Functional Analysis in Applied Mathematics and Engineering: First Mid term exam (model solution)

(1) (i) (a) Given a metric space (X, d) and an operator $T : X \to X$, provide the definition of *contraction mapping.* [1].

Solution. A map $T : X \to X$ is called contraction mapping if there exists a constant $c \in (0, 1)$ such that $d(T(x), T(y)) \le cd(x, y)$.[1]

(b) Given a metric space (X, d) and an operator $T : X \to X$, provide the definition of *fixed point* for the map T. [1].

Solution. A point $x \in X$ is a fixed point for T if $T(x) = x$.

(c) State and prove the *Contraction Mapping Theorem.* [5].

Solution. Statement of the Theorem: If $T : X \to X$ is a contraction mapping on a complete metric space (X, d), then T has exactly one fixed point.[2]

Proof of the Theorem: Let $x_0 \in X$ be any point in X. We define a sequence (x_n) in X by

$$x_{n+1} = T(x_n), \qquad \text{for } n \ge 0.$$

We show that (x_n) is a Cauchy sequence.[3] If $n \ge m \ge 1$, since T is a contraction with constant c, using the triangle inequality we get

$$\begin{aligned}
d(x_n, x_m) &= d(T^n x_0, T^m x_0) \\
&\le c^m d(T^{n-m} x_0, x_0) \\
&\le c^m \left[d(T^{n-m} x_0, T^{n-m-1} x_0) + \ldots + d(T x_0, x_0) \right] \\
&\le c^m \left[\sum_{k=0}^{n-m-1} c^k \right] d(x_1, x_0) \\
&\le c^m \left[\sum_{k=0}^{+\infty} c^k \right] d(x_1, x_0) \\
&= \left(\frac{c^m}{1-c} \right) d(x_1, x_0),
\end{aligned}$$

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[1] Many students gave the definition of contraction mapping with $c \in \mathbb{R}$

[2] Many students did not write the hypothesis that X has to be a complete metric space

[3] Many students did not understand in the proof one has to construct a Cauchy sequence

which implies that (x_n) is a Cauchy sequence since $c < 1$. Since X is complete, (x_n) converges to a limit $x \in X$. By continuity of T, we get

$$Tx = T\left(\lim_{n\to+\infty} x_n\right) = \lim_{n\to+\infty} Tx_n = \lim_{n\to+\infty} x_{n+1} = x,$$

which shows that x is a fixed point. Finally, let $x, y \in X$ be two fixed points, then

$$0 \le d(x,y) = d(Tx,Ty) \le cd(x,y).$$

Since $c < 1$, we have $d(x,y) = 0$, so $x = y$ and the fixed point is unique.

(ii) (a) Given a family $\mathcal{F}$ of functions from a metric space space (X, d_X) and (Y, d_Y), provide the definition of equicontinuity. [1].

Solution. The family $\mathcal{F}$ is said equicontinuous if for all $\varepsilon > 0$ there is a $\delta > 0$ such that if $d_X(x,y) < \delta$ then $d_Y(f(x), f(y)) < \varepsilon$ for all $f \in \mathcal{F}$.[4]

(b) State (without proof) the *Ascoli-Arzelá Theorem.* [2].

Solution. Let $\mathcal{F}$ be a family of continuous functions on a compact metric space K. Then, $\overline{\mathcal{F}}$ is compact if and only if $\mathcal{F}$ is equicontinuous and bounded.[5]

(iii) For any $n \in \mathbb{N}$, let $f_n : [0,\infty) \to \mathbb{R}$ be the sequence of functions defined by

$$f_n(x) = \frac{nx}{1+n^2x^2}.$$

(a) Show that $f_n \to 0$ pointwise in $[0,\infty)$. [2].

Solution. If $x = 0$ then $f_n(0) = 0$ for all $n \in \mathbb{N}$. If $x \neq 0$, then

$$0 \le f_n(x) \le \frac{nx}{n^2x^2} = \frac{1}{nx} \to 0$$

as $n \to +\infty$.

(b) Is $\{f_n\}_n$ uniformly convergent on $[0,\infty)$? Justify your answer in detail. [3].

Solution. The uniform convergence of f_n to 0 on $[0,+\infty)$ would hold, by definition, if

$$\|f_n\|_\infty = \sup_{x\ge 0} |f_n(x)| \to 0$$

as $n \to +\infty$. To compute the supremum of f_n on $[0,+\infty)$, differentiate f_n:

$$f_n'(x) = \frac{1}{(1+n^2x^2)^2}(n(1+n^2x^2) - 2n^3x^2) = \frac{n(1-n^2x^2)}{(1+n^2x^2)^2},$$

[4] Many students wrote the definition of equi-continuity for a single function

[5] Many students did the mistake to state that all the space of continuous functions is compact if and only it is bounded, closed and equi-continuous. This statement is of course false!

and the only stationary point of f_n on $[0,+\infty)$ is $x_n = 1/n$. Such a point is a maximum point. On x_n, f_n achieves the value

$$f_n(x_n) = f_n(1/n) = 1/2,$$

and this proves

$$\|f_n\|_\infty = \sup_{x\geq 0} |f_n(x)| = \frac{1}{2},$$

hence f_n cannot converge to zero uniformly on $[0,+\infty)$.

