ALEA, Lat. Am. J. Probab. Math. Stat. 9, 231–267 (2012)

Which random walks are cyclic?

Davide Gabrielli and Carla Valente

Università dell'Aquila,
via Vetoio loc. Coppito,
67100 L'Aquila, Italy.
E-mail address: gabriell@univaq.it, carla.valente@gmail.com
URL: http://mat.univaq.it/~gabriell/

Abstract. A cyclic random walk is a random walk whose transition probabilities/rates can be written as a superposition of the empirical measures of a family of finite cycles. This identifies a convex set of models. We discuss the problem of characterization of cyclic random walks in some special cases showing that it is related to several remarkable and classical results. In particular we introduce the notion of balanced measure and show that a translation invariant random walk on $\mathbb{Z}^d$ is cyclic if and only if its transition probability is balanced. The characterization of the extremal elements is obtained using the Carathéodory's Theorem of convex analysis. We then show that a random walk on a finite set is cyclic if and only if at every vertex the outgoing flux of the transition graph is equal to the ingoing flux. The extremal elements are characterized by the Birkhoff-Von-Neumann Theorem. Finally we consider the discrete torus and discuss when the cyclic decomposition can be done using only homotopically trivial cycles or elementary cycles associated to edges and two dimensional faces. While in one dimension this is equivalent to require some geometric properties of a discrete vector field associated to the transition rates, in two dimension this is not the case. In particular we give a simple characterization of the polyhedron of the rates admitting a cyclic decomposition with elementary cycles. The proof is based on a discrete Hodge decomposition, elementary homological algebra and the Helly's Theorem of convex analysis. Finally we discuss a natural discretization procedure of smooth divergence free continuous vector fields and an application to random walks in random environments.

1. Introduction

A discrete/continuous time cyclic random walk is a random walk whose transition probabilities/rates can be written as a superposition of empirical measures of a

Received by the editors July 27, 2011; accepted April 5, 2012.
2000 *Mathematics Subject Classification.* 82B41, 05C81, 52C99.
Key words and phrases. Random walks, convex analysis, discrete geometry.
D.G. acknowledges the financial support of PRIN 2009 protocollo n.2009TA2595.02.

family of finite cycles. This definition identifies a convex set of models. The aim of this paper is to characterize this set in some special cases. This is important since once you known that a random walk is cyclic then several properties and results can be deduced. For example the following results have been obtained using some kind of cyclic decomposition: in Komorowski et al. (2012) and Komorowski and Olla (2003) a central limit theorem and in Deuschel and Kösters (2008) a quenched central limit theorem for random walks in random environments, in Sued (2005) some regularity properties of the diffusion coefficient for a mean zero exclusion process, in Varadhan (1995) a central limit theorem for a tagged particle of a mean zero exclusion process, in Mathieu (2006) some bounds on the transition probabilities. In different papers some slightly different cyclic decompositions are used. In any case all of them correspond to define, starting from the Markov model, a weighted oriented graph for which a cyclic decomposition holds. Our results can be interpreted as results concerning cyclic decompositions of weighted oriented graphs. In this sense they can be applied in any case independently from the special rule associating a oriented weighted graph to a Markov model. There is an huge amount of results concerning cyclic decompositions of graphs, see for example Bang-Jensen and Gutin (2009), Bollobás (1998) and Diestel (2010). The majority of them are of combinatorial type. Our results have instead a geometric flavour and are the following.

We consider translational invariant Markov chains on the lattice $\mathbb{Z}^d$. A model is identified by a probability measure on $\mathbb{Z}^d$ which determines the weight to be associated to the different jumps. We introduce the notion of balanced measure. An Hyperplane on $\mathbb{R}^d$ containing the origin determines two half spaces. We can then compute the average distance from the hyperplane for the measure restricted to the two half spaces. If the two averaged distances are equal (possibly also $+\infty$) and if this happens for any hyperplane then we say that the measure is balanced. Clearly any mean zero measure is balanced. A translational invariant Markov chain is cyclic if and only if the corresponding measure determining the distribution of the jumps is balanced. A characterization of the extremal elements is obtained using the classic Carathéodory's Theorem of convex analysis whose statement is recalled in section 3. The result for mean zero measures could be deduced by the results in Karr (1983) and Winkler (1988). Here we give an independent proof and extend the validity of a cyclic decomposition to the class of balanced measures. This result is not a special case of the Choquet-Bishop-de Leeuw theorem Phelps (2001) since compactness is missing. See von Weizsäcker and Winkler (1979/80) and von Weizsäcker and Winkler (1980) for a general discussion of problems of this type.

On a finite set a Markov model is determined by an oriented weighted graph. An oriented weighted graph is said to be balanced if for any vertex the ingoing weight is equal to the outgoing weight. We show that a Markov model on a finite set is cyclic if and only if the corresponding oriented weighted graph is balanced. In the case of discrete time Markov chains, a characterization of the extremal elements is obtained using the classic Birkhoff-Von Neumann Theorem whose statement is recalled in section 3. The proofs of these results are elementary but are useful in the discussion of the next issue. In Bertini et al. (2012) we will discuss and apply a generalization of this result valid for infinite graphs with a condition of vanishing flux towards infinity.

On a finite graph we can consider a restricted class of cycles taking into account some topological obstructions. More precisely we can consider homotopically trivial cycles or elementary cycles associated to one and two dimensional faces of some cellular decomposition. We discuss this problem in the cases of the one dimensional and the two dimensional discrete torii. While in one dimension the validity of restricted cyclic decomposition of this type is equivalent to require some geometric properties of a discrete vector field constructed starting from the weights, this is not the case in two dimensions. In particular we give a simple characterization of the models that admit a cyclic decomposition using elementary cycles. Similar computations can be done also in higher dimensions. In these cases it is necessary a more detailed discussion of the homological structure and a more involved geometric construction. We will discuss this generalization in a separated paper Gabrielli and Valente (2012) together with a discussion of the case of infinite grids.

We discuss also some applications. First we introduce a simple and natural discretization procedure for a continuous smooth divergence free vector field. The result is a discrete divergence free vector field on the lattice. Then, using the previous results, we give a condition to obtain a weighted oriented graph admitting an elementary cyclic decomposition. Finally we apply the above construction to deduce a quenched Central Limit Theorem for a random walk in random environments using the results in Deuschel and Kösters (2008).

The paper is organized as follows. In section 2 we fix notation and state our main results. The section is subdivided into four subsections according to the three different frameworks that we discuss plus a subsection for applications. In section 3 we recall some classic results and prove some original results that are useful in the proofs of our main results. In section 4 we collect the proofs of our main results in the case of a translational invariant Markov chain on the lattice $\mathbb{Z}^d$. In section 5 we collect the proofs of our main results in the case of a Markov model on a finite set. In section 6 we collect the proofs of our main results concerning cyclic decompositions using restricted classes of cycles. In section 7 we discuss the applications.

2. Notation and main results

In this section we state all the main results of the paper and fix notation.

2.1. *Cyclic random walks on* $\mathbb{Z}^d$. An element $x \in \mathbb{Z}^d$ has coordinates $(x_1, \dots, x_d)$. We call $e^{(i)}$ the versors of the canonical basis of $\mathbb{R}^d$. This means that we have $e^{(i)}_j := \delta_{i,j}$, where δ is the Kronecker delta. We denote by $x \cdot y := \sum_{i=1}^d x_i y_i$ the Euclidean scalar product among $x, y \in \mathbb{R}^d$. With $|x| := \sqrt{x \cdot x}$ we denote the Euclidean norm of x.

We denote by $\mathcal{M}^{\leq 1}$ the set of positive measures on $\mathbb{Z}^d$ with total mass less or equal to 1 endowed with the topology of weak convergence. Since in all the cases we will consider a tightness condition will be always satisfied, the weak convergence will be equivalent to the pointwise convergence. In $\mathcal{M}^{\leq 1}$ there is a natural partial order structure. Given $p, q \in \mathcal{M}^{\leq 1}$ then $p \preceq q$ when $p(x) \leq q(x)$ for any $x \in \mathbb{Z}^d$. The support of $p \in \mathcal{M}^{\leq 1}$ is defined as

$$\mathcal{S}(p) := \left\{ x \in \mathbb{Z}^d \, : \, p(x) > 0 \right\} .$$

By $\mathcal{M}^1 \subseteq \mathcal{M}^{\leq 1}$ we denote the subset of probability measures and by $\mathcal{M}_0 \subseteq \mathcal{M}^{\leq 1}$ the subset of mean zero measures. This is the set of measures p such that

$$\sum_{x\in\mathbb{Z}^d} p(x)|x_i| < +\infty\,, \qquad \forall i\,, \tag{2.1}$$

and moreover $\sum_{x\in\mathbb{Z}^d} p(x)x = 0$.

Given $A \subseteq \mathbb{R}^d$ we denote by $co(A)$ its convex hull and by $aff(A)$ its affine hull. These are defined as

$$co(A) := \left\{ x \in \mathbb{R}^d \,:\, x = \sum_{i=1}^n \alpha_i x^i;\, x^i \in A,\, \alpha_i \geq 0,\, \sum_{i=1}^n \alpha_i = 1,\, n \in \mathbb{N} \right\},$$

$$aff(A) := \left\{ x \in \mathbb{R}^d \,:\, x = \sum_{i=1}^n \alpha_i x^i;\, x^i \in A,\, \alpha_i \in \mathbb{R},\, \sum_{i=1}^n \alpha_i = 1,\, n \in \mathbb{N} \right\}.$$

Let $z^0, z^1, \dots, z^{n-1}$ be distinct elements of $\mathbb{Z}^d$ and consider also $z^n := z^0$. Define

$$C := (z^0, z^1, \dots, z^{n-1}, z^n)$$

as the finite cycle on $\mathbb{Z}^d$ that, starting from z^0, visits sequentially all the elements z^i in increasing order with respect to the index i. We say that such a cycle C has cardinality $|C| = n$. Given a finite cycle C we define its empirical measure $p^C \in \mathcal{M}^1$ as

$$p^C := \frac{1}{|C|} \sum_{i=1}^{|C|} \delta_{z^i - z^{i-1}}\,, \tag{2.2}$$

where δ_x is the delta measure concentrated at x. The cycles of cardinality 1 are of the type $C = (z^0, z^0)$ and the corresponding empirical measure is δ_0. Empirical measures of finite cycles are called purely cyclic measures. It is clear that p^C defined in (2.2) depends only on the values of the displacement vectors $w^i := z^i - z^{i-1}$ and not on their relative order. Note also that for any cycle C we have

$$\sum_{i=1}^{|C|} w^i = \sum_{i=1}^{|C|} (z^i - z^{i-1}) = 0\,.$$

As a consequence we have

$$\sum_{x\in\mathbb{Z}^d} p^C(x)x = \frac{1}{|C|} \sum_{i=1}^{|C|} w^i = 0\,,$$

so that every purely cyclic measure has mean zero.

Consider a time homogeneous, discrete time random walk $\{X_n\}_{n\in\mathbb{N}}$ on $\mathbb{Z}^d$ with translation invariant transition probabilities determined by $p \in \mathcal{M}^1$ by the following identification

$$\mathbb{P}(X_{n+1} = y | X_n = x) := p(y - x)\,. \tag{2.3}$$

A translation invariant discrete time random walk on $\mathbb{Z}^d$ is called purely cyclic if the measure p in (2.3) is purely cyclic. This means that there exists a finite cycle C such that $p = p^C$.

As far as p^C is concerned, we can naturally introduce an equivalence relation $\sim$ between cycles. Let C be a cycle and let $\{w^1, \dots, w^{|C|}\}$ be the corresponding set of displacement vectors. Note that in the set of displacement vectors of a cycle, a vector w may appear more than once. For this reason we will also denote the set of displacement vectors as

$$\{(w^1, n_1), \ldots, (w^k, n_k)\},$$

to indicate that the vector w^i appears $n_i \in \mathbb{N}$ times. Consider now another cycle C' and let $\{v^1, \ldots, v^{|C'|}\}$ be the corresponding set of displacement vectors. We say that $C \sim C'$ if

$$\{w^1, \ldots, w^{|C|}\} = \{v^1, \ldots, v^{|C'|}\}.$$

The equivalence class to which the cycle C belongs is denoted by $[C]$ and is determined by the set of the displacement vectors of any representant C. We can then naturally use the following identification

$$[C] \equiv \{(w^1, n_1), \ldots, (w^k, n_k)\}. \tag{2.4}$$

We denote by $\mathcal{C}$ the countable set of equivalence classes of cycles, endowed with the discrete topology. Note that if $\{(w^1, n_1), \ldots, (w^k, n_k)\} \in \mathcal{C}$ then

$$\sum_{i=1}^{k} n_i w^i = 0. \tag{2.5}$$

Conversely if (2.5) holds then there exist natural numbers n_i' such that $n_i' \leq n_i$ and moreover $\{(w^1, n_1'), \ldots, (w^k, n_k')\} \in \mathcal{C}$. We cannot use directly the numbers n_i since according to our definition a cycle is self-avoiding. Observe that if $C \sim C'$ then $p^C = p^{C'}$ and we can therefore use the notation $p^{[C]}$ to denote p^C where C is any representant of $[C]$. The converse is not necessarily true. Indeed let $l \in \mathbb{N}$, $l \geq 2$, and consider any cycle C satisfying (2.4) and any cycle C' with

$$[C'] = \{(w^1, ln_1), \ldots, (w^k, ln_k)\}.$$

Then clearly it holds $[C] \neq [C']$ but nevertheless $p^C = p^{C'}$.

Definition 2.1. A probability measure p on $\mathbb{Z}^d$ is called cyclic if there exists a probability measure ρ on $\mathcal{C}$ such that $p = p^\rho$ where p^ρ is defined as

$$p^\rho := \sum_{[C] \in \mathcal{C}} \rho([C]) p^{[C]}. \tag{2.6}$$

A translation invariant discrete time random walk on $\mathbb{Z}^d$ is called cyclic if the measure p in (2.3) is cyclic.

The meaning of (2.6) is that for any $x \in \mathbb{Z}^d$ we have

$$p^\rho(x) = \sum_{[C] \in \mathcal{C}} \rho([C]) p^{[C]}(x), \tag{2.7}$$

i.e. we require just pointwise convergence. By monotone convergence Theorem (2.7) implies

$$p^\rho(A) = \sum_{[C] \in \mathcal{C}} \rho([C]) p^{[C]}(A), \qquad \forall A \subseteq \mathbb{Z}^d.$$

Equivalently we have a probability measure on $\mathcal{C} \times \mathbb{Z}^d$ that gives weight $\rho([C])p^{[C]}(z)$ to the pair $([C], z)$ and p^ρ is its $\mathbb{Z}^d$ marginal. Definition (2.6) can also be interpreted as follows. For any sequence $\mathcal{C}_n \subseteq \mathcal{C}$ such that $|\mathcal{C}_n| < +\infty$, $\mathcal{C}_n \subseteq \mathcal{C}_{n+1}$ and $\mathcal{C} = \cup_n \mathcal{C}_n$ it holds

$$p^\rho = \lim_{n \to +\infty} \sum_{[C] \in \mathcal{C}_n} \rho([C]) p^{[C]}, \tag{2.8}$$

where the limit is in the weak sense. This follows immediately by pointwise convergence since a tightness condition is clearly verified.

If the measure p^ρ is integrable, i.e. if (2.1) holds with p^ρ instead of p, then by Fubini Theorem we get

$$\sum_{x\in\mathbb{Z}^d} p^\rho(x)x = \sum_{x\in\mathbb{Z}^d}\left(\sum_{[C]\in\mathcal{C}}\rho([C])p^{[C]}(x)\right)x = \sum_{[C]\in\mathcal{C}}\rho([C])\left(\sum_{x\in\mathbb{Z}^d} xp^{[C]}(x)\right) = 0\,.$$

This means that every measure p^ρ that is integrable has mean zero.

