Functional Analysis in Applied Mathematics and Engineering:
Final exam: part 1 - 03/02/2017

FULL NAME:____

MATRICOLA:____

(1) (i) Let (K,d) be a compact metric space and let $C(K)$ be the linear space of continuous functions $f : K \to \mathbb{R}$.

(a) Define the $\|\cdot\|_\infty$ norm on $C(K)$. [1]

(b) Let $\mathcal{F} \subset C(K)$. Define the notion of equicontinuity for $\mathcal{F}$. [1]

(c) Let $\mathcal{F} \subset C(K)$ be bounded and equicontinuous. Prove that $\mathcal{F}$ is precompact in the normed space $(C(K), \|\cdot\|_\infty)$. [5]

(ii) State (without proof) the *contraction mapping theorem.* [2]

(iii) For any $n \in \mathbb{N}$, let $f_n : [0,\infty) \to \mathbb{R}$ be the sequence of functions defined by

$$f_n(x) = nxe^{-nx}.$$

(a) Show that $f_n \to 0$ pointwise in $[0,\infty)$ as $n \to +\infty$. [3]

(b) Is $\{f_n\}_n$ uniformly convergent on $[0,\infty)$? Justify your answer in detail. [3]

(2) (i) Given a measurable function $f : \mathbb{R}^d \to [0,+\infty]$, provide the definition of the Lebesgue integral of f. [2]

(ii) State (without proof) Beppo Levi's theorem. [2]

(iii) Let $p \in [1,+\infty)$. By invoking various results proven in the course, prove that compactly supported continuous functions on $\mathbb{R}^d$ are dense in $L^p(\mathbb{R}^d)$. [5]

(iv) For any $n \in \mathbb{N}$, let $f_n : [0,1] \to \mathbb{R}$ be the sequence of functions defined by

$$f_n(x) = \frac{1}{1+n\sqrt{x}}.$$

(a) Prove that $f_n \to 0$ in $L^1([0,1])$ as $n \to +\infty$. [3]

(b) Is $f_n \to 0$ in $L^\infty([0,1])$? Justify your answer. [3]