Functional Analysis in Applied Mathematics and Engineering:

Final exam: part 2 - 03/02/2017

Full name:____

Matricola:____

(3) (i) Let X be a Banach space and let $\mathcal{B}(X)$ be the linear space of bounded linear operator from X to X.

(a) Define the *operator norm* on $\mathcal{B}(X)$. [1]

(b) Prove that $\mathcal{B}(X)$ is a Banach space if equipped with the operator norm. [4]

(ii) Let (T_n) be a sequence in $\mathcal{B}(X)$.

(a) Define the notion of *uniform convergence* for (T_n). [1]

(b) Define the notion of *strong convergence* for (T_n). [1]

(iii) Let $X = C([0,1])$ and let $(T_n) \subset \mathcal{B}(X)$ be defined by

$$(T_n f)(x) = \int_0^1 \sin(n\pi x) f(x) dx.$$

(a) Prove that T_n converges strongly to zero. [3]

(b) Prove that T_n does not converge uniformly to zero. [3]

(iv) Let (x_n) be a sequence in a Banach space X. Define the notion of weak convergence for (x_n). [2]

(4) (i) Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space.

(a) Define the norm $\| \cdot \|$ induced by the inner product $\langle \cdot, \cdot \rangle$. [1]

(b) For all $x, y \in H$ prove that $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$. [4]

(ii) Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space.

(a) Define the notion of *orthonormal set* in H. [1]

(b) Let (e_n) be an orthonormal sequence in H and let $x \in H$. Prove that

$$\sum_{n=1}^{+\infty} |\langle e_n, x \rangle|^2 \leq \|x\|^2.$$

[4]

(c) On the Hilbert space $H = L^2([0,\pi])$, let $e_n \in H$ be defined by $e_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx)$. Prove that the set $\{e_n \,:\, n \in \mathbb{N}\}$ is orthonormal. [4]

(d) Say under which additional property an orthonormal set is called orthonormal *basis*. [1]