Given H an hyperplane in $\mathbb{R}^d$ we call H^+ and H^- the two closed half-spaces determined by H. Given $A, B \subseteq \mathbb{R}^d$ we define

$$d(A,B) := \inf_{\{x\in A\,,\, y\in B\}} |x-y|\,.$$

Definition 2.2. A measure $p \in \mathcal{M}^{\leq 1}$ is called balanced if for any hyperplane $H \ni 0$ we have

$$\sum_{x\in H^+} p(x)d(x,H) = \sum_{x\in H^-} p(x)d(x,H)\,. \tag{2.9}$$

We call $\mathcal{B} \subseteq \mathcal{M}^1$ the set of balanced probability measures.

The main result of this subsection is the following.

Theorem 2.3. *An element of $\mathcal{M}^1$ is cyclic if and only if it is balanced.*

We underline that in (2.9) both sides can also be infinite. Note also that $\mathcal{M}_0 \cap \mathcal{M}^1 \subseteq \mathcal{B}$. Indeed let n be a versor normal to H then we have $H^+ = \{x : x\cdot n \geq 0\}$ and $H^- = \{x : x \cdot n \leq 0\}$. Given $x \in H^+$ it holds $d(x,H) = x\cdot n$ and likewise given $x \in H^-$ it holds $d(x,H) = -x\cdot n$. If $p \in \mathcal{M}_0 \cap \mathcal{M}^1$ then we have

$$0 = \sum_{x\in\mathbb{Z}^d} p(x)\,(x\cdot n) = \sum_{x\in H^+\cap\mathbb{Z}^d} p(x)d(x,H) - \sum_{x\in H^-\cap\mathbb{Z}^d} p(x)d(x,H)\,.$$

Indeed the same computation shows that when p is integrable then it is balanced if and only if it belongs to $\mathcal{M}_0 \cap \mathcal{M}^1$. This corresponds to verify condition (2.9) just for the d hyper-planes orthogonal to the vectors $e^{(i)}$ of the canonical basis. The validity of (2.9) for the others hyperplanes will follow. When p is not integrable the balancing condition (2.9) has to be checked for any hyperplane H, since the validity of (2.9) with respect to the hyper-planes orthogonal to the vectors of the canonical basis does not imply the validity of (2.9) for any H. Consider for example the measure p defined by $p(2i,-i) = p(-i,2i) = \frac{3}{\pi^2 i^2}$, $i \geq 1$ and giving weight zero to all the remaining elements of $\mathbb{Z}^2$ (see figure 2.1). This measure satisfies (2.9) with respect to the 2 hyper-planes orthogonal to the vectors $e^{(1)}, e^{(2)}$ but it does not satisfy the balancing condition for example with respect to the hyperplane H in figure 2.1. This measure is not cyclic. Indeed it is not possible to generate a cycle using vectors in $\mathcal{S}(p)$ since $0 \notin co(\mathcal{S}(p))$.

A simple example of a balanced not integrable probability measure on $\mathbb{Z}^2$ is $p(x) = \frac{c}{|x|^{\frac{5}{2}}}$ where c is a suitable normalization constant. Another example is given

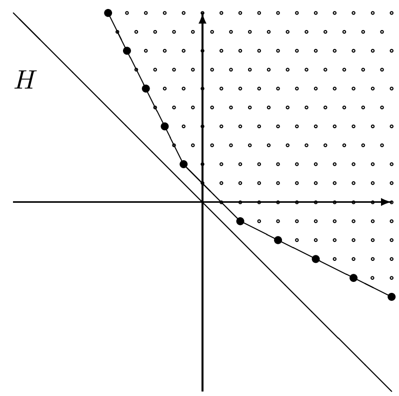

FIGURE 2.1. An example in $\mathbb{Z}^2$ of a measure p that satisfies condition (2.9) with respect to the 2 hyper-planes orthogonal to $e^{(1)}$ and $e^{(2)}$ but it is not balanced and consequently also not cyclic. With $\bullet$ we denote elements of $\mathcal{S}(p)$, the dashed region represents $co(\mathcal{S}(p))$. Condition (2.9) is not satisfied for the hyperplane H.

by a probability measure on $\mathbb{Z}^2$ such that

$$\sum_{x_1 \geq 0} p((x_1,0))|x_1| = \sum_{x_1 \leq 0} p((x_1,0))|x_1|$$
$$= \sum_{x_2 \geq 0} p((0,x_2))|x_2| = \sum_{x_2 \leq 0} p((0,x_2))|x_2| = +\infty\,.$$

Then the above conditions guarantee that the measure is balanced and consequently cyclic, whatever are the values assumed by the measure outside from the coordinate axis.

We discuss now in detail the one dimensional case. In one dimension a balanced probability measure p either belongs to $\mathcal{M}_0 \cap \mathcal{M}^1$ or satisfies

$$\sum_{x=1}^{+\infty} xp(x) = \sum_{x=1}^{+\infty} xp(-x) = +\infty\,. \tag{2.10}$$

We now sketch an iterative procedure whose generalization is at the core of the proof of Theorem 2.3 and will be discussed more in detail during its proof. This procedure generates a cyclic decomposition of p satisfying (2.10). We remark that this case corresponds to a not integrable balanced measure. Given a measure $p^l \in \mathcal{M}^{\leq 1}$ satisfying (2.10) we define

$$\begin{cases} x_+^l := \min\left\{x \geq 1 \,:\, p^l(x) > 0\right\}\,, \\ x_-^l := \max\left\{x \leq -1 \,:\, p^l(x) > 0\right\}\,. \end{cases}$$

We need to distinguish two cases

$$\begin{cases} p^l(x_+^l)x_+^l + p^l(x_-^l)x_-^l \geq 0\,, & (a)\,, \\ p^l(x_+^l)x_+^l + p^l(x_-^l)x_-^l < 0\,, & (b)\,. \end{cases}$$

We consider the equivalence class of cycles

$$[C^l] := \left\{(x_+^l, n_+^l)\,,\,(x_-^l, n_-^l)\right\}\,,$$

where n_+^l and n_-^l are determined requiring that $\frac{n_+^l}{n_-^l}$ is an irreducible fraction and moreover $\frac{n_+^l}{n_-^l} = -\frac{x_-^l}{x_+^l}$. The corresponding weight will be

$$\rho([C^l]) := \begin{cases} \frac{p^l(x_-^l)(n_+^l+n_-^l)}{n_-^l} & (a)\,, \\ \frac{p^l(x_+^l)(n_+^l+n_-^l)}{n_+^l} & (b)\,. \end{cases}$$

If we define $p^{l+1} := p^l - \rho([C^l])p^{[C^l]}$ then $p^{l+1} \in \mathcal{M}^{\leq 1}$ and still satisfies (2.10) so that we can iterate the procedure starting from $p^1 := p - p(0)\delta_0$. By construction we have $|\left\{\mathcal{S}(p^{l+1}) \cap [x_-^l, x_+^l]\right\}| \leq 1$ and condition (2.10) guarantees that $\lim_{l\to+\infty} x_-^l = -\infty$ and $\lim_{l\to+\infty} x_+^l = +\infty$. As a consequence we get

$$p = p(0)\delta_0 + \sum_{l=1}^{+\infty} \rho([C^l])p^{[C^l]}\,,$$

that is a cyclic decomposition like (2.6).

We call irreducible an equivalence class $[C] = \{w^1, \dots w^{|C|}\}$ such that for any non empty $\{v^1, \dots v^k\} \subset \{w^1, \dots w^{|C|}\}$, where the inclusion is strict, it holds $\sum_{i=1}^k v^i \neq 0$.

In lemma 4.1, stated and proved in section 4, we will characterize $\mathcal{C}^*$ a subset of the irreducible equivalence classes of cycles. The second main result of this section is the following.

Theorem 2.4. *The set of extremal elements of $\mathcal{B}$ is $\left\{p^{[C]}\right\}_{[C]\in\mathcal{C}^*}$. Moreover for any element $p \in \mathcal{B}$ there exists a probability measure ρ on $\mathcal{C}^*$ such that*

$$p = \sum_{[C]\in\mathcal{C}^*} \rho([C])p^{[C]}\,. \tag{2.11}$$

We give also a different characterization of cyclic measures. We denote by $\overrightarrow{\mathcal{M}_0}$ the monotone non-decreasing weak closure of $\mathcal{M}_0$. This means that $p \in \overrightarrow{\mathcal{M}_0}$ if there exists a sequence $p_n \in \mathcal{M}_0$ such that $p_n \succeq p_{n-1}$ and $p = \lim_{n\to+\infty} p_n$ in the weak sense. Then we have the following alternative characterization.

Theorem 2.5.

$$\mathcal{B} = \overrightarrow{\mathcal{M}_0} \cap \mathcal{M}^1\,.$$

The monotone requirement is necessary since it is easy to see that $\overline{\mathcal{M}_0} \cap \mathcal{M}^1 = \mathcal{M}^1$, where $\overline{\mathcal{M}_0}$ is the weak closure of $\mathcal{M}_0$.

In this subsection we restricted our discussion to probability measures but the arguments could be adapted to positive measures having finite mass.

2.2. *Cyclic random walks on a finite set.* Both a discrete time and a continuous time Markov chain on a finite or countable set V can be described in terms of a oriented weighted graph (V, E, r). Here V is the set of vertices, $E \subseteq V \times V$ is the set of oriented edges and $r : E \to \mathbb{R}^+$ is the weight function. For simplicity here and hereafter we consider the case $(x, x) \notin E$ for any $x \in V$, the general case can be easily handled. The set of edges E is determined by the weight function by the requirement that $(x, y) \in E$ if and only if $r(x, y) > 0$. For this reason we can also denote a weighted graph simply by the pair (V, r). The set of edges having positive weight is denoted by $E(r)$. There is a natural partial order among weighted graphs having the same set of vertices. We say that $(V, r) \preceq (V, r')$ if $r(x, y) \leq r'(x, y)$ for any $(x, y) \in E(r)$. The set of all non negative weights $r : V \times V \to \mathbb{R}^+$ is denoted by W.

In the case of a discrete time random walk we fix the weights in such a way that

$$r(x, y) := \mathbb{P}(X_{n+1} = y | X_n = x) .$$

In this case clearly the normalization condition

$$\sum_{\{y \in V : (x,y) \in E(r)\}} r(x, y) = 1 , \qquad \forall x \in V , \tag{2.12}$$

has to be satisfied. In the case of continuous time Markov chains we identify the weight $r(x, y)$ with the rate of jump from x to y.

A cycle C, on an oriented graph (V, E), is a finite sequence $C := (x^0, x^1, \ldots, x^{n-1}, x^0)$ of distinct elements $x^i \in V$ such that for any $i = 0, \ldots, n-1$ we have $(x^i, x^{i+1}) \in E$ where the sum in the indices is modulo n. We say that an edge (x, y) belongs to the cycle C and write $(x, y) \in C$ if there exists an i such that $(x, y) = (x^i, x^{i+1})$. In this framework it is natural to identify two cycles if they contain the same set of edges. More precisely given two cycles $C = (x^0, x^1, \ldots, x^{n-1}, x^0)$ and $C' = (y^0, y^1, \ldots, y^{n-1}, y^0)$, we say that $C \sim C'$ if there exists a $j \in \{0, 1, \ldots, n-1\}$ such that $y^i = x^{i+j}$, $i = 0, 1, \ldots, n-1$, where the sum in the indices is modulo n. As before we call $[C]$ the equivalence class of cycles having C as a representant. We call also $\mathcal{C}$ the set of equivalence classes of cycles.

We associate to the cycle C the weighted graph, that we call a purely cyclic weighted graph, determined by the weight function

$$r^C(x, y) := \begin{cases} 1 & \text{if } (x, y) \in C , \\ 0 & \text{otherwise} . \end{cases} \tag{2.13}$$

Note that if $C \sim C'$ then $r^C = r^{C'}$ so that we can use the notation $r^{[C]}$ without ambiguities. Moreover a purely cyclic weighted graph uniquely determine an element of $\mathcal{C}$.

Definition 2.6. We call a weighted graph (V, r) cyclic if there exists a positive measure ρ on $\mathcal{C}$ such that

$$r = \sum_{[C] \in \mathcal{C}} \rho([C]) r^{[C]} . \tag{2.14}$$

Definition 2.7. We will say that the weight r is balanced at $x \in V$ if it satisfies the condition

$$\sum_{\{y : (y,x) \in E(r)\}} r(y, x) = \sum_{\{y : (x,y) \in E(r)\}} r(x, y) . \tag{2.15}$$

The weight r is balanced if (2.15) holds at every $x \in V$.

We use the term balanced with two different meanings, one determined by definition 2.2 and the other one determined by definition 2.7. There is no risk of confusion since one refers to elements of $\mathcal{M}^{\leq 1}$ and the other one to elements of W. The main result of this subsection is the following.

Theorem 2.8. *A finite weighted graph (V, r) is cyclic if and only if it is balanced.*

Remark 2.9. Note that all the results of this paper can be formulated in terms of cyclic decompositions of weighted graphs without any reference to Markov models. A natural way to associate a weighted graph to a Markov model is the one discussed at the beginning of this subsection but it is clearly not the only one. For example given a positive measure π on V and a random walk with jump rates $c(x,y)$ we can associate to this pair a weighted graph with weights

$$r(x,y) := \pi(x)c(x,y)\,. \tag{2.16}$$

The definition of centered random walks in Mathieu (2006) is based for example on the properties of a weighted graph defined in this way. All the results of the present paper can be clearly applied also in this case. Note that the validity of the balancing condition of Definition 2.7 for the rates in (2.16) is equivalent to require that π is invariant for the rates c.

2.3. *Cyclic random walks on a finite graph with topology.* We consider the d dimensional continuous torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ of side length 1. Any element of the torus is an equivalence class that we identify with any of its representant. On the continuous torus we embed a $d-$dimensional discrete torus with mesh $\frac{1}{N}$. This is a graph whose set of vertices is

$$V_N := \left\{\mathbb{T}^d \ni x = (x_1,\dots,x_d)\,:\, x_1,\dots,x_d = 0, \frac{1}{N},\dots,\frac{N-1}{N}\right\}. \tag{2.17}$$

The set of oriented edges E_N contains all the pairs (x,y), with $x,y \in V_N$ and such that $d(x,y) \leq 1/N$, where the distance is on $\mathbb{T}^d$. For simplicity we consider a cubic grid but all the arguments can be repeated also in the case of a grid with $N_1 \times N_2 \times \cdots \times N_d$ sites.

We define $E'_N \subset E_N$ the collection of oriented edges $(x,y) \in E_N$ such that there exist $\hat{x},\hat{y} \in \mathbb{R}^d$ in the same equivalence classes respectively of x,y and such that $\hat{y} = \hat{x} + e^{(i)}/N$ for some i. Note in particular that for example

$$\Big(((N-1)/N, x_2,\dots,x_d),(0,x_2,\dots,x_d)\Big) \in E'_N\,.$$

This is the exact meaning that we give to expressions like $(x,y) \in E_N$ with $y = x \pm e^{(i)}/N$.

