Functional Analysis in Applied Mathematics and Engineering:
Third Mid term exam - 12/01/2018

FULL NAME:____

MATRICOLA:____

(1) (i) Let $(H, (\cdot, \cdot))$ be a Hilbert space.

(a) Let $x \in H$ and let $\varphi : H \to \mathbb{C}$ be defined by

$$\varphi(y) = (x, y).$$

Prove that $\varphi \in H^*$ and compute $\|\varphi\|_{H^*}$. [2]

Solution. If $x = 0$ there is nothing to prove. Otherwise, in order to prove that $\varphi \in H^*$ we need to prove that it is a linear and bounded functional. Linearity follows from the properties of the scalar product, while for the boundedness we argue as follows:
Let $y \in H$, then by using Cauchy-Schwarz inequality we get

$$|\varphi(y)| = |(x, y)| \leq \|x\|\|y\|,$$

then φ is bounded. To compute the norm we recall that

$$\|\varphi\|_{H^*} = \sup_{y \in H, y \neq 0} \frac{|\varphi(y)|}{\|y\|},$$

then by choosing $y = x$ it easily follows that $\|\varphi\|_{H^*} = \|x\|$.

(b) State (without proof) *Riesz' representation Theorem.* [2]

Solution. If φ is a bounded linear functional on a Hilbert space H, then there is a unique vector $y \in H$ such that

$$\varphi(x) = \langle y, x \rangle \qquad \text{for al } x \in H.$$

(ii) Let A be a bounded linear operator on the Hilbert space H.

(a) State the property that defines the *adjoint operator* A^*. [2]

Solution. The defining property of the adjoint $A^* \in \mathcal{B}(H)$ of an operator $A \in \mathcal{M}(H)$ is that

$$\langle x, Ay \rangle = \langle A^* x, y \rangle \qquad \text{for all } x, y \in H.$$

(b) Prove that A^* is well defined and is a bounded operator. [2]

Solution. To prove that A^* exists and is uniquely defined, we have to show that for every $x \in H$, there is a unique vector $z \in H$, depending linearly on x, such that

$$\langle z, y\rangle = \langle x, Ay\rangle \qquad \text{for all } x, y \in H.$$

For fixed x, the map φ_x defined by

$$\varphi_x(y) = \langle x, Ay\rangle$$

is a bounded linear functional on H, with $\|\varphi_x\| \leq \|A\|\|x\|$. By the Riesz representation theorem, there is a unique $z \in H$ such that $\varphi_x(y) = \langle z, y\rangle$. This z satisfies the required property, so we set $A^*x = z$.

We need to prove then that A^* is a bounded linear operator. The linearity of A^* is left as an exercise. Regarding the boundedness, we have that

$$\begin{aligned}\|A^*x\| = \sup_{y\in H, \|y\|\leq 1} |(A^*x, y)| &= \sup_{y\in H, \|y\|\leq 1} |(x, Ay)| \\ &\leq \sup_{y\in H, \|y\|\leq 1} \|x\|\|Ay\| \\ &\leq \|A\|\|x\|.\end{aligned}$$

Then, A^* is a bounded linear operator.

(c) Prove that $\overline{\mathrm{ran} A} = (\ker A^*)^{\perp}$. [4]

Solution. See the proof of Theorem 7.18 of the Lecture Notes.

(iii) State (without proof) *Banach-Alaoglu's theorem.* [2]

Solution. The closed unit ball of a Hilbert space is weakly compact.

(2) (i) Let A be a bounded linear operator on the Hilbert space H.

(a) Define the resolvent set and the spectrum of A. [1]

Solution. The resolvent set of A, denoted by $\rho(A)$, is the set of complex numbers λ such that $A - \lambda\mathbb{I}$ is invertible. The spectrum of A is the set $\sigma(A) = \mathbb{C} \setminus \rho(A)$.

(b) Define the point spectrum, the continuous spectrum, and the residual spectrum of A. [1]

Solution. The point spectrum of A is the set of complex numbers λ such that $A - \lambda\mathbb{I}$ is not $1:1$. The continuous spectrum is the set of complex numbers λ such that $A - \lambda\mathbb{I}$ is $1:1$ but not onto and $\mathrm{Ran}(A - \lambda\mathbb{I})$ is dense in H. The residual spectrum is the set of complex numbers λ such that $A - \lambda\mathbb{I}$ is $1:1$ but not onto and $\text{ran}(A - \lambda\mathbb{I})$ is not dense in H.

