A Weierstrass Type Representation for Minimal Surfaces in Hyperbolic Space with Mean Curvature Equal to One

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ABSTRACT

The subject of this paper is to give a Weierstrass type representation for mean curvature one surfaces in the hyperbolic space. This representation depends on the hyperbolic Gauss map. Some known examples are described and a new one, associated to the minimal Bonnet surface is construct with this representation.

Key words: surfaces, constant mean curvature, hyperbolic space.

INTRODUCTION

A Weierstrass type formula for surfaces of prescribed mean curvature in $\mathbb{R}^3$ was given by Kenmotsu in 1979. In 1987, R. Bryant studied the surfaces of mean curvature one in hyperbolic space as local projections of null curves in the space of the 2×2 Hermitian symmetric matrices with its Cartan-Killing metric. Recently, Umehara (in 93, 95 and 96), Yamada and Rossman produced an explicit tool to construct examples of these surfaces.

In this note we announce how to describe surfaces in $\mathbb{H}^3$ with mean curvature one in a similar manner as the minimal surfaces in $\mathbb{R}^3$ that means, given by an integral formula. It is already well known that these surfaces have a hyperbolic holomorphic Gauss map; a suitable choice of a second function will give us a Weierstrass type representation. Some solutions of this problem can be constructed by using this representation.

Let $\mathbb{L}^4 = \{x = (x_0, x_1, x_2, x_3) \in \mathbb{R}^4\}$ be the Lorentz space with the inner product

$$\langle x, y\rangle = -x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3.$$

The hyperbolic space is the submanifold

$$\mathbb{H}^3 = \{x \in \mathbb{L}^4 \mid \langle x, x\rangle = -1, x_0 > 0\}.$$

In $\mathbb{H}^3$ we will consider the induced orientation from $\mathbb{L}^4$ for which the vectors v_1, v_2, v_3 in $T_p\mathbb{H}^3$ form a positive oriented basis iff $\{p, v_1, v_2, v_3\}$ forms a positive oriented basis of $\mathbb{L}^4$.

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Let $X: M \rightarrow \mathbb{H}^3$ be an isometric immersion of an orientable Riemann surface M in the hyperbolic space and $N(p)$ the oriented unitary normal vector at $p \in M$. In local isothermal coordinates $z = u + iv$ we have $\|X_u\| = \|X_v\| = \lambda$, $\langle X_u, X_v \rangle = 0$, and N is such that

$$\{X(p), \frac{1}{\lambda}X_u, \frac{1}{\lambda}X_v, N(p)\}$$

is a positive basis of $T_p\mathbb{L}^4$.

We will consider the map

$$\Phi: \mathbb{H}^3 \longrightarrow D$$
$$(x_0, x_1, x_2, x_3) \mapsto \left(1, \frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_3}{x_0}\right)$$

and the vector $\Phi_*(N(p))$ where

$$D = \{(x_0, x_1, x_2, x_3) \mid x_0 = 1, x_1^2 + x_2^2 + x_3^2 < 1\}.$$

This map is the natural isometry between $\mathbb{H}^3$ and the Klein model for the hyperbolic space given by the unitary disc with the appropriated metric.

The boundary of D can be identified with the Riemann two sphere S^2.

DEFINITION. The *hyperbolic Gauss map of an immersion* $X: M \longrightarrow \mathbb{H}^3$ is the map $n: M \longrightarrow \partial D$ given by

$$n(p) = \Phi(X(p)) + t\Phi_*(N(p))$$

where $t > 0$ and $n(p) \in \partial D$.

Taking $z = u + iv$ isothermal parameters in $U \subset \mathbb{C}$ we have the diagram:

$$\begin{array}{ccc} M & \xrightarrow{n} & \partial D \approx S^2 \\ \downarrow & & \downarrow \Pi \\ U \subset \mathbb{C} & \xrightarrow{h} & \mathbb{C} \end{array}$$

with Π the stereographic projection; from the definition it follows that

$$n = \frac{1}{x_0 + N_0}(X + N)$$

and from the above diagram we have

$$n(z) = \left(1, \frac{2\,\mathfrak{Re}\,h}{|h|^2 + 1}, \frac{2\,\mathfrak{Im}\,h}{|h|^2 + 1}, \frac{|h|^2 - 1}{|h|^2 + 1}\right),$$

and n is holomorphic if and only if h is holomorphic.

We observe that $\|n_u\| = \|n_v\| = 0$ if and only if the immersion is umbilical and $H = 1$; in this case we have a horosphere and the hyperbolic Gauss map n is constant.