The embedding of the discrete torus on the continuous one defines in a natural way a cellular decomposition of $\mathbb{T}^d$. Every cell of maximal dimension is a d dimensional hyper-cube of side length $\frac{1}{N}$ having vertices in V_N. Every 2 dimensional cell of the cellular decomposition is a square having vertices

$$\Big\{x, x+e^{(i)}/N, x+e^{(j)}/N, x+e^{(i)}/N+e^{(j)}/N\Big\}\,,$$

for some $x \in V_N$ and some $i \neq j$. Every 2 dimensional cell can be oriented in 2 possible ways. An orientation is a choice among the two possible elementary cycles

$$\begin{cases} \big(x, x+e^{(i)}/N, x+e^{(i)}/N+e^{(j)}/N, x+e^{(j)}/N, x\big)\,, \\ \big(x, x+e^{(j)}/N, x+e^{(i)}/N+e^{(j)}/N, x+e^{(i)}/N, x\big)\,, \end{cases} \tag{2.18}$$

associated to the 2 dimensional cell (see figure 2.2 cases (b) and (c)). We call F_N the set of oriented two dimensional cells of the cellular decomposition. Conventionally the two possible orientations are called clockwise and anticlockwise. If $f \in F_N$

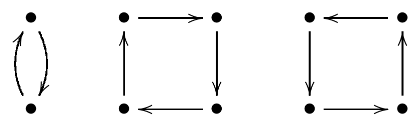

FIGURE 2.2. Elementary cycles associated to one dimensional cells (a) and two dimensional cells (b) and (c), respectively clockwise and anticlockwise oriented.

we call f^c the element of F_N associated to the same 2 dimensional cell but corresponding to the opposite orientation. We call $F'_N \subset F_N$ the subset of anticlockwise oriented 2 dimensional cells, this means the cells with associated the orientation

$$\left(x, x + e^{(\min\{i,j\})}/N, x + e^{(i)}/N + e^{(j)}/N, x + e^{(\max\{i,j\})}/N, x\right) .$$

The discrete torus is a finite graph so that cycles and equivalence classes of cycles are defined as in subsection 2.2. In particular we still call $\mathcal{C}$ the set of equivalence classes of cycles. Given a cycle $C = (z^0, z^1, \ldots, z^{n-1}, z^0)$ on the discrete torus we associate to it the displacement vectors $w^1, \ldots w^n$ defined by

$$w^i := \begin{cases} \frac{e^{(j)}}{N} & \text{if } z^i = z^{i-1} + \frac{e^{(j)}}{N} , \\ -\frac{e^{(j)}}{N} & \text{if } z^i = z^{i-1} - \frac{e^{(j)}}{N} . \end{cases} \tag{2.19}$$

Given a cycle $C = (z^0, z^1, \ldots, z^{n-1}, z^0)$ on the discrete torus we can naturally associate to it also a continuous closed curve on $\mathbb{T}^d$. This is the projection on $\mathbb{T}^d$ of a continuous curve on $\mathbb{R}^d$ whose parametrization $\{\hat{x}(s)\}_{s\in[0,1]}$ is defined by the positions $\hat{x}(0) = z^0$ and

$$\dot{\hat{x}}(s) = nw^i , \qquad s \in \left[\frac{i-1}{n}, \frac{i}{n}\right) .$$

Since the projection is a closed curve on $\mathbb{T}^d$ then $\hat{x}(1) - \hat{x}(0)$ is a vector with integer coefficients. A continuous closed curve on $\mathbb{T}^d$ is called homotopically trivial if it can be continuously deformed to a single point. More precisely a continuous closed curve $\{x(s)\}_{s\in[0,1]}$ is homotopically trivial if there exists a continuous map $X : [0,1] \times [0,1] \to \mathbb{T}^d$ such that $X(s,0) = x(s)$, $X(s,1) = x^*$ for any $s \in [0,1]$, where $x^* \in \mathbb{T}^d$ is a fixed element and $X(1,t) = X(0,t)$ for any t. The discrete cycle C will be called homotopically trivial if the associated continuous curve on the torus is homotopically trivial. It is easy to see that this condition is equivalent to require that

$$\sum_{i=1}^{n} w^i = 0 , \tag{2.20}$$

where w^i are the displacement vectors of the cycle. By $\mathcal{C}^* \subseteq \mathcal{C}$ we denote the subset of equivalence classes associated to homotopically trivial cycles.

We define also a subset $\mathcal{C}^e \subseteq \mathcal{C}^*$ of equivalence classes of elementary cycles. Elementary cycles are naturally associated to one and two dimensional cells of the cellular decomposition. Given $x, y \in V_N$ such that $d(x,y) = 1/N$ we call $[C_{\{x,y\}}]$

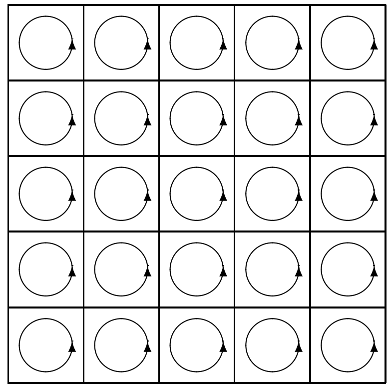

FIGURE 2.3. Elements of F'_N of the two dimensional torus. Any pair is oriented in agreement. Opposite sides of the square are identified.

the equivalence class containing the elementary cycle $C_{\{x,y\}} := (x, y, x)$ (see Figure 2.2 case (a)).

As already discussed to every element $f \in F_N$ corresponds a cycle of the type (2.18). We call $[C_f]$ the corresponding equivalence class. The set of equivalence classes of elementary cycles $\mathcal{C}^e$ is defined as

$$\mathcal{C}^e := \left(\left\{[C_{\{x,y\}}]\right\}_{(x,y)\in E'_N}\right) \cup \left(\left\{[C_f]\right\}_{f\in F_N}\right) .$$

Given $(x, y) \in E_N$ and $f \in F_N$ we write $(x, y) \in f$ if the oriented edge (x, y) belongs to any representant of $[C_f]$. A pair of elements of F_N associated to adjacent cells is said to be oriented in agreement if no elements of E_N belong to both. On the two dimensional torus every pair of elements of F'_N is oriented in agreement (see figure 2.3). This is possible since the two dimensional torus is an orientable surface.

Given a continuous time random walk on the discrete torus its transition rates identify a weight $r \in W$. The discrete torus is a finite graph and a decomposition like (2.14) exists if and only if the balancing condition of Definition 2.7 holds. We investigate under which conditions decompositions like

$$r = \sum_{[C]\in\mathcal{C}^*} \rho([C])r^{[C]} , \tag{2.21}$$

or

$$r = \sum_{[C]\in\mathcal{C}^e} \rho([C])r^{[C]} , \tag{2.22}$$

hold. Recall that ρ is a positive measure and $r^{[C]}$ is the element of W associated to any cycle in $[C]$ like in (2.13). Let us call $R^* \subseteq W$ and $R^e \subseteq W$ the sets of weights for which respectively a decomposition like (2.21) and like (2.22) holds. Clearly it holds $R^e \subseteq R^*$.

Before proceeding we need to introduce some notions and terminology. We refer to Desbrun et al. (2008), Lovasz (2004), Mercat (2001) for more details. We call Λ^0 the $|V_N|-$dimensional vector space of the real functions $f : V_N \to \mathbb{R}$.

We call a function $\phi : E_N \to \mathbb{R}$ a discrete vector field if it satisfies the condition

$$\phi(x,y) = -\phi(y,x)\,, \qquad \forall (x,y) \in E_N\,. \tag{2.23}$$

We call Λ^1 the $|E'_N|-$dimensional vector space of discrete vector fields. We endow Λ^1 of the scalar product $\langle \cdot, \cdot \rangle_1$ defined as

$$\langle \phi, \phi' \rangle_1 := \sum_{(x,y)\in E'_N} \phi(x,y)\phi'(x,y)\,. \tag{2.24}$$

Due to (2.23) the value of (2.24) does not depend on the specific choice of E'_N. On W we define a projection operator that associates to a weight r a discrete vector field $\phi^r \in \Lambda^1$ defined as

$$\phi^r(x,y) := r(x,y) - r(y,x).$$

Conversely to any $\phi \in \Lambda^1$ we canonically associate a weight $r^\phi \in W$ defined as

$$r^\phi(x,y) := [\phi(x,y)]_+\,,$$

where $[\cdot]_+$ denotes the positive part of a real number

$$[a]_+ := \begin{cases} a & \text{if } a \geq 0\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Note that $\min\left\{r^\phi(x,y), r^\phi(y,x)\right\} = 0$ and $r^\phi \in R(\phi)$, where

$$R(\phi) := \{r \in W \,:\, \phi^r = \phi\}\,.$$

We stress that $R(0)$ coincides with the set of symmetric rates, i.e. such that $r(x,y) = r(y,x)$. Given a positive weight r we can decompose it into the sum of two positive weights as

$$r = s + r^{\phi^r}\,. \tag{2.25}$$

Clearly $s \in R(0)$ and it is called the symmetric part of r. We can deduce

$$R(\phi) = \left\{r \,:\, r = s + r^\phi\,,\ s \in R(0)\right\} = r^\phi + R(0)\,. \tag{2.26}$$

A simple consequence of (2.26) is that there exists a minimal element, with respect to the partial order $\preceq$, on $R(\phi)$ that is r^ϕ.

We call a two chain a map $\psi : F_N \to \mathbb{R}$ that satisfies the additional property

$$\psi(f) = -\psi(f^c)\,, \qquad \forall f \in F_N\,.$$

We call Λ^2 the $|F'_N|-$dimensional vector space of the 2-chains.

We define the co-boundary operator $\delta : \Lambda^i \to \Lambda^{i+1}$, $i = 0, 1$ as follows. Given a function $f \in \Lambda^0$ we define $\delta f \in \Lambda^1$ as

$$\delta f(x,y) := f(y) - f(x)\,, \qquad (x,y) \in E_N\,.$$

A discrete vector field obtained in this way is called a gradient. Given $\phi \in \Lambda^1$ we define $\delta\phi \in \Lambda^2$ as

$$\delta\phi(f) := \sum_{\{(x,y)\in E_N\,:\,(x,y)\in f\}} \phi(x,y)\,, \qquad f \in F_N\,.$$

The value $\delta\phi(f)$ is called the value of the circulation of the vector field ϕ around f.

We define also a boundary operator $d : \Lambda^i \to \Lambda^{i-1}$, $i = 1, 2$. Given $\psi \in \Lambda^2$ we define $d\psi \in \Lambda^1$ as

$$d\psi(x,y) := \sum_{\{f\in F_N\,:\,(x,y)\in f\}} \psi(f)\,, \quad (x,y) \in E_N\,. \tag{2.27}$$

Given $\phi \in \Lambda^1$ we define $d\phi \in \Lambda^0$ as

$$d\phi(x) := \sum_{\{y \in V_N \,:\, (x,y) \in E_N\}} \phi(x,y)\,, \qquad x \in V_N\,. \tag{2.28}$$

The r.h.s of (2.28) is called the discrete divergence at x of ϕ.

It is easy to verify according to our definitions that for any $\psi \in \Lambda^2$ it holds

$$d(d\psi) = 0\,. \tag{2.29}$$

The one dimensional case is elementary and will be considered separately. Here we restrict to dimensions $d \geq 2$. We denote by $d\Lambda^2 \subseteq \Lambda^1$ the set of vector fields ϕ for which there exists a $\psi \in \Lambda^2$ with $\phi = d\psi$. A characterization of $d\Lambda^2$ will be discussed in terms of a discrete Hodge decomposition in section 3. It can be summarized as follows. A discrete vector field ϕ belongs to $d\Lambda^2$ if and only if $d\phi(x) = 0$ for any x and

$$\sum_{x \in V_N} \phi\left(x, x + e^{(i)}/N\right) = 0\,, \qquad i = 1, \dots, d\,.$$

We have the following result.

Lemma 2.10. *A necessary condition to have $r \in R^*$ is that $\phi^r \in d\Lambda^2$. The same holds for R^e.*

We have also the following Lemma where the metrics in W is the Euclidean one.

Lemma 2.11. *The sets R^e and R^* are closed sets in W.*

As we will see, an immediate consequence of the above Lemma is the following Corollary. It says that $\phi^r \in d\Lambda^2$ is not a sufficient condition to have $r \in R^*$ (and consequently also $r \in R^e$). We add a condition of positivity since in this way we get a stronger statement.

Corollary 2.12. *There exists a weight $r \in W$ such that $\phi^r \in d\Lambda^2$, $r(x,y) > 0$ for any $(x,y) \in E_N$ but nevertheless $r \notin R^*$.*

2.3.1. *One dimensional torus.* In the one dimensional case there are no two dimensional faces and

$$\mathcal{C}^* = \mathcal{C}^e = \left\{[C_{(x,x+1/N)}]\right\}_{x \in V_N}\,,$$

so that $|\mathcal{C}^*| = N$. Moreover we have $|\mathcal{C}| = N+2$. To the previous equivalence classes we need to add the two equivalence classes having as representants the cycles

$$C_+ := (0, 1/N, 2/N, \dots, (N-1)/N, 0)$$

and

$$C_- := (0, (N-1)/N, (N-2)/N, \dots, 1/N, 0)\,.$$

The following Theorem says that the validity of decompositions of the type (2.21) and (2.22) is equivalent to a geometric property of the vector field ϕ^r. More precisely (2.21) and (2.22) hold if and only if ϕ^r is zero.

Theorem 2.13. *It holds a decomposition of the type (2.21) if and only if $\phi^r = 0$. Moreover this happens if and only if the rates r are reversible with respect to the uniform measure.*

Let

$$m := \min_{(x,y)\in E_N} r(x,y)\,.$$

The following Theorem says that also the validity of a decomposition like (2.14) is still equivalent to a geometric property of the vector field ϕ^r. More precisely (2.14) holds if and only if ϕ^r is divergence free. Moreover the model is so simple that we can characterize completely the associated measures ρ.

Theorem 2.14. *A decomposition like* (2.14) *holds if and only if the discrete vector field ϕ^r is constant or equivalently has zero divergence $d\phi^r = 0$. Moreover if $\phi^r = c$ then all the measures on $\mathcal{C}$ for which* (2.14) *holds are parameterized by the real parameter $a \in [0, m]$ as*

$$\begin{cases} \rho([C_{\{x,y\}}]) = \min\{r(x,y), r(y,x)\} - a\,, \\ \rho([C_+]) = [c]_+ + a\,, \\ \rho([C_-]) = [-c]_+ + a\,. \end{cases} \tag{2.30}$$

By a constant discrete vector field $\phi = c$ we mean $\phi(x,y) = c$ for any $(x,y) \in E'_N$.

2.3.2. *Two dimensional torus.* Every $(x,y) \in E'_N$ belongs to only two elements of F_N and moreover one is clockwise oriented and the other one is anticlockwise oriented. We call f_+ the element of F'_N such that $(x,y) \in f_+$ and f_- the element of F'_N such that $(y,x) \in f_-$. Let $\phi \in d\Lambda^2$ and consider any $\psi \in \Lambda^2$ such that $d\psi = \phi$. As will be explained more in detail in section 3, any other ψ' such that $d\psi' = \phi$ differs from ψ by an additive constant, this means that there exists a real constant c such that

$$\psi'(f) = \psi(f) + c\,, \qquad \forall f \in F'_N\,.$$

The main result of this subsection, Theorem 2.15 below, will not depend on the specific choice of c. To every $(x,y) \in E'_N$ we associate the closed interval $I(x,y)$ of $\mathbb{R}$ defined as

$$I(x,y) = [i_1(x,y), i_2(x,y)] := [\min\{\psi(f_-), \psi(f_+)\}, \max\{\psi(f_-), \psi(f_+)\}]\,.$$

Given $I_1, \ldots, I_n$ a collection of closed intervals of $\mathbb{R}$ we call $\mathcal{P}(I_1, \ldots, I_n) \subseteq \mathbb{R}^n$ the closed unbounded convex polyhedron defined as follows. An element $s = (s_1, \ldots, s_n) \in \mathbb{R}^n$ belongs to $\mathcal{P}(I_1, \ldots, I_n)$ if and only if the inequalities

$$s_i + s_j \geq d(I_i, I_j)\,, \qquad i,j = 1, \ldots, n\,, \tag{2.31}$$

are satisfied. Note that in (2.31) we are considering also the cases $i = j$ that imply $s_i \geq 0$. See Figure 2.4 for a simple example in dimension $n = 2$. Two important properties of this polyhedron are the following. The first one is that if $s \in \mathcal{P}(I_1, \ldots, I_n)$ and $s' \geq s$ then $s' \in \mathcal{P}(I_1, \ldots, I_n)$. The second one is that $0 \in \mathcal{P}(I_1, \ldots, I_n)$ if and only if $d(I_i, I_j) = 0$ for any i, j. By the Helly's Theorem recalled in section 3 this is equivalent to $\cap_i I_i \neq \emptyset$. In this case clearly $\mathcal{P}(I_1, \ldots, I_n) = (\mathbb{R}^+)^n$.