(2) (i) Given a measurable function $f : \mathbb{R}^d \to [0,+\infty]$, provide the definition of the Lebesgue integral of f. [2]

Solution. The Lebesgue integral of f is defined by[6]

$$\int f dx = \sup\left\{\int \phi dx \,:\, 0 \leq \phi \leq f,\, \phi \text{ simple}\right\}.$$

(ii) (a) State (without proof) *Fatou's lemma.* [2]

Solution. Let $f_n : \mathbb{R}^d \to \mathbb{R}$ be a sequence of nonnegative measurable functions. Then,[7]

$$\int \left(\liminf_{n\to+\infty} f_n(x)\right) dx \leq \liminf_{n\to+\infty} \int f_n(x)dx.$$

(b) Find a sequence $f_n : \mathbb{R} \to [0,+\infty)$ for which the strict inequality holds in Fatou's lemma. [3]

Solution. An example is $f_n(x) = 1_{[n,n+1)}$. Indeed, for all $x \in \mathbb{R}$ one has that $f_n(x) = 0$ for all $n > x$, which implies that $f_n(x) \to 0$ pointwise as $n \to +\infty$.[8] Hence

$$\int \left(\liminf_{n\to+\infty} f_n(x)\right) dx = \int 0dx = 0.$$

Moreover,

$$\int f_n(x)dx = \int_n^{n+1} 1dx = 1$$

[6] Many students here provided other definitions, such $f \in L^p$, f measurable; some others also provided the definition of integral of a sign changing function, which was not required.

[7] The assumption f nonnegative is crucial, otherwise the statement is not true in general, see the counter example $f_n = -1/n$ discussed in class.

[8] Other examples could be provided such as $f_n(x) = n1_{[0,1/n]}(x)$, leading to a similar outcome. In any case, the convergence of f_n to zero almost everywhere *must be proven*! I removed fractions of points in some cases in which this property was stated but not proven.

for all $n \in \mathbb{N}$. Hence,

$$\liminf_{n\to+\infty} \int f_n(x)dx = 1 > 0.$$

(iii) Let $f, g \in L^p(\mathbb{R}^d)$ with $p \in (1, +\infty)$. Prove the Hölder inequality

$$\|f\,g\|_1 \leq \|f\|_p \, \|g\|_{p'},$$

with p' such that $\frac{1}{p} + \frac{1}{p'} = 1$. [5]

Solution. The statement is trivial if either f or g are zero almost everywhere. Otherwise, we clearly have $\|f\|_{L^p} > 0$ and $\|g\|_{L^q} > 0$. For a fixed $\alpha > 0$ we have

$$|f(x)g(x)| = \left|\frac{f(x)}{\alpha}\right| |\alpha g(x)| \leq \frac{1}{p}\left|\frac{f(x)}{\alpha}\right|^p + \frac{1}{q}|\alpha g(x)|^q,$$

where we have used Young's inequality. By integrating the above inequality on $\mathbb{R}^d$ we get

$$\|fg\|_{L^1(\mathbb{R}^d)} \leq \frac{1}{p}\frac{1}{\alpha^p}\|f\|^p_{L^p(\mathbb{R}^d)} + \frac{1}{q}\alpha^q\|g\|^q_{L^q(\mathbb{R}^d)}.$$

We now choose α such that the two terms on the above right hand side are equal, namely

$$\alpha := \frac{\|f\|^{\frac{1}{q}}_{L^p}}{\|g\|^{\frac{1}{p}}_{L^q}},$$

which yields

$$\|fg\|_{L^1(E)} \leq \frac{1}{p}\frac{\|g\|_{L^q}}{\|f\|^{p/q}_{L^p}}\|f\|^p_{L^p} + \frac{1}{q}\frac{\|f\|_{L^p}}{\|g\|^{q/p}_{L^q}}\|g\|^q_{L^q},$$

and the definition of p and q implies the last term above equals $\|f\|_{L^p}\|g\|_{L^q}$.[9]

(iv) Let $p \in [1, +\infty)$ and $\alpha \in \mathbb{R}$. Prove that the function

$$f(x) = \begin{cases} \frac{\sin(|x|)}{|x|^\alpha} & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1, \end{cases}$$

is in $L^p(\mathbb{R}^d)$ for $\alpha < \frac{d+p}{p}$. [3].

Solution. Compute

$$\begin{aligned}\int_{\mathbb{R}^d} |f(x)|^p dx &= \int_{\{|x|\leq 1\}} \frac{|\sin(x)|^p}{|x|^{\alpha p}} dx \\ &= \int_0^1 r^{d-1} \int_{\{|x|=r\}} \frac{|\sin x|^p}{r^{\alpha p}} dr d\sigma,\end{aligned}$$

[9]Some students opted for an alternative procedure in which the functions are normalized in L^p and L^q respectively, and the proof actually works much faster. Clearly, such alternative procedure (without mistakes) was still implying the 5 points.

where we have used polar coordinates. Here C_d is a suitable positive constant depending on the dimension d. Now, the above integrand is possibly singular at $r = 0$, and is continuous at any other point $r \in (0, 1]$. By Taylor expanding the sin function near 0 we get $\sin x = x + o(|x|)$. This implies, near $x = 0$,

$$r^{d-1} \frac{|\sin x|^p}{r^{\alpha p}} \sim r^{d-1+p-\alpha p}.$$

Now,

$$\int_0^1 r^{d-1+p-\alpha p} dr = \lim_{\varepsilon \to 0} \int_\varepsilon^1 r^{d-1+p-\alpha p} dr = \frac{1}{d+p-\alpha p} r^{d+p-\alpha p}|_{r=\varepsilon}^{r=1},$$

which equals $\frac{1}{d+p-\alpha p}(1 - \varepsilon^{d+p-\alpha p})$, and the limit of this quantity as $\varepsilon \to 0$ is finite if $d + p - \alpha p > 0$, which is equivalent to $\alpha < \frac{d+p}{p}$.[10]

[10]No one completed this proof correctly. I still gave some points for those who got a condition by estimating the sin term by 1 in absolute value