The hyperbolic Gauss map behaves as the classical Gauss map for minimal surfaces in an euclidean space, that is, we have the following theorem first proved by Bryant:

THEOREM 1. *Let $n: M \longrightarrow \partial D$ be the hyperbolic Gauss map of a surface $X: M \longrightarrow \mathbb{H}^3$, n non constant. The map $n: M \longrightarrow \partial D$ is conformal iff the immersion X either has mean curvature H constant and equal to one (in which case n preserves the orientation) or X is totally umbilic (in which case n reverses the orientation).*

We can now state our main result:

THEOREM 2. *Let* $X\colon M \longrightarrow \mathbb{H}^3$ *be a non-umbilic immersion in* $\mathbb{H}^3$ *with mean curvature one,* $X(z) = (x_0(z), x_1(z), x_2(z), x_3(z))$ *and*

$$n(z) = \left(1, \frac{2\,\mathfrak{Re}\,h}{1+|h|^2}, \frac{2\,\mathfrak{Im}\,h}{1+|h|^2}, \frac{|h|^2-1}{|h|^2+1}\right)$$

its hyperbolic Gauss map. The real functions $\phi_1(z) = x_0(z) + x_3(z)$ *and* $\phi_2(z) = x_0(z) - x_3(z)$ *and the complex function* $\phi_3(z) = x_1(z) + ix_2(z)$ *satisfy*

$$\begin{cases} \phi_1\phi_2 = 1 + |\phi_3|^2 \\ \frac{\partial\phi_1}{\partial z} = h\frac{\partial\overline{\phi}_3}{\partial z} \\ \frac{\partial\phi_2}{\partial z} = \frac{1}{h}\frac{\partial\phi_3}{\partial z} \end{cases} \qquad (*)$$

Conversely, given a holomorphic non-constant function $h\colon U \subset \mathbb{C} \longrightarrow \mathbb{C}$*, two real functions* ϕ_1 *and* $\phi_2 (\phi_2 > 0)$ *and a complex function* ϕ_3 *satisfying* $(*)$ *in the simply connected domain* U*, then*

$$X(z) = \left(\frac{\phi_1(z)+\phi_2(z)}{2}, \mathfrak{Re}\,\phi_3(z), \mathfrak{Im}\,\phi_3(z), \frac{\phi_1(z)-\phi_2(z)}{2}\right)$$

defines a conformal immersion in $\mathbb{H}^3$ *with constant mean curvature one and hyperbolic Gauss map* n *given by* h *as above.*

SKETCH OF PROOF. First of all we observe that

$$X(z) = (x_0, x_1, x_2, x_3) \in \mathbb{H}^3 \Longleftrightarrow -x_0^2 + x_1^2 + x_2^2 + x_3^2 = -1 \Longleftrightarrow \phi_1\phi_2 = 1 + |\phi_3|^2;$$

from the first equivalence it also follows that if $\phi_2 = x_0 - x_3$ then $\phi_2 > 0$.

Given ϕ_1, ϕ_2, ϕ_3 as above we have $\langle X_z, n\rangle = 0$ if and only if

$$\frac{\partial\phi_1}{\partial z} + |h|^2\frac{\partial\phi_2}{\partial z} - h\frac{\partial\overline{\phi}_3}{\partial z} - \overline{h}\frac{\partial\phi_3}{\partial z} = 0 \qquad (1)$$

The assumption on the mean curvature gives us

$$H = 1 \Longleftrightarrow \langle X_z, n_{\overline{z}}\rangle = 0,$$

and as h is nonconstant $(h_z \neq 0)$ it follows

$$\frac{\partial\phi_3}{\partial z} = h\frac{\partial\phi_2}{\partial z}. \qquad (2)$$

Returning with this last equation in (1) we have

$$\frac{\partial\phi_1}{\partial z} = h\frac{\partial\overline{\phi}_3}{\partial z}.$$

Conversely, from (*) it follows that $\langle X_z, X_z\rangle = 0$ and we have isothermal parameters. We can show that the hyperbolic Gauss map of the immersion X is given by h. From the fact that h is holomorphic we have $H = 1$; h non constant gives us a non-umbilic immersion. □

REMARKS

1. The compatibility condition for the two partial differential equations in (*) is the same and writes

$$\mathfrak{Im}\{\overline{h}\Delta\phi_3\} = 0. \qquad (4)$$

This follows from the fact that each differential equation of (*) is as

$$\frac{\partial \phi}{\partial z} = F(z)$$

or as

$$\begin{cases} \frac{\partial \phi}{\partial u} = 2F_1(u,v) \\ \frac{\partial \phi}{\partial v} = -2F_2(u,v) \end{cases}$$

with

$$z = u + iv, F(z) = F_1(u,v) + iF_2(u,v), \partial/\partial z = \frac{1}{2}(\partial/\partial u - i\partial/\partial v)\,.$$

The integrability condition for the system is:

$$\frac{\partial F_1}{\partial v} = -\frac{\partial F_2}{\partial u} \Longleftrightarrow \mathfrak{Im}\left\{\frac{\partial F}{\partial \bar{z}}\right\} = 0$$

Returning to the system (*), each equation will have its integrability condition respectively given by:

$$\mathfrak{Im}\left\{h\frac{\partial^2 \overline{\phi}_3}{\partial \bar{z}\partial z}\right\} = 0$$

and

$$\mathfrak{Im}\left\{\frac{1}{|h|^2}\frac{\overline{h\partial^2 \overline{\phi}_3}}{\partial \bar{z}\partial z}\right\} = -\frac{1}{|h|^2}\,\mathfrak{Im}\left\{h\frac{\partial^2 \overline{\phi}_3}{\partial \bar{z}\partial z}\right\} = 0$$