The main result of this subsection is the following. It says that, differently from the one dimensional case, the validity of a decomposition like (2.22) is not simply equivalent to require a geometric property of the vector field ϕ^r, but involves instead also the symmetric part s.

Theorem 2.15. *Let r such that $\phi^r = \phi \in d\Lambda^2$. We have $r \in R^e$ if and only if*

$$s \in \mathcal{P}\left(\{I(x,y)\}_{(x,y)\in E'_N}\right)\,. \tag{2.32}$$

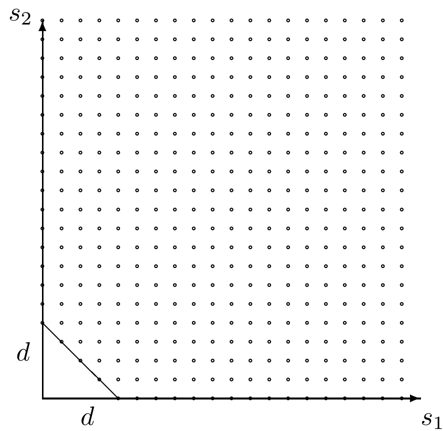

FIGURE 2.4. The dashed region represents the polyhedron $\mathcal{P}(I_1, I_2) \subseteq (\mathbb{R}^+)^2$ when $d(I_1, I_2) = d > 0$. In this case $0 \notin \mathcal{P}(I_1, I_2)$.

In (2.32) s is the symmetric part of r (see (2.25)) and the intervals are constructed using any $\psi \in \Lambda^2$ such that $d\psi = \phi$. The polyhedron obtained is independent on the specific choice.

2.3.3. *Other topologies.* We can generalize Theorem 2.15 to surfaces different from the two dimensional torus and to cellular decompositions different from the cubic one. More precisely consider an unoriented graph embedded on a compact surface without boundary. Vertices are associated to points of the surface, edges are associated to continuous self avoiding curves on the surface connecting vertices. Two different curves may intersect only on vertices. Cutting the surface along edges we obtain a finite number of two dimensional cells homeomorphic to two dimensional balls. Every edge belong to only two different two dimensional cells. From this we construct an oriented graph (V, E) whose vertices coincide with the vertices of the unoriented graph and whose oriented edges are obtained splitting any unoriented edge $\{x, y\}$ into (x, y) and (y, x). Elementary cycles can be defined also in this case. They are naturally associated to one dimensional cells and oriented two dimensional cells and are defined like in the previous subsection. Also the boundary and co-boundary operators are defined in a similar way. We avoid formal definitions, see Desbrun et al. (2008), Lovasz (2004) and Mercat (2001) for more details. We need to distinguish two cases, when the surface is orientable or not. If the surface is orientable and $\phi \in d\Lambda^2$ then $\psi \in \Lambda^2$ such that $\phi = d\psi$ is defined up to an additive constant. If the surface is non orientable and $\phi \in d\Lambda^2$ then $\psi \in \Lambda^2$ such that $\phi = d\psi$ is uniquely determined.

In the orientable case we can fix $F'_N \subseteq F_N$ choosing an orientation for any two dimensional cell in such a way that any pair of adjacent elements of F'_N is oriented in agreement. We fix also $E'_N \subseteq E_N$ choosing arbitrarily one among the two possible orientations for any edge. For any $(x, y) \in E'_N$ there exists only one element $f_+ \in F'_N$ such that $(x, y) \in f_+$ and one element $f_- \in F'_N$ such that $(y, x) \in f_-$. The corresponding interval $I(x, y)$ is defined as in the case of the two dimensional torus. We then have the following Theorem.

Theorem 2.16. *Consider a weighted oriented graph constructed starting from a finite cellular subdivision of a compact orientable surface without boundary and let $r \in W$ such that $\phi^r = \phi \in d\Lambda^2$. Then we have $r \in R^e$ if and only if*

$$s \in \mathcal{P}\left(\{I(x,y)\}_{(x,y)\in E'_N}\right) .$$

Given $J_1, \ldots, J_n$ a collection of closed subsets of $\mathbb{R}$ we call $\mathcal{P}'(J_1, \ldots, J_n) \subseteq \mathbb{R}^n$ the closed unbounded convex polyhedron defined as follows. An element $s = (s_1, \ldots, s_n) \in \mathbb{R}^n$ belongs to $\mathcal{P}'(J_1, \ldots, J_n)$ if and only if the inequalities

$$s_i \geq d(0, J_i)\,, \qquad i = 1, \ldots, n\,,$$

are satisfied. Note that if $J_1, \ldots, J_n$ are closed intervals then the triangle inequality immediately implies $\mathcal{P}'(J_1, \ldots, J_n) \subseteq \mathcal{P}(J_1, \ldots, J_n)$.

In the case of a non orientable surface we fix $F'_N \subseteq F_N$ choosing for any two dimensional cell arbitrarily one among the two possible orientations. Note that in this case is not possible to select the orientations in such a way that any pair of adjacent elements of F'_N is oriented in agreement. Likewise we fix $E'_N \subseteq E_N$ choosing arbitrarily one among the two possible orientations for any 1 dimensional cell.

For a non orientable surface, fixed $(x,y) \in E'_N$, two possible situations are possible. The first case is when there exist one element $f_+ \in F'_N$ such that $(x,y) \in f_+$ and one $f_- \in F'_N$ such that $(y,x) \in F'_N$. When this happens we define $J(x,y) := I(x,y)$ as for orientable surfaces. The second case is when either do not exist elements of F'_N to which (x,y) belongs or there are $f_1, f_2 \in F'_N$ associated to adjacent two dimensional cells such that $(x,y) \in f_i$, $i = 1, 2$. We then define

$$J(x,y) := \Big(-\infty, \min\left\{\psi(f_1), \psi(f_2)\right\}\Big] \cup \Big[\max\left\{\psi(f_1), \psi(f_2)\right\}, +\infty\Big)\,,$$

where depending on the cases either $(y,x) \in f_i$ or $(x,y) \in f_i$, $i = 1, 2$. We have the following Theorem.

Theorem 2.17. *Consider a weighted oriented graph constructed starting from a finite cellular subdivision of a compact non-orientable surface without boundary and let $r \in W$ such that $\phi^r = \phi \in d\Lambda^2$. Then we have $r \in R^e$ if and only if*

$$s \in \mathcal{P}'\left(\{J(x,y)\}_{(x,y)\in E'_N}\right) .$$

2.4. *Applications.* Finally we discuss some applications. In particular we concentrate on the elementary decomposition for the two dimensional torus. We start from a smooth continuous vector field on $\mathbb{T}^2$ and consider its Hodge decomposition. The natural continuous counterpart of a discrete vector field on $d\Lambda^2$ is a vector field obtained as the orthogonal gradient of a smooth potential function ψ. We introduce then a natural discretization procedure. The value of the discrete vector field on the edge $(x,y) \in E_N$ is the value of the flux of the continuous vector field across the dual edge of (x,y) (see Remark 6.3 for the definition) with a normal vector oriented in agreement with (x,y). The result of the discretization procedure is an element of $d\Lambda^2$. Then we apply Theorem 2.15 to this very general example and obtain a condition for the validity of the elementary decomposition in terms of the variation of the potential function ψ.

Using the above framework we then construct a periodic random environment on $\mathbb{Z}^d$. Theorem 2.15 gives a condition on the strength of the noise to be added in such a way that an elementary decomposition holds almost surely. Using the results in Deuschel and Kösters (2008) we can then deduce a quenched Central Limit Theorem.

All the applications are discussed in an informal way but the claims could be easily transformed into Theorems.

3. Preliminary notions and results

We start recalling some classic definitions of convex analysis. See for example Gruber (2007) for more details.

Given $w^1, \ldots, w^k$ distinct elements of $\mathbb{Z}^d$, with $2 \leq k \leq d+1$, we will say that they are in general position if the vectors

$$\begin{cases} d^1 := w^2 - w^1 \\ \vdots \\ d^{k-1} := w^k - w^1 \end{cases} \tag{3.1}$$

are linearly independent. By convention a single vector $w^1 \in \mathbb{Z}^d$ will be always considered to be in general position. It is easy to see that this definition does not depend on the specific order among the vectors.

Given $A \subseteq \mathbb{R}^d$ we denote by A^0 its relative interior part. This is defined as

$$A^0 := \{x \in A \,:\, \exists \epsilon > 0 \ s.t. \ B_\epsilon(x) \cap aff(A) \subseteq A\} \,,$$

where $B_\epsilon(x)$ is the Euclidean ball of radius ϵ centered at x.

Given $\underline{w} := (w^1, \ldots, w^k) \in (\mathbb{Z}^d)^k$ in general position then $co(\{w^1, \ldots, w^k\})$ is a $(k-1)-$dimensional simplex and consequently for any $y \in co(\{w^1, \ldots, w^k\})$ there exists a unique element $\mu := (\mu_1, \ldots, \mu_k)$ of

$$\mathbb{S}^k := \left\{ \mu = (\mu_1, \ldots, \mu_k) \ : \ \mu_i \geq 0 \ , \ \sum_{i=1}^k \mu_i = 1 \right\} ,$$

such that

$$\sum_{i=1}^k \mu_i w^i = y \,. \tag{3.2}$$

When $y \in \big(co(\{w^1, \ldots, w^k\})\big)^0$ the coefficients μ_i satisfy in addition the strict inequalities $0 < \mu_i < 1$.

We recall that on $\mathbb{R}^d$, as shown in Rubin and Wesler (1958), $co(A)$ coincides with the set of elements $x \in \mathbb{R}^d$ that can be written as $x = \int_{\mathbb{R}^d} y d\mu(y)$, where μ is any probability measure such that there exists a measurable set $A' \subseteq A$ such that $\mu(A') = 1$. In general this equivalence is false.

Let us recall the following basic result of convex analysis (see for example Gruber (2007))

Theorem 3.1. **[Carathéodory]** *Let $A \subseteq \mathbb{R}^d$. Then for any $x \in co(A)$ there exist $x^1, \ldots, x^k \in A$ in general position such that $x \in co(\{x^1, \ldots, x^k\})^0$.*

Remember that since the vectors are in general position then necessarily $k \leq d+1$.

We state and prove the following characterization of the convex hull that we could not find in the literature.

Lemma 3.2. *Let $x \in \mathbb{R}^d$ and $S \subseteq \mathbb{R}^d$. Then $x \in co(S)$ if and only if for any hyperplane $H \ni x$ it holds*

$$H^+ \cap S \neq \emptyset \qquad \text{and} \qquad H^- \cap S \neq \emptyset \,. \tag{3.3}$$

Proof: Note that it holds $(H^+ \cap S) \cup (H^- \cap S) = S$. First we suppose that $x \in co(S)$ and show that (3.3) holds. Assume by contradiction that for example there exists H such that $H^+ \cap S = S$ and $H^- \cap S = \emptyset$. This implies also that $H^- \cap co(S) = \emptyset$. As a consequence we have

$$x = x \cap co(S) \subseteq H^- \cap co(S) = \emptyset \,,$$

a contradiction.

Conversely we assume that (3.3) holds for any hyperplane H and show that this implies $x \in co(S)$. Assume by contradiction that $x \notin co(S)$. Then (see Theorem 4.4 in Gruber (2007)) there exists a separating hyperplane $\tilde{H}$ among the two disjoint convex sets $\{x\}$ and $co(S)$ for which it holds for example $\tilde{H}^+ \cap S = \emptyset$ and $x \in \tilde{H}^+$. If H is the hyperplane parallel to $\tilde{H}$ and containing x then we have $H^+ \subseteq \tilde{H}^+$ that implies $H^+ \cap S = \emptyset$, a contradiction. □

We recall the following basic result of convex analysis (see for example Gruber (2007))

Theorem 3.3. **[Helly]** *Consider $\{I_\alpha\}_{\alpha \in \mathcal{A}}$ a collection of compact convex subsets of $\mathbb{R}^d$. It holds $\bigcap_{\alpha \in \mathcal{A}} I_\alpha \neq \emptyset$ if and only if for any $(\alpha_1, \ldots, \alpha_{d+1}) \in \mathcal{A}^{d+1}$ it holds $\bigcap_{i=1}^{d+1} I_{\alpha_i} \neq \emptyset$.*

We defined the vector spaces $\Lambda^0, \Lambda^1, \Lambda^2$ associated to the discrete torus but it is clear that such vector spaces can be defined for any cellular complex. Correspondingly the following general result holds (see for example Desbrun et al. (2008), Lovasz (2004) and Mercat (2001)).

Theorem 3.4. **[Discrete Hodge Decomposition]** *For any finite cellular complex it holds the following orthogonal decomposition*

$$\Lambda^1 = \delta\Lambda^0 \oplus d\Lambda^2 \oplus \Lambda^1_H \,,$$

where Λ^1_H is called the subspace of harmonic one forms.

We briefly discuss the ideas behind the proof of this Theorem considering the case of the 2 dimensional torus. The elements of

$$\delta\Lambda^0 := \left\{\phi \in \Lambda^1 \ : \ \exists f \in \Lambda^0 \ s.t.\ \phi = \delta f\right\} \,,$$

are called potentials or gradient vector fields. The dimension of the vector space Λ^0 is $|V_N|$. It is easy to see that the kernel of the co-boundary operator δ on Λ^0 coincides with the constant functions, and in particular is a subspace of dimension 1. By the general identity

$$dim\left(\Lambda^0\right) = dim\left(Ker(\delta_{|\Lambda^0})\right) + dim\left(\delta\Lambda^0\right) \,, \tag{3.4}$$

we deduce that $dim(\delta\Lambda^0) = |V_N| - 1 = N^2 - 1$, where $dim(\cdot)$ denotes the dimension. The orthogonal complement of $\delta\Lambda^0$ in Λ^1 is easily characterized. The subspace $\delta\Lambda^0$ of Λ^1 is spanned by the elements $\{\delta\, \mathbb{I}_x\}_{x \in V}$, where $\mathbb{I}_x$ is the characteristic function of $x \in V_N$. We deduce that an element $\phi \in \Lambda^1$ belongs to the orthogonal complement of $\delta\Lambda^0$ if and only if for any $x \in V_N$ it holds

$$\langle \phi, \delta\, \mathbb{I}_x \rangle_1 = -d\phi(x) = 0 \,.$$

This means that the orthogonal complement of $\delta\Lambda^0$ is the set of divergence free discrete vector fields also called circulations.

