mathbb{R}_+; L^s(\mathbb{R}^d))$ whenever the point $Q = (\frac{1}{r}, \frac{1}{s})$ belongs to the triangle spanned by the vertices

$$(0,1), \quad (0, \frac{1}{\gamma}), \quad (\frac{1}{\hat{r}}, \frac{1}{\hat{r}}), \qquad \text{with } \hat{r} := \gamma + \frac{1}{d}\,.$$

Let us illustrate this : $\longrightarrow$

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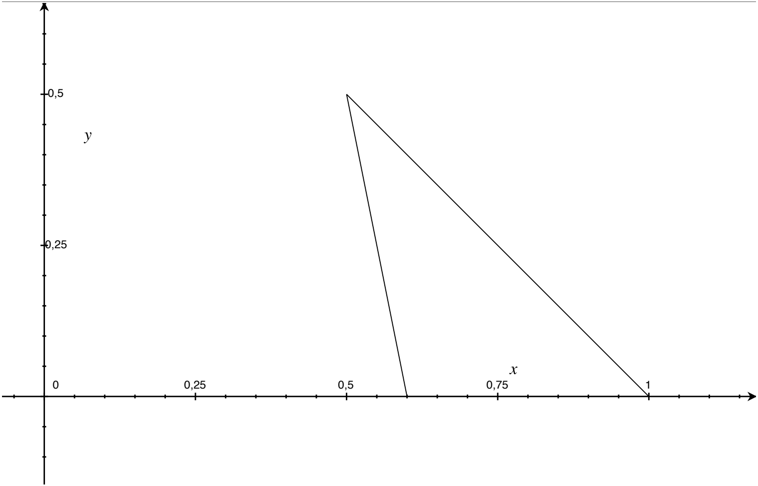

Coordinates are $\frac{1}{s}$ (horizontal) and $\frac{1}{r}$ (vertical). In this example, $d = 3$ and $\gamma = \frac{5}{3}$ (mono-atomic gas). Conservations of mass and energy correspond to the horizontal edge ; the new estimate extends them to the whole triangle.

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The drawback of (4) is that it does not give any information about the velocity. So far, it is controlled only through the energy estimate :

$$\sup_{t>0} \int_{\mathbb{R}^d} \rho |v|^2 \, dy \leq 2E_0.$$

To go forward, we consider the integrated tensor.

$$\mathcal{A}(y) := \int_0^\tau A(t,y)\, dt$$

where $A = \rho v \otimes v + p I_d$. We have $\mathcal{A}(y) = \pi(y) I_d + \mathcal{A}_{kin}(y)$ where

$$\pi = \int_0^\tau p(t,y)\, dt, \qquad \mathcal{A}_{kin} := \int_0^\tau \rho v \otimes v \, dt \qquad \text{(kinetic part).}$$

Then we apply Inequality (7) of Lesson #5,

$\longrightarrow \dots$

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$$\|(\det \mathcal{A})^{\frac{1}{d}}\|_{\frac{d}{d-1}} \leq c_d \left(\|m(0,\cdot)\|_{\mathcal{M}} + \|m(\tau,\cdot)\|_{\mathcal{M}}\right). \tag{6}$$

We have

$$\det \mathcal{A} = \sum_{k=0}^{d} \pi^{d-k} \sigma_k \left(\mathcal{A}_{kin}\right)$$

where

$$\sigma_0 = 1, \qquad \sigma_1 = \mathsf{Tr}\,, \ \ldots \quad \sigma_d = \det.$$

Every term in the sum if non-negative. For instance

$$\sigma_1 \left(\mathcal{A}_{kin}\right) = \int_0^\tau \rho |v|^2.$$

For the determinant, we use *Andreev's Formula* [1] :

$$\det\left[\int_I f_j(t)g_k(t)\,dt\right]_{j,k=1}^N$$
$$= \quad \frac{1}{N!}\int_I dt_1\cdots\int_I dt_N\,\det[f_j(t_\ell)]_{j,\ell=1}^N\det[g_j(t_\ell)]_{k,\ell=1}^N.$$

We apply this with $N=d$, $f=\rho v$ and $g=v$, to obtain

$$\det\mathcal{A}_{kin}=\frac{1}{d!}\int\cdots\int_{(0,\tau)^d}\rho(t_1)\cdots\rho(t_d)\,[\det(v(t_1),\dots,v(t_d))]^2\,dt_1\cdots dt_d,$$

where we omitted the argument y (the same for all factors).

1. C. Andréief (!). Note sur une relation entre les intégrales définies des produits de fonctions. *Mém. Soc. Sci. Phys. et Nat. Bordeaux*, Ê**2** (1886), 110–123.