(c) Prove that if λ belongs to the residual spectrum of A then $\bar{\lambda}$ is an eigenvalue of A^*. [2]

Solution. Let λ be in the residual spectrum of A. Then $\text{ran}(A - \lambda\mathbb{I})$ is not dense in H. Hence, $\overline{\text{ran}(A - \lambda\mathbb{I})} \subsetneq H$ is a proper, closed linear subspace of H. By the orthogonal projection theorem, there exists $z \neq 0$ such that $z \in \overline{\text{ran}(A - \lambda\mathbb{I})}^{\perp}$. Since $\overline{\lambda}\mathbb{I}$ is the adjoint of $\lambda\mathbb{I}$, the result in exercise 1-(ii)-(c) implies that $z \in \ker(A^* - \overline{\lambda}\mathbb{I})$. Since $z \neq 0$ this shows that $A^* - \overline{\lambda}\mathbb{I}$ is not $1:1$, and therefore $\overline{\lambda}$ is an eigenvalue of A^*.

(ii) Let $H = L^2([0,1])$ and $M : H \to H$ defined by

$$(Mf)(x) = xf(x), \qquad x \in [0,1].$$

(a) Prove that M has no eigenvalues. [2]

Solution. Let $\lambda \in \mathbb{C}$ be an eigenvalue for M. Then, there exists $f \in L^2([0,1])$, f not equal to zero almost everywhere, such that $Mf = \lambda f$. By definition of M this means

$$(x - \lambda)f(x) = 0 \qquad \text{almost everywhere on } [0,1].$$

Since f is not equal to zero almost everywhere, the latter implies that $(x-\lambda) = 0$ for almost every $x \in [0,1]$, which is a contradiction (independently on whether or not λ belongs to $[0,1]$!). Hence, M has not eigenvalues.

(b) Prove that $\sigma(M) = [0,1]$. [3]

Solution. Since M has no eigenvalues, the only possibility for $\lambda \in \mathbb{C}$ to be an element of the spectrum is that the equation

$$(M - \lambda\mathbb{I})f = g$$

has no solution $f \in L^2([0,1])$ for some $g \in L^2([0,1])$. Formally, the only possible candidate solution is given by

$$f(x) = \frac{g(x)}{x - \lambda}.$$

Now, if $\lambda \in [0,1]^c$ then the function $\frac{1}{x-\lambda}$ is uniformly bounded on $[0,1]$ because $|x - \lambda| \geq c$ for some positive constant c. Hence

$$\int_0^1 |f(x)|^2 dx = \int_0^1 \frac{1}{(x-\lambda)^2} |g(x)|^2 dx \leq \frac{1}{c} \int_0^1 |g(x)|^2 dx$$

and the latter term is finite in view of $g \in L^2$. This proves that $[0,1]^c \subset \rho(M)$ or equivalently $\sigma(M) \subset [0,1]$. On the other hand, if $\lambda \in [0,1]$ then the are examples of $g \in L^2$ such that the above f is not in L^2, for example take $g \equiv 1$. We get in this case $f(x) = \frac{1}{x-\lambda}$ which is clearly not in $L^2([0,1])$ because

$$\int_0^1 |f(x)|^2 dx = \int_0^1 \frac{1}{(x-\lambda)^2} dx = +\infty.$$

(iii) Let $H = \ell^2(\mathbb{N})$ and let $S : H \to H$ be the *left-shift operator*

$$(Tx)_n = x_{n+1}, \qquad k = 1, 2, 3, \ldots \qquad x = (x_k)_{k=1}^{+\infty}.$$

Consequently, $[0,1] \subset \sigma(M)$.

(a) Prove that $\|T\| \leq 1$. [1]

Solution.

$$\|Tx\|_{\ell^2}^2 = \sum_{n=1}^{+\infty} (Tx)_n^2 = \sum_{n=1}^{+\infty} x_{n+1}^2 = \sum_{n=2}^{+\infty} x_n^2 \leq \sum_{n=1}^{+\infty} x_n^2 = \|x\|_{\ell^2}^2,$$

therefore $\|T\| = \sup_{\|x\|_{\ell^2} \leq 1} \|Tx\|_{\ell^2} \leq 1$.

(b) Let $\lambda \in (-1,1)$. Prove that λ is an eigenvalue of T. [3]

Solution.

(c) Prove that $\sigma(T) = [-1,1]$. [1]

Solution.