We have also that $dim(\Lambda^2) = |F'_N| = N^2$. It is easy to see that the kernel of the boundary operator d on Λ^2 is the one dimensional subspace of the constant 2 forms

$$Ker(d_{|\Lambda^2}) = \{\psi \,:\, \psi(f) = c\,,\, \forall f \in F'_N\,,\, c \in \mathbb{R}\}\,.$$

By the formula analogous to (3.4) we deduce $dim(d\Lambda^2) = N^2 - 1$. To show a part of the orthogonal decomposition in Theorem 3.4 it is then enough to show that $d\psi$ is divergence free for any $\psi \in \Lambda^2$. This is exactly the content of formula (2.29). Elements of $d\Lambda^2$ are called $0-$homologous circulations. The orthogonal complement $(d\Lambda^2)^\perp$ is characterized as follows. Given $g \in F'_N$ we define $\psi_g \in \Lambda^2$ as

$$\psi_g(f) := \begin{cases} +1 & \text{if } f = g\,, \\ -1 & \text{if } f = g^c\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Clearly $d\Lambda^2$ is spanned by $\{d\psi_g\}_{g\in F'_N}$. An element $\phi \in \Lambda^1$ belongs to $(d\Lambda^2)^\perp$ if and only if for any $g \in F'_N$ it holds

$$\langle \phi, d\psi_g\rangle_1 = \delta\phi(g) = 0\,. \tag{3.5}$$

This means that the rotation of ϕ around any $g \in F'_N$ is zero. An element $\phi \in \Lambda^1$ that satisfies condition (3.5) for any $g \in F'_N$ is called rotation free. Clearly any gradient vector field is rotation free.

Finally if we define

$$\Lambda^1_H := (\delta\Lambda^0)^\perp \cap (d\Lambda^2)^\perp\,, \tag{3.6}$$

the orthogonal decomposition is proved. By (3.6) we have that the elements of Λ^1_H are rotation free circulations. Such a kind of discrete vector fields are called harmonic. From the dimensional counting we have $dim\left(\Lambda^1_H\right) = 2$.

Let us define $\phi_i \in \Lambda^1$, $i = 1, 2$, as

$$\phi_i(x, y) := \begin{cases} +1 & \text{if } y = x + e^{(i)}/N \\ -1 & \text{if } y = x - e^{(i)}/N \\ 0 & \text{otherwise}\,. \end{cases} \tag{3.7}$$

As it is easy to check ϕ_1 and ϕ_2 are linearly independent rotation free circulations. Since $dim\left(\Lambda^1_H\right) = 2$ we can identify

$$\Lambda^1_H = \{c_1\phi_1 + c_2\phi_2\ ,\ c_i \in \mathbb{R}\}\ .$$

Consider a square matrix M whose rows and columns are labeled by a finite set V and call $M(x, y)$ the element corresponding to row x and column y. The matrix M is called bi-stochastic if its elements are non negative and moreover

$$1 = \sum_{y'\in V} M(x, y') = \sum_{x'\in V} M(x', y)\,, \qquad \forall x, y \in V\,.$$

These conditions identify a convex compact subset called the Birkhoff polytope. To every element $\pi \in Sym(V)$, the permutation group on V, we can associate a $|V| \times |V|$ matrix M_π called the permutation matrix. It is defined as

$$M_\pi(x, y) := \begin{cases} 1 & \text{if } y = \pi(x)\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{3.8}$$

It is clear that $|V| \times |V|$ matrices having positive elements and whose rows and columns are labeled by elements of V, are in bijection with weighted oriented graphs on V. The bijection identifies a matrix M and a graph (V, r) when $M(x, y) = r(x, y)$ for any $x, y \in V$. For example the weighted graph corresponding to (3.8) has weights $r^\pi(x, y) := 1$ if $y = \pi(x)$ and zero otherwise. The following result is classic (see for example Gruber (2007)).

Theorem 3.5. **[Birkhoff-Von-Neumann]** *The set of $|V| \times |V|$ bi-stochastic matrices is convex and compact. Its extremal elements are the permutation matrices on V.*

4. Cyclic random walks on $\mathbb{Z}^d$

Recall the definition of irreducible equivalence class of cycles stated just before Theorem 2.4. The following Lemma identifies an important class of irreducible equivalence classes of cycles.

Lemma 4.1. *Consider $\underline{w} = (w^1, \ldots, w^k) \in (\mathbb{Z}^d)^k$ in general position and such that $0 \in co(\{w^1, \ldots, w^k\})^0$. Then there exists an unique collection of strictly positive natural numbers $n_1, \ldots, n_k$ such that*

$$[C] = \{(w^1, n_1), \ldots, (w^k, n_k)\}$$

is an irreducible element of $\mathcal{C}$. We will call $\mathcal{C}^*$ the set of irreducible elements of $\mathcal{C}$ obtained in this way.

Proof: If $y \in co(\{w^1, \ldots, w^k\})^0 \cap \mathbb{Z}^d$ then the element $\mu \in \mathbb{S}^k$, uniquely determined by (3.2) has the coordinates μ_i that are rational numbers. Indeed if we call A the $(d + 1) \times k$ matrix whose i-column is $(w^i_1, \ldots, w^i_d, 1)$, then A has rank k. Every $k \times k$ sub-matrix containing the last row has determinant different from zero. This follows easily by the fact that the vectors are in general position. Moreover μ is the unique solution of the linear system of $d + 1$ equations in the k variables μ_i

$$\sum_{j=1}^{k} A_{i,j}\mu_j = \delta_{i,d+1} + (1 - \delta_{i,d+1})y_i \,, \qquad i = 1, \ldots, d + 1 \,, \tag{4.1}$$

where $\delta_{i,j}$ is the Kronecker delta. The unique solution to (4.1) can be obtained by Cramer formula applied to k equations among which the last one. The matrices to be used have integer coefficients so that the solution is a vector of rational numbers. By construction they are also strictly positive. This means that if we fix $y = 0$ in (3.2) we obtain rational values for the μ_i. Let us write $\mu_i = \frac{a_i}{b_i}$ where a_i and b_i are natural numbers and $\frac{a_i}{b_i}$ is an irreducible fraction for every i. Let $b := lcm\{b_1, \ldots, b_k\}$ the least common multiple and define the natural numbers

$$n_i := b\mu_i \,, \qquad i = 1, \ldots, k \,.$$

We have

$$\sum_{i=1}^{k} n_i w^i = b \sum_{i=1}^{k} \mu_i w^i = 0 \,. \tag{4.2}$$

We show now that

$$\sum_{i=1}^{k} m_i w^i \neq 0 \,, \tag{4.3}$$

when the strict inclusion

$$\{(w^1, m_1), \ldots, (w^k, m_k)\} \subset \{(w^1, n_1), \ldots, (w^k, n_k)\},$$

holds. Indeed, if (4.3) is false then

$$\sum_{i=1}^{k} \frac{m_i}{(\sum_{j=i}^{k} m_j)} w^i = 0, \tag{4.4}$$

and since the vectors w^i are in general position we deduce

$$\frac{m_i}{\sum_{j=i}^{k} m_j} = \mu_i = \frac{a_i}{b_i}. \tag{4.5}$$

This implies

$$lcm\{b_1, \ldots, b_k\} \leq \sum_{j=i}^{k} m_j < \sum_{j=i}^{k} n_j = b = lcm\{b_1, \ldots, b_k\},$$

and we get a contradiction. Equations (4.2) and (4.3) imply that if we define

$$[C] := \left\{(w^1, n_1), \ldots, (w^k, n_k)\right\},$$

then $[C] \in \mathcal{C}$ and moreover it is irreducible.

It remains to show that the numbers n_i are uniquely characterized. Let us suppose that $\{(w^1, m_1), \ldots, (w^k, m_k)\}$ is an irreducible element of $\mathcal{C}$, then we want to show that necessarily $m_i = n_i$. Clearly for such a collection of integer numbers m_i, equations (4.4) and (4.5) hold. We deduce $\sum_{j=i}^{k} m_j = lb$, with $l \in \mathbb{N}$ and $l \geq 1$. When $l \geq 2$ then $\{(w^1, m_1), \ldots, (w^k, m_k)\}$ is not irreducible. When $l = 1$ then $m_i = n_i$ and we obtain $[C]$. □

Not all irreducible equivalence classes of cycles belong to $\mathcal{C}^*$. Indeed consider

$$w^1 := (-5, -5), \ w^2 := (1, 1), \ w^3 := (2, 2),$$

vectors in $\mathbb{Z}^2$ and the equivalence class $\left\{(w^1, 1), (w^2, 1), (w^3, 2)\right\}$. It is easy to see that such equivalence class is irreducible nevertheless it does not belong to $\mathcal{C}^*$.

Proof of Theorem (2.3). First we show that if $p \in \mathcal{B}$ then it is cyclic. Since $p \in \mathcal{B}$ it holds (2.9) for any H. Equation (2.9) can hold only if (3.3) holds with $S = \mathcal{S}(p)$. By Lemma 3.2 we deduce that $0 \in co(\mathcal{S}(p))$. We can then apply Carathéodory's Theorem 3.1 and deduce that there exists $\underline{w} = (w^1, \ldots, w^k)$ in general position with $w^i \in \mathcal{S}(p)$ and such that $0 \in co(\{w^1, \ldots, w^k\})^0$. Let $[C] \in \mathcal{C}^*$ be the corresponding irreducible equivalence class of cycles as constructed in Lemma 4.1. The corresponding purely cyclic measure is

$$p^{[C]} = \sum_{i=1}^{k} \mu_i(\underline{w}) \delta_{w^i} := q_1. \tag{4.6}$$

Here and hereafter we call $\mu(\underline{w})$ the unique element of $\mathbb{S}^k$ determined by (3.2) with $y = 0$ when the elements of $\underline{w}$ are in general position and moreover $0 \in co\left(\{x^1, \ldots, x^k\}\right)^0$.

Let

$$m_1 := \min\left\{\frac{p(w^i)}{q_1(w^i)}, i = 1, \ldots, k\right\} > 0. \tag{4.7}$$

We have that $m_1 q_1 \preceq p$. This implies that $p_1 := p - m_1 q_1$ belongs to $\mathcal{M}^{\leq 1}$ and moreover, since q_1 is a mean zero probability measure, p_1 is balanced. Note also that by construction there exists $x \in \mathcal{S}(p)$ such that $p_1(x) = 0$. Such an x is a vector w^i that minimizes (4.7).

We want now iterate this procedure. More precisely let p_i be a balanced measure. Then $0 \in co(\mathcal{S}(p_i))$. As before we can apply Carathéodory's Theorem identifying vectors $\underline{w} = (w^1, \dots, w^k)$ in general position, determine the corresponding element of $\mathcal{C}^*$ and define like in (4.6) the corresponding purely cyclic measure q_{i+1}. Then we define

$$m_{i+1} := \min \left\{ \frac{p_i(w^j)}{q_{i+1}(w^j)} \, , \, j = 1, \dots, k \right\} > 0 \, , \tag{4.8}$$

and finally call

$$p_{i+1} := p_i - m_{i+1} q_{i+1} \, , \tag{4.9}$$

that is still a balanced element of $\mathcal{M}^{\leq 1}$. When $|\mathcal{S}(p)| < +\infty$, after a finite number l of iterations of the above procedure we obtain $p_l = 0$. This follows directly by the fact that $|\mathcal{S}(p_{i+1})| \leq |\mathcal{S}(p_i)| - 1$. As a consequence we get

$$p = \sum_{i=1}^{l} m_i q_i \, . \tag{4.10}$$

Recalling that the probability measures q_i are purely cyclic, equation (4.10) is exactly a representation of p of the type

$$p = p^\rho = \sum_{[C] \in \mathcal{C}^*} \rho([C]) p^{[C]} \, , \tag{4.11}$$

where the measure ρ gives weight m_i to the unique element of $\mathcal{C}^*$ associated to q_i, $i = 1, \dots, l$.

When $|\mathcal{S}(p)| = +\infty$ we need to implement the iterative procedure in a suitable way. Let us introduce the family of cubes

$$\Lambda_n := \left\{ x \in \mathbb{Z}^d \, : \, \max_i |x_i| \leq n \right\} \, .$$

Given a balanced p_i we define

$$n_i := \inf \left\{ n \in \mathbb{N} \, : \, 0 \in co\left(\mathcal{S}(p_i) \cap \Lambda_n\right) \right\} \, .$$

Clearly $n_i < +\infty$, since $0 \in co(\mathcal{S}(p_i))$ and consequently there exists a finite number of elements in $\mathcal{S}(p_i)$ whose convex envelope contains the origin. We can then define the measure q_{i+1} as in (4.6) using vectors $\underline{w}$ in general position and belonging to $\Lambda_{n_i} \cap \mathcal{S}(p_i)$. Defining p_{i+1} like in (4.8) and (4.9) we have $\mathcal{S}(p_{i+1}) \subset \mathcal{S}(p_i)$ from which we deduce $n_{i+1} \geq n_i$. We obtain in this way an increasing family of cubes $\Lambda_{n_i} \subseteq \Lambda_{n_{i+1}}$ and a decreasing family of measures $p_{i+1} \preceq p_i$. We now show that necessarily

$$\begin{cases} \lim_{i \to +\infty} \Lambda_{n_i} = \mathbb{Z}^d \, , \\ \lim_{i \to +\infty} p_i = 0 \, , \end{cases} \tag{4.12}$$

where in the second limit it is enough to show just pointwise convergence. As a consequence, taking the limit $j \to +\infty$ in

$$p_j = p - \sum_{i=1}^{j} m_i q_i \, ,$$

we obtain

$$p = \sum_{i=1}^{+\infty} m_i q_i \,. \tag{4.13}$$

This identifies p with p^ρ like in (4.11), where the probability measure ρ on $\mathcal{C}^*$ gives weight m_i to the unique element of $\mathcal{C}^*$ associated to q_i.

We need then to show (4.12). Both sequences are monotone and then the limits exist. Let us suppose by contradiction that $\lim_{i\to+\infty} n_i = n^* < +\infty$. Note that in this case for any i we have $\mathcal{S}(q_i) \subseteq \Lambda_{n^*}$ and moreover

$$|\mathcal{S}(p_{i+1}) \cap \Lambda_{n^*}| \leq |\mathcal{S}(p_i) \cap \Lambda_{n^*}| - 1 \,.$$

In particular after a finite number of iterations, say j we have that $\mathcal{S}(p_j) \cap \Lambda_{n^*} = \emptyset$. This implies $n_{j+1} > n^*$ and we get a contradiction.

Finally let us suppose by contradiction that $\lim_{i\to+\infty} p_i = p^* \neq 0$. Then, by monotone convergence Theorem, p^* is balanced and consequently $0 \in co(\mathcal{S}(p^*))$. Let us call $w^1, \dots, w^k$ a finite collection of vectors in general position belonging to $\mathcal{S}(p^*)$ and such that $0 \in co(\{w^1, \dots, w^k\})^0$. Consider $j \in \mathbb{N}$ such that $\{w^1, \dots, w^k\} \subseteq \Lambda_{n_j - 1}$. By construction we have $\mathcal{S}(p^*) \subset \mathcal{S}(p_j)$ and moreover $0 \notin co\left(\mathcal{S}(p_j) \cap \Lambda_{n_j-1}\right)$. This is a contradiction.