In conclusion, we obtain a list of estimates. From that with σ_1 :

$$\int_{\mathbb{R}^d} \int_0^\tau p\, dt \cdot \left(\int_0^\tau \rho |v|^2 dt \right)^{\frac{1}{d-1}} dy \leq k_d \sqrt{8 M_0 E_0}\,. \tag{7}$$

to that with σ_d,

$$\| \, (\det \mathcal{A}_{kin})^{\frac{1}{d}} \, \|_{L^{\frac{d}{d-1}}} \leq c_d \sqrt{8 M_0 E_0}\,, \tag{8}$$

where $\det \mathcal{A}_{kin}$ is given by the formula above. Somehow, (8) estimates the torsion of the curves $t \mapsto v(t, y)$.

As usual the estimates are valid with $\tau = +\infty$.

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Compressible gas in a bounded domain

When the physical domain ω is a bounded open set in $\mathbb{R}^d$, we invoque Theorem 4 (L#5). The right-hand side of its Functional Inequality involves the total masses of $m = \rho v$ and A, over $(0, \tau) \times \omega$. As usual

$$\|m\|_{\mathcal{M}} = \int_{\Omega_\tau} |\rho v| \, dy \, dt \leq \int_0^\tau \sqrt{2M_0 E_0} \, dt = \tau \sqrt{2M_0 E_0} \,.$$

On the other hand, because $|A|_{\rm op} \leq \operatorname{Tr} A$ over $\mathbf{Sym}_d^+$,

$$\|A\|_{\mathcal{M}} \leq \int_{\Omega_\tau} (\rho |v|^2 + dp) \, dy \, dt$$

For a polytropic gas (adiabatic constant γ)

$$\rho |v|^2 + dp \leq \max(2, d(\gamma - 1)) \left(\frac{1}{2} \rho |v|^2 + \rho e \right)$$

gives $\|A\|_{\mathcal{M}} \leq \max(2, d(\gamma - 1)) \, \tau E_0$. Remark that $d(\gamma - 1) = 2$ for a mono-atomic gas.

We thus obtain the estimate

Proposition 2

An admissible flow of a polytropic gas (either isentropic or not) in an open bounded domain $\omega \subset \mathbb{R}^d$ satisfies

$$\int_0^\tau dt \int_\omega \mathrm{dist}(y,\partial\omega)^{1+\frac{1}{d}} \rho^{\frac{1}{d}} p \, dy \leq K M_0^{\frac{1}{2d}} \sqrt{E_0} \, \left(R\sqrt{M_0} + \tau\sqrt{E_0} \right)^{1+\frac{1}{d}} \tag{9}$$

where $R = \max \mathrm{dist}(y,\partial\omega)$ is the radius of ω, and $K = K(d,\gamma)$ is an absolute constant.

This estimate is the *only one where the bound does depend upon the length τ* of the time interval. Predictible ! The gas cannot disperse and the left-hand side has to grow at least linearly as τ increases.

It seems that (9) is not accurate : the right-hand side is superlinear in τ, while the left-hand side is expected to behave linearly, say if ρ, p admit limits ρ_∞, p_∞ as $\tau \to +\infty$.

We can improve it by splitting a given time interval $(0,\tau)$ into ℓ equal parts, and to apply (9) on each one :

$$\int_{(k-1)\frac{\tau}{\ell}}^{k\frac{\tau}{\ell}} dt \int_\omega \mathrm{dist}(y,\partial\omega)^{1+\frac{1}{d}} \rho^{\frac{1}{d}} p\, dy \le K M_0^{\frac{1}{2d}} \sqrt{E_0} \left(R\sqrt{M_0} + \frac{\tau}{\ell}\sqrt{E_0} \right)^{1+\frac{1}{d}} .$$

Summing from $k=1$ to $k=\ell$, we obtain

$$\int_0^\tau dt \int_\omega \mathrm{dist}(y,\partial\omega)^{1+\frac{1}{d}} \rho^{\frac{1}{d}} p\, dy \le \ell K M_0^{\frac{1}{2d}} \sqrt{E_0} \left(R\sqrt{M_0} + \frac{\tau}{\ell}\sqrt{E_0} \right)^{1+\frac{1}{d}} .$$

When τ is large, we can choose an integer which balances the terms in the parenthesis :

$$\ell \sim \frac{\tau\sqrt{E_0}}{R\sqrt{M_0}} .$$

We infer

$$\int_0^\tau dt \int_\omega \mathrm{dist}(y,\partial\omega)^{1+\frac{1}{d}} \rho^{\frac{1}{d}} p \, dy =_{\tau\to+\infty} O\left(\tau (RM_0)^{\frac{1}{d}} E_0\right).$$

This means that in average,

$$\int_\omega \mathrm{dist}(y,\partial\omega)^{1+\frac{1}{d}} \rho^{\frac{1}{d}} p \, dy$$

behaves as an

$$O\left((RM_0)^{\frac{1}{d}} E_0\right)$$

when $\tau \to +\infty$.

Heat-conducting gas

This is the situation where the third equation (3), that of energy balance, incorporates a diffusion $\operatorname{div}(\kappa\nabla\vartheta)$ in its right-hand side. This term expresses Fourier's law of heat diffusion ; we speak of the Euler–Fourier system.