Consequently, the two compatibility conditions are verified if and only if, locally,

$$\mathfrak{Im}\{\overline{h}\,\Delta\phi_3\} = 0$$

2. An integral formula can be written:

$$X = \left(\mathfrak{Re}\int_{z_0}^{z}\left(h\frac{\partial \overline{\phi}_3}{\partial z} + \frac{1}{h}\frac{\partial \phi_3}{\partial z}\right)dz,\ \mathfrak{Re}\,\phi_3,\ \mathfrak{Im}\,\phi_3,\ \mathfrak{Re}\int_{z_0}^{z}\left(h\frac{\partial \overline{\phi}_3}{\partial z} - \frac{1}{h}\frac{\partial \phi_3}{\partial z}\right)dz\right).$$

3. Calling (Hopf, 1983)

$$\psi = \frac{1}{2}\left[(h_{11} - h_{22}) - 2ih_{12}\right]$$

we know that the Codazzi equations can be written as

$$\frac{\partial}{\partial \bar{z}}(\lambda^2\psi) = \lambda^2\frac{\partial H}{\partial z}\,.$$

Consequently H is constant if and only if $\lambda^2\psi$ is holomorphic. We can verify that (*) implies this last condition (Proposition 2 in Bryant's paper), and we have the Codazzi equations; with some calculations we can also verify the Gauss equation.

EXAMPLES

To get examples we have to find solutions of

$$\mathfrak{Im}\{\overline{h}\,\Delta\phi_3\} = 0$$

and a linear combination of this solutions in order to have

$$\phi_1\phi_2 = 1 + |\phi_3|^2,$$

that is, in order to have the corresponding immersion in $\mathbb{L}^4$ contained in $\mathbb{H}^3$.

1. Given the function h we can search solutions as

$$\phi_3 = h(z)F(|z|^2)$$

with F a one real variable differentiable function. We have an immersion with constant mean curvature one solving (*) with $h(z) = z$, $F_\alpha(t) = t^\alpha$ and

$$\phi_3(z) = h(z)\left[AF_\alpha(|z|^2) + BF_\beta(|z|^2)\right].$$

The condition $\phi_1\phi_2 = 1 + |\phi_3|^2$ is verified under the restrictions:

$$\alpha + \beta = -1, \quad AB\left(\frac{2\alpha+1}{\alpha(\alpha+1)}\right)^2 = 1, \quad 2\alpha + 1 \neq 0.$$

These surfaces are called "catenoid cousins".

2. The system (*) also admits solutions as

$$\phi_3(z) = F(z)\overline{G(z)}$$

with F, G and h holomorphic functions. In this case if

$$F'(z) = h(z)G'(z) \tag{5}$$

the integrability condition (4) is verified.

The surfaces called "Enneper Cousins" are corresponding to

$$\phi_3 = AF_1\overline{G}_1 + BF_2\overline{G}_2,$$
$$h(z) = \tan h\lambda z, \quad G_1'(z) = \cos h\lambda z, \quad G_2'(z) = z\cos h\lambda z, \quad AB = |\lambda|^6, \quad \lambda \in \mathbb{C}.$$

More details can be found in Umehara and Yamada (1993, 1996).

3. By taking

$$h(z) = \tan h\left(\frac{\sqrt{5}}{2}z\right) = \frac{\sin h(\alpha_1 z) + \sin h(\alpha_2 z)}{\cos h(\alpha_1 z) + \cos h(\alpha_2 z)},$$

with $\alpha_1 = \dfrac{\sqrt{5}-1}{2}$ and $\alpha_2 = \dfrac{\sqrt{5}+1}{2}$ and

$$\phi_3 = AF_1\overline{G}_1 + BF_2\overline{G}_2,$$

we can obtain the "Bonnet Cousins" corresponding to the solutions:

$$\begin{aligned}
F_1(z) &= \frac{1}{\alpha_1}\cos h(\alpha_1 z) + \frac{1}{\alpha_2}\cos h(\alpha_2 z)\\
G_1(z) &= \frac{1}{\alpha_1}\sin h(\alpha_1 z) + \frac{1}{\alpha_2}\sin h(\alpha_2 z)\\
F_2(z) &= \frac{1}{\alpha_1}\sin h(\alpha_1 z) - \frac{1}{\alpha_2}\sin h(\alpha_2 z)\\
G_2(z) &= \frac{1}{\alpha_1}\cos h(\alpha_1 z) - \frac{1}{\alpha_2}\cos h(\alpha_2 z),\\
AB &= \frac{1}{(\alpha_2^2 - \alpha_1^2)^2}.
\end{aligned}$$

With these three examples we have hyperbolic mean curvature one surfaces associated to euclidean minimal surfaces whose coordinate curves are planar curvature lines, as we show in *Hyperbolic Mean*

Curvature One Surfaces associated to Minimal Surfaces with Planar Curvature Lines, in preparation by B. Nelli and M. Elisa Galvão.

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