We now prove that if p is cyclic then $p \in \mathcal{B}$. By hypothesis (recall (2.8) we have

$$p = \lim_{n\to+\infty} \sum_{[C]\in\mathcal{C}_n} \rho([C]) p^{[C]} \,, \tag{4.14}$$

and for any fixed n the sum on the r.h.s. of (4.14) satisfies (2.9) for any H. We get immediately the validity of (2.9) for p by applying the monotone convergence Theorem. □

Proof of Theorem (2.4). The extremality of $[C] \in \mathcal{C}^*$ follows directly by the irreducibility. The validity of (2.11) has been implicitly proved during the proof of Theorem 2.3. □

Proof of Theorem (2.5). To show the identification it is enough to show that $\overrightarrow{\mathcal{M}_0} \cap \mathcal{M}^1$ coincides with the set of cyclic measures.

Let p be a cyclic probability measure, then clearly $p \in \mathcal{M}^1$. Moreover it holds (4.14) and this is the monotone limit assuring $p \in \overrightarrow{\mathcal{M}_0}$.

Conversely take $p \in \overrightarrow{\mathcal{M}_0} \cap \mathcal{M}^1$, we need to show that p is cyclic. By definition $p \in \mathcal{M}^1$ and there exists a non decreasing sequence $p^n \in \mathcal{M}_0$ such that $p = \lim_{n\to+\infty} p^n$. Since $p^n \in \mathcal{M}_0$ then also $\Delta^n := p^n - p^{n-1} \in \mathcal{M}_0$, where we defined $p^0 := 0$. This means that we can write $\Delta^n = \sum_{[C]\in\mathcal{C}^*} \rho^n([C]) p^{[C]}$ for suitable measures ρ^n. This follows by the fact that any element of $\mathcal{M}_0$ is balanced and then by Theorem 2.3 is cyclic. Since we have $p = \sum_{n=1}^{+\infty} \Delta^n$ we get (2.6) with $\rho([C]) = \sum_{n=1}^{+\infty} \rho^n([C])$ and p on the l.h.s.. □

5. Cyclic random walks on a finite set

The following results are elementary but useful for the forthcoming results.

Proof of Theorem 2.8 Any purely cyclic graph is balanced. This implies that a necessary condition for the validity of (2.14) is that r is balanced. Let us now show the other implication. Let

$$m_1^* := \min_{(x,y)\in E(r)} r(x,y) > 0$$

and let $(z^0, z^1) \in E(r)$ such that $r(z^0, z^1) = m_1^*$. Due to the balancing condition 2.7 and the definition of m_1^* there exists an edge $(z^1, z^2) \in E(r)$ with $r(z^1, z^2) \geq m_1^*$. By the same argument, if $z^2 \neq z^0$, there exists $(z^2, z^3) \in E(r)$ such that $r(z^2, z^3) \geq m_1^*$. We can iterate this procedure up to the first time we visit twice a vertex of V. Since V is finite this happens after at most $|V|$ iterations. We obtain in this way a sequence $z^0, z^1, \dots, z^{n-1}$ of distinct elements of V such that $(z^i, z^{i+1}) \in E(r)$ and moreover a z^n such that $z^n = z^j$ for some $0 \leq j \leq n-1$. We call C_1 the cycle $C_1 := (z^j, z^{j+1}, \dots, z^n)$. We also call $m_1 := \min_{i=j,\dots,n-1} r(z^i, z^{i+1}) \geq m_1^* > 0$. Clearly we have $(V, m_1 r^{[C_1]}) \preceq (V, r)$. As a consequence we have that $r - m_1 r^{[C_1]} \in W$, it is still balanced and moreover it holds $|E\left(r - m_1 r^{[C_1]}\right)| \leq E(r) - 1$. This last inequality implies that after a finite number (at most $|E(r)|$) of iterations of the above procedure we obtain

$$r = \sum_{i=1}^{l} m_i r^{[C_i]}\,,$$

that is the decomposition (2.14) with the measure ρ that gives weight m_i to $[C_i] \in \mathcal{C}$ □

Remark 5.1. The validity of the balancing condition 2.7 implies that the uniform measure $\pi(x) := \frac{1}{|V|}$ satisfies the stationary condition

$$\pi(x) \sum_{\{y\,:\,(x,y)\in E(r)\}} r(x,y) = \sum_{\{y\,:\,(y,x)\in E(r)\}} \pi(y) r(y,x)\,.$$

If the Markov chain is irreducible, this is the unique invariant measure.

In the case of a continuous time Markov chain the set of balanced rates is convex but not compact. In the case of discrete time Markov chains, in addition to the balancing condition there are also the conditions (2.12). The constraints $r \geq 0$, (2.12) and (2.15) for every x, identify the Birkhoff polytope. The extremal elements of the Birkhoff polytope are then characterized by the Birkhoff-Von-Neumann Theorem 3.5. Theorem 3.5 together with the classical statement of Krein-Milmann Theorem Gruber (2007), Phelps (2001) implies that, given any bi-stochastic matrix M, we can decompose it like

$$M = \sum_{\pi\in Sym(V)} m_\pi M_\pi\,,$$

where m is a probability measure on $Sym(V)$. Written in terms of weights this equation becomes

$$r = \sum_{\pi\in Sym(V)} m_\pi r^\pi\,. \tag{5.1}$$

Recall the classical result that every permutation can be decomposed into disjoint cycles. It is easy to see that in terms of weights this means that for every $\pi \in Sym(V)$ we can write

$$r^\pi = \sum_i r^{[C_i^\pi]}\,, \tag{5.2}$$

where the C_i^π constitutes a family of disjoint cycles such that every element of V belongs to one of them. The cycle containing the element $x^0 \in V$ can be written as $(x^0, \pi(x^0), \pi^2(x^0), \dots, \pi^l(x^0), x^0)$, where with π^m we denote the composition of $m-$times the element $\pi \in Sym(V)$ and l is the minimal integer such that $\pi^{l+1}(x^0) = x^0$. Putting together (5.1) and (5.2) we obtain a special decomposition like (2.14).

We finish the section discussing the case of an infinite graph. We simply reinterpret the results of subsection 2.1 in terms of an infinite weighted graphs with vertices $V = \mathbb{Z}^d$ and edges $E = \mathbb{Z}^d \times \mathbb{Z}^d$. The weighted graph corresponding to the translation invariant Markov chain on $V = \mathbb{Z}^d$ defined by (2.3), gives weight $r(x,y) = p(y-x)$ to the edge (x,y). Let us suppose that p is a cyclic measure. Consider C_i a cycle representant of the equivalence class in $\mathcal{C}^*$ corresponding to the cyclic measure q_i in (4.13). On $\mathbb{Z}^d$ there is defined a shift operator τ_x that acts naturally on cycles by

$$\tau_x(x^0, x^1, \dots, x^{n-1}, x^0) := (x + x^0, x + x^1, \dots, x + x^{n-1}, x + x^0)\,.$$

Define the family of cycles $\{\tau_x C_i\}_{x \in \mathbb{Z}^d}^{i \in \mathbb{N}}$ in $\mathbb{Z}^d$. Let ρ be the positive measure on $\mathcal{C}$ that gives weight m_i to $[\tau_x C_i]$ for any x. The results in subsection 2.1 imply the validity of the decomposition (2.14) with this specific measure ρ. In this case (2.14) becomes

$$r(x,y) = p(y-x) = \sum_{i \in \mathbb{N}} \sum_{z \in \mathbb{Z}^d} m_i r^{[\tau_z C_i]}(x,y)\,, \qquad \forall\ (x,y)\,.$$

6. Cyclic random walks on a finite graph with topology

We discuss here the general case of a $d \geq 2$ dimensional torus. More detailed results for the specific cases $d = 1, 2$ will be discussed separately in some subsections.

We start observing that formula (2.25) can be written as

$$r = r^{\phi^r} + \sum_{(x,y) \in E'_N} s(x,y) r^{[C_{\{x,y\}}]}\,.$$

A subset $A \subseteq W$ is monotone non decreasing if $r \in A$ and $r \preceq r'$ implies $r' \in A$. The subsets R^e and R^* are in general not monotone subsets of W. Nevertheless $R^e \cap R(\phi)$ and $R^* \cap R(\phi)$ are non decreasing for any fixed ϕ. This is the content of the next lemma.

Lemma 6.1. *Consider r, and r' belonging to W and such that $r \preceq r'$ and $\phi^r = \phi^{r'}$. We have that if $r \in R^*$ then also $r' \in R^*$. The same happens for R^e.*

Proof: We prove the statement for R^*. The proof for R^e is the same. By assumption there exists a decomposition like (2.21) for r. We then have

$$r' = \sum_{[C] \in \mathcal{C}^*} \rho([C]) r^{[C]} + (r' - r)\,. \tag{6.1}$$

By the hypotheses of the lemma $h := r' - r$ belongs to W and moreover $\phi^h = 0$. Clearly $R(0) \subseteq R^*$ and moreover for any $h \in R(0)$ we have

$$h = \sum_{(x,y) \in E'_N} h(x,y) r^{[C_{\{x,y\}}]}\,. \tag{6.2}$$

Putting together (6.1) and (6.2) we get $r' \in R^*$. $\square$

Proof of Lemma 2.10 Since we discussed the discrete Hodge decomposition in the two dimensional case we will prove this Lemma also in the two dimensional case. The proof in the general case is analogous. Consider $C = (z^0, z^1, \dots, z^n)$ with $z^n = z^0$ a cycle such that $[C] \in \mathcal{C}^*$. Since C is homotopically trivial it holds (2.20). Componentwise (2.20) is written as

$$\begin{cases} |\{(x, x+e^{(1)}/N) \in C\}| - |\{(x+e^{(1)}/N, x) \in C\}| = 0\,, \\ |\{(x, x+e^{(2)}/N) \in C\}| - |\{(x+e^{(2)}/N, x) \in C\}| = 0\,, \end{cases}$$

that implies

$$\begin{cases} \langle \phi^{r^{[C]}}, \phi_1 \rangle_1 = 0\,, \\ \langle \phi^{r^{[C]}}, \phi_2 \rangle_1 = 0\,, \end{cases}$$

where the vector fields ϕ_i are defined in (3.7). This means that $\phi^{r^{[C]}} \in \left(\Lambda^1_H\right)^\perp$. Moreover we know that $r^{[C]}$ satisfies the balancing condition 2.7. This implies that $\phi^{r^{[C]}}$ is a divergence free discrete vector field i.e. it is a circulation. This is equivalent to say that it belongs to $(\delta\Lambda^0)^\perp$. By Theorem 3.4 we obtain $\phi^{r^{[C]}} \in d\Lambda^2$ for any $[C] \in \mathcal{C}^*$. Given $r \in R^*$ having a decomposition like (2.21), by linearity of the projection of weights onto vector fields we have

$$\phi^r = \sum_{[C]\in\mathcal{C}^*} \rho([C]) \phi^{r^{[C]}}\,. \tag{6.3}$$

Since $d\Lambda^2$ is a vector subspace, it is closed under linear combinations and consequently the right hand side of (6.3) belongs to $d\Lambda^2$. Since $\mathcal{C}^e \subseteq \mathcal{C}^*$ the condition $\phi^r \in d\Lambda^2$ is also necessary for the validity of (2.22). $\square$

Proof of Lemma 2.11 We prove the statement for R^*. The proof for R^e is the same. We need to prove that for any sequence $r_n \in R^*$ converging to some element $r \in W$ we have necessarily $r \in R^*$. Observe that for any $(x,y) \in E_N$ we have $r_n(x,y) \geq 0$ and consequently $\lim_{n\to+\infty} r_n(x,y) = r(x,y) \geq 0$. Since for every n it holds $r_n \in R^*$, we can write

$$r_n = \sum_{[C]\in\mathcal{C}^*} \rho_n([C]) r^{[C]}\,,$$

for some positive measures ρ_n. Let $M := \max_{(x,y)\in E_N} r(x,y)$. Consider any $[C] \in \mathcal{C}^*$ and take $(x,y) \in C$. For n large enough we deduce

$$0 \leq \rho_n([C]) \leq \sum_{[C]\in\mathcal{C}^*} \rho_n([C]) r^{[C]}(x,y) = r_n(x,y) \leq 2M\,.$$

The last inequality follows by the fact that r_n converges to r and $r(x,y) \leq M$. Since $\rho_n([C])$ takes values on a compact set we can then extract a converging subsequence $\rho_{n_j}([C])$. By a finite Cantor diagonalizing argument, there exists a subsequence (that we still call n_j) such that $\rho_{n_j}([C])$ is converging for any $[C] \in \mathcal{C}^*$. Let us call $\rho([C]) \geq 0$ the corresponding limits. Since we have a finite sum we get

$$r = \lim_{j\to+\infty} r_{n_j} = \lim_{j\to+\infty} \sum_{[C]\in\mathcal{C}^*} \rho_{n_j}([C]) r^{[C]} = \sum_{[C]\in\mathcal{C}^*} \rho([C]) r^{[C]}$$

This is equivalent to say $r \in R^*$. $\square$

The bi-dimensional example illustrated in figure 6.5 shows that the condition $\phi^r \in d\Lambda^2$ is not sufficient neither for (2.21) nor for (2.22). Indeed in this case

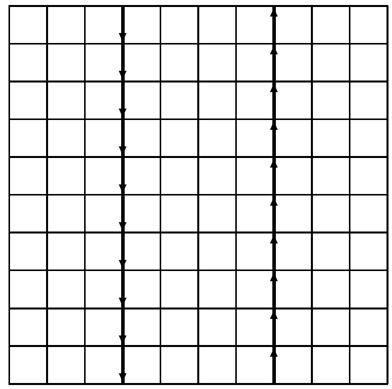

FIGURE 6.5. An $r \in W$ for the two dimensional torus such that $\phi^r \in d\Lambda^2$ nevertheless $r \notin R^*$. Positive unitary weights are represented by boldfaced arrows. Opposite sides of the square are identified.

$\phi^r \in d\Lambda^2$ but the oriented weighted graph (V_N, r) contains no homotopically trivial cycles. Consequently decompositions of the type (2.21) and (2.22) are not possible.

Recalling Lemma 6.1 and the fact that r^ϕ is the minimal element of $R(\phi)$, it is natural to study the following problem. Given $\phi \in d\Lambda^2$ a discrete vector field, under which conditions $r^\phi \in R^*$, or $r^\phi \in R^e$? When $r^\phi \in R^*$ then by Lemma 6.1 we deduce that $R(\phi) \subseteq R^*$. When $r^\phi \in R^e$ then by Lemma 6.1 we deduce that $R(\phi) \subseteq R^e \subseteq R^*$. On the other side if $r^\phi \notin R^*$ then by Lemma 2.11 all the elements of W in a neighborhood of r^ϕ will also not belong to R^*. Note that inside such a neighborhood there will be also rates such that $r(x,y) > 0$ for any $(x,y) \in E_N$. The same happens for R^e.

Proof of Corollary 2.12 The proof follows directly from the above argument and the fact that there exists a $\phi \in d\Lambda^2$ such that $r^\phi \notin R^*$. In dimension two this is exactly the example in Figure 6.5. Similar examples can be clearly constructed in any dimension $d \geq 2$. $\square$

We will give a complete characterization of vector fields such that $r^\phi \in R^e$ in dimension one and two. A characterization of vector fields such that $r^\phi \in R^*$ seems to be an interesting combinatorial problem.

6.1. *One dimensional torus.* The balancing condition (2.15) at site $x \in V_N$ can be written as

$$r(x, x+1/N) - r(x+1/N, x) = r(x-1/N, x) - r(x, x-1/N)\,,$$

and corresponds to require that the discrete vector field ϕ^r has divergence zero. Note that in one dimension a discrete vector field is divergence free if and only if it is constant.

