Functional Analysis in Applied Mathematics and Engineering:

Second Mid term exam - 07/12/2018

Model Solution

(1) (i) Let $(X, \|\cdot\|)$ be a Banach space and let $T : X \to X$ be a *linear* map.

(a) Say when T is called *bounded* and define the *operator norm* of T. [1,5]

Solution. T is called bounded if there exists $C > 0$ such that $\|Tx\| \le \|x\|$ for all $x \in X$. The operator norm of T is the nonnegative real number $\|T\| = \sup_{x\neq 0} \frac{\|Tx\|}{\|x\|}$.

(b) Prove that T is bounded if and only if T is continuous. [3,5]

Solution. Let T be bounded. Then there exists $C \ge 0$ such that $\|Tx\| \le \|x\|$ for all $x \in X$. Let $x, y \in X$. The linearity of T implies $\|Tx - Ty\| \le C\|x - y\|$. Therefore T is Lipschitz continuous, and hence continuous.

Viceversa, suppose T is continuous. Hence, T is continuous at the point 0. Therefore (by definition of continuity), for $\varepsilon = 1$ there exists $\delta > 0$ such that $\|x\| < \delta$ implies $\|Tx\| < 1$. Now, let $x \neq 0$ and set $z = \delta \frac{x}{2\|x\|}$. Since $\|z\| = \delta/2 < \delta$ we have $\|Tz\| < 1$. Consequently, in view of the linearity of T and of the positive homogeneity of norms, we have

$$\|Tx\| = \frac{2\|x\|}{\delta}\|Tz\| < \frac{2\|x\|}{\delta},$$

and the definition of boundedness for T holds with $C = 2/\delta$.

(c) Say when T is called a *compact* operator. [1]

Solution. T is called a compact operator if for every bounded set $B \subset X$ we have that $T(B)$ is precompact in X.

(ii) Let $(X, \|\cdot\|) = (C([0,1]), \|\cdot\|_\infty)$, and consider the linear operator $T : X \to X$

$$(Tf)(x) = \int_0^x t^2 f(y)dy.$$

Prove that T is a compact operator. [2]

Solution. Let $B = \{f \in X \ : \ \|f\|_\infty \le 1\}$. For all $f \in B$ and for all $x \in [0,1]$ we have

$$|(Tf)(x)| \le \int_0^x |t^2 f(t)|dt \le \int_0^1 |f(t)|dt \le \|f\|_\infty \le 1,$$

because $t^2 \leq 1$ for all $t \in [0,1]$ and by monotonicity of the integral. Therefore,

$$\|Tf\|_\infty = \sup_{x\in[0,1]} |(Tf)(x)| \leq 1$$

and therefore the set $T(B)$ is bounded. Moreover, let $f \in B$ and let $x, y \in [0,1]$. Additivity of integrals implies

$$|(Tf)(x-y)| \leq \int_x^y |t^2 f(t)|dt \leq \int_x^y |f(t)|dt \leq |x-y|$$

because once again $t^2 \leq 1$ for all $t \in [0,1]$. Hence, $T(B)$ is equicontinuous. From Arzelá-Ascoli's theorem the set $T(B)$ is relatively compact. Hence, T is compact.

(iii) (a) Let $(X, \|\cdot\|)$ be a Banach space and let $(x_n)_n$ be a sequence in X. Define the concept of *weak convergence* for the sequence $(x_n)_n$. [1]

Solution. The sequence x_n is said to be weakly convergent to x if $\phi(x_n)$ converges to $\phi(x)$ as $n \to +\infty$ for all $\phi \in X^*$ where X^* is the dual space of X, that is the normed space of all bounded linear functionals on X.

(b) Let $X = L^2([0,2\pi])$ and let $f_n \in X$ defined as $f_n(x) = \sin(nx)$. Prove that f_n converges weakly to zero in L^2. [4]

Solution. From a known result on the weak convergence in L^2, we need to prove that for all $g \in L^2([0,2\pi])$ we have

$$\int_0^{2\pi} f_n(x)g(x)dx \to 0, \qquad \text{as } n \to +\infty.$$

Now, let $\varepsilon > 0$. Since continuous functions are dense in L^2, there exists a continuous function h on $[0,2\pi]$ such that $\|g-h\|_{L^2} < \varepsilon$. Moreover, since polynomial are dense in $C([0,2\pi])$ with respect to the $\|\cdot\|_\infty$ norm, there exists a polynomial p on $[0,2\pi]$ such that $\|h-p\|_\infty < \varepsilon$. This implies

$$\|g-p\|_{L^2} \leq \|g-h\|_{L^2} + \|h-p\|_{L^2} < \varepsilon + \left(\int_0^{2\pi} |h(x)-p(x)|^2 dx\right)^{1/2}$$
$$\leq \varepsilon + (2\pi)^{1/2}\|h-p\|_\infty < \varepsilon(1+(2\pi)^{1/2}).$$

