Functional Analysis in Applied Mathematics and Engineering:
Final exam: part 3 - 03/02/2017

Full name:______________________________

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(1) (i) Let A be a bounded linear operator on a Hilbert space H.

(a) State the property that defines the adjoint operator A^*. [1]

(b) Prove that A^* is uniquely defined. [2]

(c) Let $H = \ell^2(\mathbb{N})$. Define the left-shift operator $S : H \to H$ by

$$S(x)_k = \begin{cases} 0 & \text{if } k = 1 \\ x_{k-1} & \text{if } k \geq 2. \end{cases}$$

Find S^* (justify your answer). [2]

(ii) State *and* prove the *uniform boundedness theorem.* [5]

(iii) Let $H = L^2([0,1])$ and consider the sequence $f_n \in H$

$$f_n(x) = \begin{cases} \sqrt{n}x & \text{if } 0 \leq x \leq 1/n \\ 0 & 1/n \leq x \leq 1. \end{cases}$$

(a) Prove that f_n in uniformly bounded in H. [2]

(b) Prove that f_n converges weakly to zero in H as $n \to +\infty$. [3]

(2) (i) Let A be a bounded linear operator on a Hilbert space H.

(a) Define the *resolvent set* and the *spectrum* of A. [1]

(b) Define the *point spectrum*, the *continuous spectrum*, and the *residual spectrum* of A. [2]

(c) Prove that if λ belongs to the residual spectrum of A then $\overline{\lambda}$ is an eigenvalue of A^* (invoke general properties of adjoint operators). [2]

(ii) Prove that a compact operator on a Hilbert space maps weakly converging sequences into strongly converging sequences. [4]

(iii) Let $K : L^2([0,1]) \to L^2([0,1])$ be the *Volterra operator*

$$(Kf)(x) = \int_0^x f(y)dy.$$

(a) Prove that 0 is in the continuum spectrum of K. [3]

(b) Prove that there are no other elements in the spectrum apart from 0. [3]