ESTIMATES ON THE DEPTH OF THE ASSOCIATED GRADED RING

ANNA GUERRIERI
(joint work with M.E. Rossi)

Dipartimento di Matematica
Universitá di L'Aquila
67100 L'Aquila, Italy

Exeter - 13-16 August , 1998

NOTATION

Let $(R, \mathfrak{m})$ be a Noetherian local ring[a], $\dim R = d > 0$ and $I \subseteq R$ a proper ideal.

The most common among the blow-up algebras are the *Rees algebra of I*

$$R[It] := R \oplus It \oplus I^2t^2 \cdots \subseteq R[t]$$

and the *associated graded ring of I*

$$\mathrm{gr}_I(R) := R/I \oplus I/I^2 \oplus I^2/I^3 \oplus \cdots \cong R[It]/IR[It].$$

[a] Later we will assume R to have some additional properties (such as: Cohen–Macaulay, with infinite residue field, etc.)

GENERAL PROBLEM

Since the work of Northcott and Rees a central problem in the theory of blow-up algebras has been to understand the Hilbert function[a] of an $\mathfrak{m}$-primary ideal of a Cohen-Macaulay (CM) local ring in relation with its Hilbert coefficients and the arithmetical invariants of the corresponding associated graded ring.

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[a]The *Hilbert function of I* is $H_I(n) := \lambda(I^n/I^{n+1})$. $H_I(n)$ is a polynomial function for large n.

“ERRATIC” BEHAVIOUR

Despite the fact that the Hilbert function of a graded CM k-algebra is well understood, little is known in the local case.

In fact, even if a given local ring is CM, its associated graded ring can be very far from being CM.

- Let $R = k[\![X, Y]\!]$ with k a field, let $I = (X^4Y, X^6, XY^5, Y^6)R$, then $\operatorname{depth} \operatorname{gr}_I(R) = 0$.
- Let $R = k[\![t^6, t^7, t^{15}]\!]$ with k a field, let $\mathfrak{m} = (t^6, t^7, t^{15})R$, then $\operatorname{depth} \operatorname{gr}_{\mathfrak{m}}(R) = 0$.

A TWOFOLD GAME[a]

Wanting to get information on the Hilbert function of $\mathrm{gr}_I(R)$ "*the game is twofold: try to get information on the Hilbert function in spite of the bad properties of* $\mathrm{gr}_I(R)$ *and try to find properties of* $(R, \mathfrak{m})$ *(and* I*) which lead to reasonable associated graded rings*".

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[a] J. Sally, *Reductions, local cohomology and Hilbert functions of local rings.* In R. Sharp, editor, *Commutative Algebra: Durham 1981*, pages 231-241. Cambridge University Press, 1982

A QUESTION ON THE ASSOCIATED GRADED RING

From now on the context will be the following:

$(R, \mathfrak{m})$ is a CM local ring of positive dimension d,

$R/\mathfrak{m}$ is infinite,

$I \subseteq R$ is $\mathfrak{m}$-primary.

In this context, $\dim \operatorname{gr}_I(R) = d$ and $\mathcal{M} = \mathfrak{m}/I \oplus I/I^2 \oplus I^2/I^3 \oplus \cdots$ is the unique maximal homogeneous ideal of $\operatorname{gr}_I(R)$. Define $\operatorname{depth} \operatorname{gr}_I(R) := \operatorname{depth} (\operatorname{gr}_I(R))_{\mathcal{M}}$.

QUESTION

Which are the structures that affect the depth of $\operatorname{gr}_I(R)$?

A KEY NOTION: THE REDUCTION OF AN IDEAL[a]

The reductions of an ideal I give a “natural pathway to the properties of I, with the advantage of having fewer generators”.

- An ideal $J \subseteq I$ is a *reduction of* I if $I^{r+1} = JI^r$ for some $r \geq 0$. A *minimal reduction of* I is a reduction of I which is minimal wrt containment.

- Minimal reductions always exist. In our context, if $J \subseteq I$ is a minimal reduction, then $J = (x_1, \ldots, x_d)$ where $\underline{x}$ is a regular sequence in R.

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[a] D.G. Northcott and D. Rees, *Reductions of ideals in local rings*, Proc. Cambridge Philos. Soc. **50** (1954), 145–158.

THE COHEN-MACAULAY CASE[a]

The associated graded ring of I is CM, i.e. $\operatorname{depth} \operatorname{gr}_I(R) = d$, if and only if there exists a minimal reduction J of I such that

$$J \cap I^{n+1} = JI^n$$

for all $n \geq 1$.

[a] P. Valabrega and G. Valla, *Form rings and regular sequences*, Nagoya Math. J. **72** (1978), 91–101.