Now let us compute via integration by parts

$$\left|\int_0^{2\pi} f_n(x)p(x)dx\right|$$
$$= \left|\int_0^{2\pi} \sin(nx)p(x)dx\right| = \left|-\frac{1}{n}\cos(nx)p(x)\Big|_{x=0}^{x=2\pi} + \frac{1}{n}\int_0^{2\pi}\cos(nx)p'(x)dx\right|$$
$$\leq \frac{2}{n}\|p\|_\infty + \frac{2\pi}{n}\|p'\|_\infty,$$

where we have used that p and p' are continuous (and hence bounded) functions on $[0, 2\pi]$. Therefore, the right hand side above tends to zero as $n \to +\infty$. Now, triangle inequality implies

$$\left|\int_0^{2\pi} f_n(x)g(x)dx\right| = \left|\int_0^{2\pi} f_n(x)p(x)dx\right| + \left|\int_0^{2\pi} f_n(x)(g(x) - p(x))dx\right|$$

$$\leq \frac{2}{n}\|p\|_\infty + \frac{2\pi}{n}\|p'\|_\infty + \int_0^{2\pi} |\sin(nx)||g(x) - p(x)|dx$$

$$\leq \frac{2}{n}\|p\|_\infty + \frac{2\pi}{n}\|p'\|_\infty + \|f_n\|_{L^2}\|g - p\|_{L^2},$$

where we have used Hoelder's inequality. Now, since $|f_n(x)| \leq 1$ for all $x \in [0, 2\pi]$ and since $\|g - p\|_{L^2} < \varepsilon(1 + (2\pi)^{1/2})$ we get

$$\lim_{n\to+\infty}\left|\int_0^{2\pi} f_n(x)g(x)dx\right| \leq \varepsilon(2\pi)^{1/2}(1 + (2\pi)^{1/2}),$$

and the latter quantity is arbitrarily small, which implies that the above limit equals zero.

(iv) State (without proof) the *open mapping theorem.* [1]

Solution. Let X and Y be Banach spaces and let $T : X \to Y$ be a bounded, linear, and invertible operator. Then T^{-1} is bounded.

(2) (i) Let H be a Hilbert space.

(a) State *and* prove *Cauchy-Schwarz inequality.* [3]

Solution. Cauchy-Schwarz inequality asserts that for all $x, y \in H$ one has $|(x, y)| \leq \|x\|\|y\|$. Proof: Let $x, y \in H$ and $\lambda, \mu \in \mathbb{C}$. Nonnegativity of the norm, linearity of the inner product with respect to the second component, and antisymmetry imply

$$0 \leq (\lambda x - \mu y, \lambda x - \mu y) = |\lambda|^2\|x\|^2 + |\mu|^2\|y\|^2 - \overline{\mu}\lambda(y, x) - \overline{\lambda}\mu(x, y).$$

Now, write the inner product (x, y) in polar form, that is $(x, y) = re^{i\phi}$, with $r = |(x, y)| \geq 0$ and $\phi \in [0, 2\pi)$. We choose $\lambda = \|y\|e^{i\phi}$ and $\mu = \|x\|$. We obtain

$$0 \leq \|y\|^2\|x\|^2 + \|x\|^2\|y\|^2 - \|x\|\|y\|e^{i\phi}|(x, y)|e^{-i\phi} - \|y\|e^{-i\phi}\|x\||(x, y)|e^{i\phi}$$

and the obvious cancellations of $e^{i\phi}$ with $e^{-i\phi}$ imply

$$2\|x\|^2\|y\|^2 \geq 2\|x\|\|y\||(x, y)|$$

which implies the assertion upon further cancellations.

(b) State *and* prove the *parallelogram rule*. [2]

Solution. The parallelogram rule reads $\|x+y\|^2+\|x-y\|^2=2\|x\|^2+2\|y\|^2$ for all $x,y\in H$. Proof: Let $x,y\in H$. By the elementary properties of the inner product we get

$$\|x+y\|^2+\|x-y\|^2=(x+y,x+y)+(x-y,x-y)$$
$$=\|x\|^2+\|y\|^2+(x,y)+(y,x)+\|x\|^2+\|y\|^2-(y,x)-(x,y)$$
$$=2\|x\|^2+2\|y\|^2.$$

(ii) Let H be a Hilbert space and let $M\subset H$ be a closed linear subspace of H.