Proof of Theorem 2.13 We want to characterize the Markov model having rates r admitting a decomposition of the type

$$r = \sum_{[C]\in\mathcal{C}^*} \rho([C]) r^{[C]} = \sum_{x\in V_N} \rho([C_{\{x,x+1/N\}}]) r^{[C_{\{x,x+1/N\}}]}\,. \tag{6.4}$$

If (6.4) holds then clearly we have that the associated discrete vector field satisfies

$$\phi^r(x,x+1/N) = \rho([C_{\{x,x+1/N\}}]) - \rho([C_{\{x,x+1/N\}}]) = 0\,.$$

Conversely given a weight r such that $\phi^r = 0$ then we can set in (6.4)

$$\rho([C_{\{x,x+1/N\}}]) = r(x,x+1/N) = r(x+1/N,x).$$

Finally the reversibility follows by the detailed balance condition

$$\frac{1}{N}r(x,x+1/N) = \frac{1}{N}r(x+1/N,x)\,, \qquad \forall x \in V_N\,,$$

that is equivalent to $\phi^r = 0$. □

Proof of Theorem 2.14 We want to characterize now the Markov models having a decomposition of the type

$$r = \sum_{[C]\in\mathcal{C}} \rho([C])r^{[C]} = \sum_{x\in V_N} \rho([C_{\{x,x+1/N\}}])r^{[C_{\{x,x+1/N\}}]} + \rho([C_+])r^{[C_+]} + \rho([C_-])r^{[C_-]}\,. \tag{6.5}$$

The discrete one dimensional torus is a finite graph and consequently this problem has been solved in Theorem 2.8. We discuss it again in terms of the discrete vector field ϕ^r. The fact that in one dimension a discrete vector field has zero divergence if and only if it is constant has already been stressed. If (6.5) holds then clearly we have

$$\begin{aligned}\phi^r(x,x+1/N) &= \rho([C_{\{x,x+1/N\}}]) - \rho([C_{\{x,x+1/N\}}]) + \rho([C_+]) - \rho([C_-])\\ &= \rho([C_+]) - \rho([C_-])\,,\end{aligned}$$

that is a constant discrete vector field. Conversely consider some rates such that for any $x \in V_N$ it holds $\phi^r(x,x+1/N) = c$ with c a constant real number. Then from (6.5) recalling the decomposition (2.25) we obtain

$$\begin{cases} r(x,x+1/N) = \rho([C_{\{x,x+1/N\}}]) + \rho([C_+]) = [c]_+ + s(x,x+1/N)\,,\\ r(x+1/N,x) = \rho([C_{\{x,x+1/N\}}]) + \rho([C_-]) = [-c]_+ + s(x+1/N,x)\,, \end{cases} \tag{6.6}$$

where we recall that s is the symmetric part of r. Clearly (6.6) implies

$$\rho([C_{\{x,x+1/N\}}]) = s(x,x+1/N) - a\,, \tag{6.7}$$

where a is a real parameter. Putting (6.7) in (6.6) we obtain also

$$\begin{cases} \rho([C_+]) = [c]_+ + a\,,\\ \rho([C_-]) = [-c]_+ + a\,. \end{cases} \tag{6.8}$$

Since the left hand sides of (6.7) and (6.8) are non-negative we obtain the constraint $a \in [0,m]$. Recalling the definition of the symmetric part s we obtain (2.30). □

Note that the decomposition (6.5) is unique only when $m = 0$.

6.2. *Two dimensional torus.* Let ρ a positive measure on $\mathcal{C}^e$. The general decomposition (2.22) reads

$$r = \sum_{(x,y)\in E'_N} \rho([C_{\{x,y\}}])r^{[C_{\{x,y\}}]} + \sum_{f\in F_N} \rho([C_f])r^{[C_f]}\,. \tag{6.9}$$

To the measure ρ in (6.9) we can also associate an element $\psi^\rho \in \Lambda^2$ defined as

$$\psi^\rho(f) := \rho([C_f]) - \rho([C_{f^c}]) \qquad f \in F_N\,. \tag{6.10}$$

Lemma 6.2. *If r is like in (6.9) then we have $\phi^r = d\psi^\rho$.*

Proof: Given $(x,y)\in E_N$ it belongs to only two elements of F_N. Moreover one is clockwise oriented and the other one is anticlockwise oriented. Let us call f_+ the element of F'_N such that $(x,y)\in f_+$ and f_- the element of F'_N such that $(x,y)\in f^c_-$. Then we have

$$\phi^r(x,y)=\Big(\rho([C_{f_+}])-\rho([C_{f^c_+}])\Big)-\Big(\rho([C_{f_-}])-\rho([C_{f^c_-}])\Big)=\psi^\rho(f_+)-\psi^\rho(f_-)\,.$$

Recalling (2.27) and (6.10) we obtain the statement of the lemma. □

The following Remarks will not be used during our proofs but will be useful in the following.

Remark 6.3. (*Duality*) We recall a well known duality relationship. To the 2 dimensional discrete torus (V_N,E_N) we associate a dual discrete torus $(\tilde V_N,\tilde E_N)$ defined as follows. The vertices of the dual graph are the elements of $\mathbb{T}^2$ having coordinates $x+\frac{1}{2N}(e^{(1)}+e^{(2)})$ with $x\in V_N$. Note that every element of $\tilde V_N$ is the center of a cell of the original cellular decomposition of $\mathbb{T}^2$. The set of oriented edges $\tilde E_N$ is constituted by the pairs (v,w) such that $v,w\in\tilde V_N$ and moreover $d(v,w)=1/N$. It is possible to define a duality map D. This map is defined both on E_N with image on $\tilde E_N$ and on $\tilde E_N$ with image in E_N. This map is injective and satisfies the involution property $D^2=-\mathbb{I}$. We use the same symbol D, both when it acts on E_N or in $\tilde E_N$, since it can be defined easily in the same way. Any element (x,y), both of E_N and of $\tilde E_N$ can be naturally represented by an arrow exiting from x and entering in y. The element $D(x,y)$ is defined as the unique element of, either $\tilde E_N$ or E_N, whose representing arrow is obtained rotating counterclockwise of $\frac{\pi}{2}$ the arrow representing (x,y) around its middle point. The dual map can be naturally extended to act on discrete vector fields. More precisely given for example $\phi\in\Lambda^1$ we define $D\phi\in\tilde\Lambda^1$ as

$$(D\phi)\,(w,z):=\phi(x,y)\,,$$

where (x,y) is the unique element of E_N such that $D(x,y)=(w,z)$. Clearly we called $\tilde\Lambda^1$ the set of discrete vector fields on the dual torus. According to these definition the notion of circulation and rotation free are dual to each other. More precisely given a discrete vector field ϕ whose discrete divergence is zero in a vertex, then $D\phi$ will have zero circulation around the corresponding dual face. Conversely given a discrete vector field ϕ that satisfies the condition of zero rotation around a face then $D\phi$ will satisfy the condition of zero discrete divergence in the corresponding dual vertex.

Remark 6.4. Given $\psi\in\Lambda^2$ it is simple to compute $\phi=d\psi$ by

$$\phi(x,y)=\psi(f_+)-\psi(f_-)\,. \tag{6.11}$$

In terms of the dual graph it can be interpreted in the following way. Consider $\tilde\psi$ the element of $\tilde\Lambda_0$ that associates to any vertex of $\tilde V_N$ the value $\psi(f)$ where $f\in F'_N$ is the oriented dual face of the fixed vertex. Then (6.11) can be written as

$$D\phi=\delta\tilde\psi\,. \tag{6.12}$$

In extended form (6.12) is

$$
\begin{cases}
\phi(x, x+e^{(1)}/N) = \\
\qquad\qquad \delta\tilde{\psi}(x+e^{(1)}/(2N)-e^{(2)}/(2N), x+e^{(1)}/(2N)+e^{(2)}/(2N))\,, \\
\phi(x, x+e^{(2)}/N) = \\
\qquad\qquad -\,\delta\tilde{\psi}(x-e^{(1)}/(2N)+e^{(2)}/(2N), x+e^{(1)}/(2N)+e^{(2)}/(2N))\,.
\end{cases}
\tag{6.13}
$$

Conversely given $\phi \in d\Lambda^2$, the determination of a ψ such that $\phi = d\psi$ requires a non local computation. More precisely using (6.12) we determine $\delta\tilde{\psi} \in \tilde{\Lambda}_1$ and then we can compute

$$
\psi(f) - \psi(f') = \sum_{i=1}^{n} \delta\tilde{\psi}(x^i, x^{i+1})\,, \tag{6.14}
$$

where $(x^i, x^{i+1}) \in \tilde{E}_N$ and x^0 is dual to f' and x^n is dual to f.

Proof of Theorem 2.15 Given $\phi \in d\Lambda^2$ there exists a $\psi \in \Lambda^2$ such that $\phi = d\psi$. Since the kernel of the boundary operator d coincides with the constant 2 forms, the elements $\psi' \in \Lambda^2$ such that $d\psi' = \phi$ are exactly of the type $\psi + c$ where c is an arbitrary constant.

Let $r \in W$ having a decomposition like (6.9) and such that $\phi^r = \phi$. By Lemma (6.2) we have $\psi^\rho = \psi + c$ for some constant c. This means

$$
\rho([C_f]) - \rho([C_{f^c}]) = \psi(f) + c\,, \qquad f \in F'_N\,. \tag{6.15}
$$

Recalling (2.25) we have that

$$
s := r - r^\phi \tag{6.16}
$$

is the symmetric part of r and belongs to $R(0)$. We want to characterize which are the elements $s \in R(0)$ that can be obtained in (6.16) when $r \in R^e \cap R(\phi)$.

The more general positive solution to (6.15) is

$$
\rho([C_f]) = [(\psi + c)(f)]_+ + m(f)\,, \qquad f \in F_N\,, \tag{6.17}
$$

where $m(f)$ are arbitrary non-negative numbers such that $m(f) = m(f^c)$. In (6.17) $\psi + c$ is the element of Λ^2 obtained by the sum of ψ and the constant 2-chain $c \in \Lambda^2$ defined by $c(f) = c$ for any $f \in F'_N$. In particular (6.17) means

$$
\rho([C_f]) = [\psi(f) + c]_+ + m(f)\,, \qquad \rho([C_{f^c}]) = [-\psi(f) - c]_+ + m(f)\,, \qquad f \in F'_N\,.
$$

Note that we can write

$$
\sum_{f \in F_N} m(f) r^{[C_f]} = \sum_{(x,y) \in E'_N} \left(\sum_{\{f \in F_N \,:\, (x,y) \in f\}} m(f) \right) r^{[C_{\{x,y\}}]}\,. \tag{6.18}
$$

Putting (6.17) in (6.9) using (6.18) we get

$$
s = \sum_{(x,y) \in E'_N} \rho'([C_{\{x,y\}}]) r^{[C_{\{x,y\}}]} + \sum_{f \in F_N} [(\psi + c)(f)]_+ r^{[C_f]} - r^\phi\,, \tag{6.19}
$$

where

$$
\rho'([C_{\{x,y\}}] := \rho([C_{\{x,y\}}]) + \sum_{\{f \in F_N \,:\, (x,y) \in f\}} m(f)\,.
$$

Since the positive numbers $m(f)$ and $\rho([C_{x,y}])$ are arbitrary, for any fixed $c \in \mathbb{R}$ (6.19) says that $s \geq s(c)$ where

$$s(c) := \sum_{f\in F_N} [(\psi + c)(f)]_+ r^{[C_f]} - r^{\phi} .$$

Recall that given $(x,y) \in E'_N$ there exist only two elements of F_N that contain (x,y). One is clockwise oriented and the other one is anticlockwise oriented. As before we call f_+ the element of F'_N such that $(x,y) \in f_+$ and f_- the element of F'_N such that $(x,y) \in f_-^c$. If $c \in \mathbb{R}$ is fixed, the condition $s \geq s(c)$ is equivalent to require for any $(x,y) \in E'_N$ (the condition for (y,x) will automatically be satisfied)

$$s(x,y) \geq [\psi(f_+) + c]_+ + [-\psi(f_-) - c]_+ - [\phi(x,y)]_+ . \tag{6.20}$$

Since $\phi = d\psi$ we have also

$$\phi(x,y) = \psi(f_+) - \psi(f_-) . \tag{6.21}$$

It holds the following identity

$$Z(a,b) := [a]_+ + [-b]_+ - [a-b]_+ = \begin{cases} \min\{|a|, |b|\} & \text{if } \operatorname{sg}(a) = \operatorname{sg}(b) , \\ 0 & \text{if } \operatorname{sg}(a) \neq \operatorname{sg}(b) , \end{cases}$$

where a and b are real numbers, $\operatorname{sg}(\cdot)$ is the sign function that associate to any real number its sign and the first identity is the definition of the function Z. Putting (6.21) in (6.20) we get for any $(x,y) \in E'_N$

$$s(x,y) \geq Z(\psi(f_+) + c, \psi(f_-) + c) .$$

It is simple to check that

$$Z(\psi(f_+) + c, \psi(f_-) + c) = d(0, c + I(x,y)) ,$$

where d is the Euclidean distance on the real line and $I(x,y)$ is the closed interval of the real line $[\min\{\psi(f_+), \psi(f_-)\}, \max\{\psi(f_+), \psi(f_-)\}] := [i_1(x,y), i_2(x,y)]$.

Let us call

$$S(c) := \{s \in R(0) \,:\, s \geq s(c)\} .$$

We showed that

$$S(c) = \{s \in R(0) \,:\, s(x,y) \geq d(0, c + I(x,y)) , (x,y) \in E'_N\} .$$

Since c is arbitrary we get that when $r \in \mathbb{R}^e \cap R(\phi)$ the corresponding symmetric part s in (6.16) satisfies

$$s \in \cup_{c\in\mathbb{R}} S(c). \tag{6.22}$$

We now write this condition in a simpler form. Indeed we will now show that $s \in R(0)$ belongs to $\cup_{c\in\mathbb{R}} S(c)$ if and only if

$$\cap_{(x,y)\in E'_N} [-i_2(x,y) - s(x,y), -i_1(x,y) + s(x,y)] \neq \emptyset . \tag{6.23}$$

Indeed it holds

$$s(x,y) \geq d(0, c + I(x,y)) \qquad \Longleftrightarrow \qquad c \in [-i_2(x,y) - s(x,y), -i_1(x,y) + s(x,y)] . \tag{6.24}$$

If $s \in \cup_c S(c)$ then, using (6.24), there exists a $c^* \in \mathbb{R}$ such that

$$c^* \in [-i_2(x,y) - s(x,y), -i_1(x,y) + s(x,y)] , \qquad \forall (x,y) \in E'_N ,$$

and (6.23) holds. Conversely assume that (6.23) holds and call c^* an element of the non empty intersection. Then still using (6.24) we get $s \in S(c^*)$.