Two important remarks :

- The conservation laws of mass and momentum remain the same, and thus we keep the same DPT as in the Euler system.
- The total energy is still conserved (or even decays) provided we have

$$\int_\omega \operatorname{div}(\kappa\nabla\vartheta)\,dy \equiv 0.$$

  This will happen in the following situations : either the domain is the whole $\mathbb{R}^d$ with suitable decay at infinity (Cauchy problem with finite mass and energy), or it is surrounded by an insulated boundary. The latter corresponds to a boundary condition $\vec{N}\cdot\nabla\vartheta = 0$.

In all these cases, the same analysis as above can be carried out and the estimates (4,7,8,9) remain valid.

Thus the Euler–Fourier system does not distinguish from the Euler system in our theory.

This contrasts with the Navier-Stokes system, where the momentum equation incorporates viscous effects,

$$\partial_t(\rho v) + \mathrm{Div}\,(\rho v \otimes v) + \nabla p = \mathrm{Div}\ T$$

where

$$T = \mu(\nabla v + \nabla v^T) + \lambda(\mathrm{div}\, v)I_d.$$

Here T contributes to the divergence-free symmetric tensor S, but the positivity is lost.

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Relativistic gas

Context : Special Relativity. The speed of light c is an upper bound for the fluid velocity. Focus on the Cauchy problem (there are no boundaries to moderate the velocity).

The conservation of mass is not any more independent from that of energy :

$$\begin{aligned} \partial_t \left(\frac{\rho c^2 + p}{c^2 - |v|^2} - \frac{p}{c^2} \right) + \mathrm{div}_y \left(\frac{\rho c^2 + p}{c^2 - |v|^2}\, v \right) &= 0, \\ \partial_t \left(\frac{\rho c^2 + p}{c^2 - |v|^2}\, v \right) + \mathrm{Div}_y \left(\frac{\rho c^2 + p}{c^2 - |v|^2}\, v \otimes v \right) + \nabla_y p &= 0. \end{aligned}$$

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The DPT is

$$S = \begin{pmatrix} \frac{\rho c^2+p}{c^2-|v|^2} - \frac{p}{c^2} & \frac{\rho c^2+p}{c^2-|v|^2}\, v \\ \frac{\rho c^2+p}{c^2-|v|^2}\, v & \frac{\rho c^2+p}{c^2-|v|^2}\, v \otimes v + pI_d \end{pmatrix},$$

whose determinant is still ρp^d.

Applying Theorem 3 (L#5),

$$\int_0^\tau dt \int_{\mathbb{R}^d} \rho^{\frac{1}{d}} p\, dy \le k_d E_0^{\frac{1}{d}} (\mu(0) + \mu(\tau)),$$

with

$$E_0 \equiv \int_{\mathbb{R}^d} \left(\frac{\rho c^2 + p}{c^2 - |v|^2} - \frac{p}{c^2} \right) dy, \qquad \mu(t) = \left\| \frac{\rho c^2 + p}{c^2 - |v|^2}\, v \right\|_{\mathcal{M}}.$$

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To estimate $\mu(t)$, we make the reasonable assumption that the pressure is a linear function of ρ. This is supported by the observation that the sound speed $\sqrt{\partial p/\partial \rho}$ must remain smaller than c, hence be bounded. Actually the Stefan–Boltzmann law for a gas at thermodynamical equilibrium yields the identity $p = a^2 \rho$ where $a^2 = c^2/3$.

Under such an assumption, we have

$$E_0 \equiv \int_{\mathbb{R}^d} \frac{\rho(c^4 + a^2|v|^2)}{c^2(c^2 - |v|^2)}\, dy, \qquad \mu(t) = \left\| \frac{\rho(c^2 + a^2)}{c^2 - |v|^2}\, v \right\|_{\mathcal{M}}.$$

We next use the inequality

$$2a^2 c|v| \le c^4 + a^2|v|^2,$$

valid whenever $|v| < c$, to give a bound of $\mu(t)$:

$$\mu(t) \le c\, \frac{a^2 + c^2}{2a^2}\, E_0.$$

We deduce

Theorem 2

Assume a linear equation of state $p = a^2\rho$ *with* $0 < a < c$. *Then an admissible flow of the Cauchy problem for the relativistic Euler system satisfies*

$$\int_0^\tau dt \int_{\mathbb{R}^d} \rho^{\frac{1}{d}} p \, dy \le k_d \, c \, \frac{a^2 + c^2}{a^2} \, E_0^{1+\frac{1}{d}}. \tag{10}$$

Remark : This estimate involves only one invariant quantity, namely the total energy E_0. This is a consequence of the equivalence between mass and energy, which implies that that there is no extra parameter M_0.

Conclusion

Compensated Integrability applies to, and gives estimates of a new kind for, gas dynamics, provided that the stress tensor is non-negative (lack of viscous forces).

Otherwise, it supports various contexts :

- Heat conduction,
- Relativistic velocities.