AN IMPORTANT MODULE

The module

$$\bigoplus_{n\geq 1} \frac{J\cap I^{n+1}}{JI^n}$$

is the first homology of the Koszul complex [a]

$$C_\bullet^J := \mathcal{K}_\bullet(x_1t, \ldots, x_dt; M_I)$$

where M_I denotes the graded $R[It]$-module $R[t]/R[It]$.

It can be shown that

$$\operatorname{depth} \operatorname{gr}_I(R) = \min \{j | H_{d-j}(\mathcal{C}_\bullet^J(n)) \neq 0 \text{ for some } n\}.$$

[a] This complex and its graded parts were first analyzed by Marley inspired by ideas that can be traced back to works of Kirby and Mehran and Huneke.

REPHRASING THE PROBLEM

Given the last observation about the depth of $\mathrm{gr}_I(R)$ the theorem of Valabrega and Valla follows immediately, in fact it may be rephrased as follows:

> The associated graded ring of I is CM if and only if there is a minimal reduction J of I such that $H_1(C^J_\bullet) = 0$.

MOVING AWAY FROM THE EXTREMAL CASE[a]

If $\lambda(H_1(C_\bullet^J)) = t$ where $0 \leq t \leq 2$ then $\operatorname{depth} \operatorname{gr}_I(R) \geq d - t$.

QUESTION

Does $\lambda(H_1(C_\bullet^J)) = t$ imply $\operatorname{depth} \operatorname{gr}_I(R) \geq d - t$ for every $t \geq 1$?

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[a] A. Guerrieri, *On the depth of the associated graded ring of an m-primary ideal*, J. of Algebra **167** (1994), no.3, 745–757.

A. Guerrieri, *On the depth of the associated graded ring*, Proc. of the AMS **123** (1995), no. 1, 11-20.

H. Wang, *Hilbert coefficients and the associated graded ring*, preprint.

C. Polini, work in progress.

CASE $t = 3$

In the paper "*Estimates on the depth of the associated graded ring*" (to appear in J. of Algebra) we prove the following:

Theorem *Let $(R, \mathfrak{m})$ be a CM local ring of dimension $d \geq 4$. Let I be an $\mathfrak{m}$-primary ideal of R and $J \subseteq I$ a minimal reduction of I such that*

$$\lambda\left(\frac{J \cap I^2}{JI}\right) = 3 \quad \textit{and} \quad J \cap I^n = JI^{n-1} \quad \textit{for all} \quad n \geq 3.$$

If R/I is a Gorenstein local ring, then $\operatorname{depth} \operatorname{gr}_I(R) \geq d - 3$.

TOOLS 1

Recently, Huckaba and Marley[a] extended the complex $\mathcal{C}_\bullet^J$ to the case of Hilbert filtrations and gave the following formula for the Hilbert coefficients of a Hilbert filtration $\mathcal{G} = \{G_n\}$ for all $i \geq 1$

$$e_i(\mathcal{G}) = \sum_{n \geq i} \binom{n-1}{i-1} [\lambda(G_n/JG_{n-1}) - \sum_{i \geq 2}^{d} (-1)^i \lambda(H_i(\mathcal{C}_\bullet^J(n, \mathcal{G})))].$$

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[a] S. Huckaba and T. Marley, *Hilbert coefficients and the depths of associated graded rings*, J. of London Math. Soc. **56** (1997), no. 2, 64–76.

TOOLS 2

We apply the formula presented before to compare the Hilbert coefficients that arise from three different filtrations:

- The I-adic filtration $\{I^n\}_{n\geq 1}$;
- The Ratliff-Rush filtration $\{\widetilde{I^n}\}_{n\geq 1}$ where $\widetilde{I^n} := \bigcup_{k\geq 1}(I^{n+k} : I^k)$;
- $\{\overline{I}^n\}_{n\geq 1}$ where $\overline{I}^n := (I^n + (x))/(x)$ and x is one of the minimal generators of J for which $I^n : x = I^{n-1}$ for $n \gg 0$ (i.e. x is superficial).

TOOLS 3

We reduce ourselves to study the case of dimension 4 by using the Principle of Sally's descent: *If J denotes the ideal generated by a superficial sequence $x_1, \ldots, x_r$, $\overline{I}$ the ideal I/J and $S = R/J$, then* $\operatorname{depth} \operatorname{gr}_I(R) \geq r+1$ *if and only if* $\operatorname{depth} \operatorname{gr}_{\overline{I}}(S) \geq 1$.

We use various criteria that allow to determine the non-negativity of the depth of $\operatorname{gr}_I(R)$:

- $\operatorname{depth} \operatorname{gr}_I(R) \geq 1$ if and only if $\widetilde{I^n} = I^n$ for all $n \geq 1$.
- $\operatorname{depth} \operatorname{gr}_I(R) \geq 1$ if and only if $H_d(\mathcal{C}^J_\bullet) = 0$.
- $\operatorname{depth} \operatorname{gr}_I(R) \geq 1$ if and only if $e_d(I) = e_d(\overline{I})$.