Since closed intervals are compact convex sets of $\mathbb{R}^1$, we can write condition (6.23) in a simpler form using the Helly's Theorem 3.3. We apply it in the special case of dimension one. We obtain that condition (6.23) holds if and only if for any pair (x, y) and (x', y') of elements of E'_N it holds

$$[-i_2(x,y) - s(x,y), -i_1(x,y) + s(x,y)] \cap \\ [-i_2(x',y') - s(x',y'), -i_1(x',y') + s(x',y')] \neq \emptyset\,.$$

This is equivalent to require

$$d\left(I(x,y), I(x',y')\right) \leq s(x,y) + s(x',y')\,. \tag{6.25}$$

Since (6.25) has to be satisfied for any pair of elements of E'_N we deduce that condition (6.22) is equivalent to

$$s \in \mathcal{P}\left(\{I(x,y)\}_{(x,y)\in E'_N}\right)\,.$$

□

Let us briefly discuss how the above results can be used on a problem concerning an infinite graph. Consider the infinite graph having vertices $\mathbb{Z}^2$ and edges coinciding with the pairs of vertices (x, y) such that $d(x, y) = 1$. Consider also a periodic weight r, i.e. a weight such that there exist N_1 and N_2 integer numbers such that

$$r(x,y) = r(x + (N_1,N_2), y + (N_1,N_2))\,, \qquad \forall (x,y)\,.$$

Then the problem of finding a periodic cyclic decomposition of r, i.e. a decomposition such that $\rho([C]) = \rho([\tau_{(N_1,N_2)}C])$, is strictly related to the problem of finding a decomposition like (2.21) or (2.22) for a finite torus with $N_1 \times N_2$ vertices. Indeed in the case of elementary decompositions (2.22) the two problems are equivalent. Apart from an irrelevant dilation, an elementary cycle on the discrete torus can be naturally interpreted as an elementary cycle on the infinite graph. It is easy to see that if (2.22) holds on the discrete torus, then

$$\sum_{[C]\in\mathcal{C}^e}\sum_{x\in\mathbb{Z}^d}\rho([C])r^{[\tau_x C]}\,, \tag{6.26}$$

is a periodic cyclic decomposition on the infinite graph with elementary cycles. In (6.26) $\mathcal{C}^e$ are the equivalence classes of elementary cycles on the discrete torus. Conversely every elementary cycle on the infinite graph can be projected onto an elementary cycle of the torus. By periodicity of the weight r you get a decomposition like (2.22). In the case of decompositions (2.21) the relation among the two problems is more subtle and we will not discuss here.

6.3. *Other topologies.* The proofs of Theorems 2.16 and 2.17 are very similar to the one of 2.15. We give only a sketch.

Proof of Theorem 2.16 The proof of this Theorem follows closely the lines of reasoning of the proof of Theorem 2.15. In particular the two basic ingredients are the following. The first one is that every edge $(x, y) \in E_N$ belongs to only two elements of F_N. The second one is that we can choose orientations in F'_N in such a way that every pair of elements on F'_N is oriented in agreement. As a consequence given $\phi \in d\Lambda^2$ the two chain ψ such that $\phi = d\psi$ is defined only up an additive constant. □

Proof of Theorem 2.17 The proof of this Theorem uses the same type of arguments of Theorem 2.15. The difference is that given $\phi \in d\Lambda^2$ then the two chain ψ such that $\phi = d\psi$ is uniquely determined. This is due to the fact that the surface is not orientable. For any $(x, y) \in E'_N$ such that there exist $f_+, f_- \in F'_N$ such that $(x, y) \in f_+$ and $(y, x) \in f_-$ the we get the constraint $s(x, y) \geq d(0, I(x, y))$. For an $(x, y) \in E'_N$ for which for example there exist $f_1, f_2 \in F'_N$ such that $(x, y) \in f_i$, $i = 1, 2$, instead of (6.20) and (6.21) we get

$$s(x, y) \geq [\psi(f_1)]_+ + [\psi(f_2)]_+ - [\psi(f_1) + \psi(f_2)]_+ .$$

Since for any pair of real numbers a, b it holds

$$[a]_+ + [b]_+ - [a + b]_+ = \begin{cases} \min\{|a|, |b|\} & \text{if } \ \mathrm{sg}(a) \neq \mathrm{sg}(b)\,, \\ 0 & \text{if } \ \mathrm{sg}(a) = \mathrm{sg}(b)\,, \end{cases}$$

we deduce in this case $s(x, y) \geq d(0, J(x, y))$. The Theorem follows. □

7. Applications

We start this section discussing a very general example. First we show how to discretize in a natural way a smooth divergence free vector field by a divergence free discrete vector field. Then we apply the results of Theorem 2.15. In dimension $d = 2$, the continuous version of the Hodge decomposition is the following. Let $u \in \mathbb{T}^2$ and $F = (F_1(u), F_2(u))$ be a smooth vector field on $\mathbb{T}^2$. The Hodge decomposition says that there exist two smooth functions f and ψ such that

$$F = \nabla f + \nabla^{\perp}\psi + a_1(1, 0) + a_2(0, 1)\,. \tag{7.1}$$

In the above formula ∇ is the gradient, $\nabla^{\perp}\psi(u) := (\psi_{u_2}(u), -\psi_{u_1}(u))$ is the orthogonal gradient and $a_i = \int_{\mathbb{T}^2} F_i(v)dv$. The definition of the orthogonal gradient can be seen as a continuum version of formula (6.13) in Remark 6.4. Decomposition (7.1) says that the continuous version of the discrete vector fields on $d\Lambda^2$ are the vector fields on $\mathbb{T}^2$ of the form

$$F = (\psi_{u_2}, -\psi_{u_1})\,, \tag{7.2}$$

for a suitable smooth function ψ. Clearly a vector field of the type (7.2) has zero divergence since

$$\nabla \cdot \nabla^{\perp}\psi = \psi_{u_2 u_1} - \psi_{u_1 u_2} = 0\,. \tag{7.3}$$

To any smooth vector field of the type (7.2) we can associate a discrete vector field $\phi_N \in d\Lambda^2$ on the discrete torus defined as

$$\begin{aligned} \phi_N(x, x + e^{(1)}/N) &:= \int_{x_2 - \frac{1}{2N}}^{x_2 + \frac{1}{2N}} \psi_{u_2}\left(x_1 + \frac{1}{2N}, y\right) dy \\ &= \psi\left(x_1 + \frac{1}{2N}, x_2 + \frac{1}{2N}\right) - \psi\left(x_1 + \frac{1}{2N}, x_2 - \frac{1}{2N}\right)\,, \end{aligned} \tag{7.4}$$

and

$$\begin{aligned} \phi_N(x, x + e^{(2)}/N) &:= -\int_{x_1 - \frac{1}{2N}}^{x_1 + \frac{1}{2N}} \psi_{u_1}\left(y, x_2 + \frac{1}{2N}\right) dy \\ &= \psi\left(x_1 - \frac{1}{2N}, x_2 + \frac{1}{2N}\right) - \psi\left(x_1 + \frac{1}{2N}, x_2 + \frac{1}{2N}\right)\,. \end{aligned} \tag{7.5}$$

In (7.4) and (7.5) we fixed the values of the discrete vector field coinciding with the fluxes of the continuous vector field $\nabla^{\perp}\psi$ respectively across the segments

$$c\left(x+e^{(1)}/(2N)-e^{(2)}/(2N)\right)+(1-c)\left(x+e^{(1)}/(2N)+e^{(2)}/(2N)\right)\,,\quad c\in[0,1]\,,$$

and

$$c\left(x-e^{(1)}/(2N)+e^{(2)}/(2N)\right)+(1-c)\left(x+e^{(1)}/(2N)+e^{(2)}/(2N)\right)\,,\quad c\in[0,1]\,,$$

with associated normal vectors respectively $e^{(1)}$ and $e^{(2)}$. According to these definitions it is easy to find a two chain $\psi_N\in\Lambda^2$ such that $\phi_N=d\psi_N$. Given $f\in F'_N$ of the form $f=(x,x+e^{(1)}/N,x+e^{(1)}/N+e^{(2)}/N,x+e^{(2)}/N,x)$ we have

$$\psi_N(f):=\psi\left(x+e^{(1)}/(2N)+e^{(2)}/(2N)\right)\,.$$

The condition of zero divergence for ϕ_N can be also directly derived by (7.3).

We us call

$$M:=\max_{u,v\in\mathbb{T}^2}|\psi(u)-\psi(v)|\,.$$

Fix a weight $r:=r^{\phi_N}+s\in R(\phi_N)$ with $s(x,y)\geq\frac{M}{2}$ for any $(x,y)\in E_N$. Since $d(I(x,y),I(x',y'))\leq M$ for any pair of elements of E'_N then

$$s(x,y)+s(x',y')\geq M\geq d(I(x,y),I(x',y'))\,,$$

so that $r\in R^e$.

As an application of this example we discuss a direct consequence for a random walk in a random environment. Consider the infinite graph with vertices $\mathbb{Z}^2$ and oriented edges coinciding with pairs of nearest neighbor vertices. Let ψ be a smooth periodic function on $\mathbb{R}^2$ with integer periods N_1 and N_2, i.e. such that $\psi(u)=\psi(u+(N_1,N_2))$ for any $u\in\mathbb{R}^2$. Let $U=(U_1,U_2)$ be a random variable uniform on $[0,N_1]\times[0,N_2]\subseteq\mathbb{R}^2$. We construct the random discrete vector field $\phi(\omega)$ defined by

$$\phi(x,x+e^{(1)};\omega):=\tau_{U(\omega)}\left[\psi\left(x_1+\frac{1}{2},x_2+\frac{1}{2}\right)-\psi\left(x_1+\frac{1}{2},x_2-\frac{1}{2}\right)\right]\,,$$
$$\phi(x,x+e^{(2)};\omega):=\tau_{U(\omega)}\left[\psi\left(x_1-\frac{1}{2},x_2+\frac{1}{2}\right)-\psi\left(x_1+\frac{1}{2},x_2+\frac{1}{2}\right)\right]\,.$$

Clearly $\phi(\omega)\in d\Lambda^2$ for any ω and moreover the law of ϕ is invariant under translations. Let also $\left\{U_{\{x,y\}}\right\}_{(x,y)\in E'_N}$ be a collection of i.i.d. random variables taking values on an interval $[a,b]$, $a,b>0$ and independent from U. Here E'_N are oriented edges of a $N_1\times N_2$ discrete torus of mesh size $N=1$. We construct the random weights

$$r(x,y;\omega):=Z(x,\omega)^{-1}\left(r^{\phi(\omega)}(x,y)+U_{\{x,y\}}\right)\,,\tag{7.6}$$

where $Z(x,\omega)$ is the normalization constant that guarantees the validity of (2.12). Note that $U_{\{x,y\}}=U_{\{x+N_1,y+N_2\}}$ so that $r(\omega)$ is a periodic environment. Let

$$M:=\max_{u,v\in[0,N_1]\times[0,N_2]}|\psi(u)-\psi(v)|\,.$$

If $a\geq\frac{M}{2}$ then the hypotheses in Deuschel and Kösters (2008) are satisfied and the discrete time random walk in random environment having transition probabilities determined by (7.6) satisfies a quenched Central Limit Theorem since it admits a periodic cyclic decomposition into elementary cycles. Since the random environment is periodic the result obtained is rather simple but the argument can be generalized to the non periodic case Gabrielli and Valente (2012).

We end the paper with some comments. When $r \in W$ is given, to determine if $r \in R^e$ we need to compute the associated discrete vector field ϕ and then a two chain ψ such that $\phi = d\psi$. Once ψ is computed we determine the corresponding polyhedron $\mathcal{P}$ and finally we can check if the symmetric part s of r belongs or not to $\mathcal{P}$. As discussed in Remark 6.4 the determination of ψ requires a non local computation. It is then useful to have some simple sufficient conditions. For any pair (x, y) and (x', y') of elements of E'_N we have

$$d(I(x,y), I(x',y')) \leq \max_{f,f' \in F'_N} |\psi(f) - \psi(f')| \,. \tag{7.7}$$

If we are able to estimate from above the right hand side of (7.7) by a constant M and $s(x,y) \geq M/2$ for any $(x,y) \in E'_N$ then $r \in R^e$. Consider $\tilde{\psi}$ the element of $\tilde{\Lambda}_0$ associated to ψ as in Remark 6.4 and use (6.14). Consider an unoriented graph having as vertices $\tilde{V}_N$ and as unoriented edges $\{x, y\}$ with $(x,y) \in \tilde{E}'_N$. To every unoriented edge we associate the weight $w(x,y) := |\phi^r(D(x,y))|$. The weight of a path is the sum of the weights of its edges. These weights introduce a metric structure on the graph. The distance among two vertices is the minimal weight among all paths connecting the two vertices. An upper bound to the right hand side of (7.7) is then the diameter of this unoriented graph. It can be estimated for example as the weight of any spanning tree.

Acknowledgements. D.G. thanks the Department of Physics of the University “La Sapienza” for the kind hospitality. We thank an anonymous referee for a very careful reading of the paper.

References

J. Bang-Jensen and G. Gutin. *Digraphs. Theory, Algorithms and Applications.* Springer-Verlag (2009). Springer Monographs in Mathematics, second edition MR2472389.

L. Bertini, A. Faggionato and D. Gabrielli. In preparation (2012).

B. Bollobás. *Modern graph theory.* Springer-Verlag, New York, (1998). Graduate Texts in Mathematics, **184** MR1633290.

M. Desbrun, E. Kanso and Y. Tong. Discrete differential forms for computational modeling. *Discrete differential geometry, Oberwolfach Semin.* **38**, 287–324 (2008). MR2405673.

J. D. Deuschel and H. Kösters. The quenched invariance principle for random walks in random environments admitting a bounded cycle representation. *Ann. Inst. Henri Poincaré Probab. Stat.* **44** (3), 574–591 (2008). MR2451058.

R. Diestel. *Graph theory.* Springer, Heidelberg (2010). Fourth edition. Graduate Texts in Mathematics **173** MR2744811.

D. Gabrielli and C. Valente. In preparation (2012).

P. M. Gruber. *Convex and discrete geometry.* Springer, Berlin (2007). Grundlehren der Mathematischen Wissenschaften **336** MR2335496.

A. F. Karr. Extreme points of certain sets of probability measures with applications. *Math. Oper. Res.* **8** (1), 74–85 (1983). MR0703827.

T. Komorowski, C. Landim and S. Olla. *Fluctuations in Markov Processes: Time Symmetry and Martingale Approximation*. Springer, Berlin (2012). Grundlehren der Mathematischen Wissenschaften **345**.

T. Komorowski and S. Olla. A note on the central limit theorem for two-fold stochastic random walks in a random environment. *Bull. Polish Acad. Sci. Math.* **51** (3), 217–232 (2003). MR1990811.

L. Lovasz. Discrete analytic functions: An exposition. *Surv. Differ. Geom.* **IX**, 241–273 (2004). MR2195410.

P. Mathieu. Carne-Varopoulos bounds for centered random walks. *Ann. Probab.* **34** (3), 987–1011 (2006). MR2243876.

C. Mercat. Discrete Riemann surfaces and the ising model. *Comm. Math. Phys.* **218** (1), 177–216 (2001). MR1824204.

R. R. Phelps. *Lectures on Choquet's theorem*. Springer-Verlag, Berlin (2001). Second edition. Lecture Notes in Mathematics **1757** MR1835574.

H. Rubin and O. Wesler. A note on convexity in Euclidean n-space. *Proc. Amer. Math. Soc.* **9**, 522–523 (1958). MR0097772.

M. Sued. Regularity properties of the diffusion coefficient for a mean zero exclusion process. *Ann. Inst. H. Poincaré Probab. Statist.* **41** (1), 1–33 (2005). MR2110444.

S. R. S. Varadhan. Self-diffusion of a tagged particle in equilibrium for asymmetric mean zero random walk with simple exclusion. *Ann. Inst. H. Poincaré Probab. Statist.* **31** (1), 273–285 (1995). MR1340041.

H. von Weizsäcker and G. Winkler. Integral representation in the set of solutions of a generalized moment problem. *Math. Ann.* **246** (1), 23–32 (1979/80). MR0554129.

H. von Weizsäcker and G. Winkler. Noncompact extremal integral representations: some probabilistic aspects pages 115–148 (1980). Functional analysis: surveys and recent results, II (Proc. Second Conf. Functional Anal., Univ. Paderborn, Paderborn, 1979) Notas Mat. **68**, MR0565403.

G. Winkler. Extreme points of moment sets. *Math. Oper. Res.* **13** (4), 581–587 (1988). MR0971911.