(a) Define the orthogonal complement $M^\perp$. [1]

Solution. The orthogonal complement is defined as

$$M^\perp=\{x\in H\ :\ (x,y)=0\ \text{ for all } y\in M\}.$$

(b) Given $x_0\in H\setminus M$, prove that there exists a point $y\in M$ such that

$$\|x_0-y\|=\min\{\|x_0-z\|\ :\ z\in M\}$$

and that such a point y is unique. [4]

Solution. Define $d=\inf_{y\in M}\|x_0-y\|$. We have that $d>0$ because x_0 and M are closed and the pointy set $\{x_0\}$ is compact. By definition of inf there exists a minimizing sequence $y_n\in M$ such that

$$d=\lim_{n\to+\infty}\|x_0-y_n\|$$

and such that

$$\|x_0-y_n\|\le d+1/n$$

for all positive integers n. Our first goal is to prove that y_n is a Cauchy sequence. To prove that, we use the parallelogram rule with the two vectors x_0-y_n and x_0-y_m with distinct $n,m\in\mathbb{N}$. We get

$$2\|x_0-y_n\|^2+2\|x_0-y_m\|^2$$
$$=\|2x_0-(y_n+y_m)\|^2+\|y_n-y_m\|^2=4\|x_0-(y_n+y_m)/2\|^2+\|y_n-y_m\|^2$$
$$\ge 4d^2+\|y_n-y_m\|^2,$$

where we have used that the vector $(y_n + y_m)/2$ belongs to M, as M is a linear subspace. Using the defining property of y_n we get

$$\|y_n - y_m\|^2 \le 2(d+1/n)^2 + 2(d+1/m)^2 - 4d^2$$
$$= 2d^2 + 4/n + 2/n^2 + 2d^2 + 4/m + 2/m^2 - 4d^2 = 4/n + 2/n^2 + 4/m + 2/m^2.$$

Clearly, the latter quantity tends to zero as $n, m \to +\infty$, which shows that the quantity $\|y_n - y_m\|$ is arbitrarily small as $n, m \to +\infty$, which means that y_n is a Cauchy sequence. Now, since H is complete, y_n converges to some $y \in M$. Since the distance function is continuous we get

$$\|x_0 - y\| = \lim_{n\to+\infty} \|x_0 - y_n\| = d,$$

which proves that the inf above is actually a minimum, and y is the point in M we were looking for. In order to prove uniqueness, assume there are two points $y_1, y_2 \in M$ such that

$$d = \|x_0 - y_1\| = \|x_0 - y_2\|.$$

Parallelogram rule applied to the pair $x_0 - y_1$, x_0, y_2 implies

$$4d^2 = 2\|x_0 - y_1\|^2 + 2\|x_0 - y_2\|^2 = 4\|x_0 - (y_1+y_2)/2\|^2 + \|y_1 - y_2\|^2 \ge 4d^2 + \|y_1 - y_2\|^2,$$

as $(y_1 + y_2)/2 \in M$. Therefore,

$$\|y_1 - y_2\|^2 \le 4d^2 - 4d^2 = 0,$$

which shows that $y_1 = y_2$, hence the point of minimal distance $y \in M$ is unique.

(iii) Let H be a separable Hilbert space and let $\mathcal{U} = \{u_n \in H \,:\, n \in \mathbb{N}\}$ be a sequence.

(a) Say when $\mathcal{U}$ is called an *orthonormal* sequence. [1]

Solution. $\mathcal{U}$ is called orthonormal sequence if $(u_n, u_m) = 0$ for all $n \neq m$ and $\|u_n\| = 1$ for all n.

(b) Let $x \in H$. Prove that

$$\sum_{n\in\mathbb{N}} |(u_n, x)|^2 \le \|x\|^2.$$

[3]

Solution. Fix a positive integer N. Set $x_N = \sum_{k=1}^{N}(u_k, x)u_k$. The elementary properties of the inner product imply

$$0 \leq (x - x_N, x - x_N) = (x, x) - \sum_{k=1}^{N}(u_k, x)(x, u_k) - \sum_{n=1}^{N}\overline{(u_n, x)}(u_n, x)$$

$$+ \sum_{k,n=1}^{N} \overline{(u_k, x)}(u_n, x)(u_k, u_n).$$

As $\mathcal{U}$ is an orthonormal sequence, (u_k, u_n) is nonzero only if $n = m$, and with some simple cancellations we get

$$0 \leq \|x\|^2 - \sum_{k=1}^{N} |(u_n, x)|^2.$$

Hence,

$$\sum_{k=1}^{N} |(u_n, x)|^2 \leq \|x\|^2.$$

This shows that the above sum is the partial sum of a convergent series. By letting N go to infinity we get the assertion.