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Polyhedral Surfaces in $\mathbb{R}^3$ with Minimality Conditions.

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Sunto. – *Prendendo spunto dalla nozione di superficie minimale classica, introduciamo sulle superfici poliedrali, condizioni che permettano il riprodursi di alcuni dei principali fenomeni del caso liscio.*

1. – Introduction.

We consider polyhedral surfaces S in $\mathbb{R}^3$ obtained as image, by a symplicial proper embedding of a regular, 2-dimensional locally finite symplicial complex, which is homeomorphic to a surface of genus g without boundary minus a closed discrete set of points.

The goal of this paper is to find metric and combinatorial conditions on S, which allow the reproduction of some of the behaviours of classical minimal surfaces in $\mathbb{R}^3$, in particular the fact that a complete minimal surface with finite total curvature has parallel ends and it is conformally equivalent to a compact Riemann surface minus a finite number of points, and that a formula of Gauss-Bonnet type holds (for a survey on classical minimal surfaces see [O]).

We proceed in the following way.

A) We pick out a notion of minimality on polyhedral surfaces; roughly speaking, the idea arises from the fact that the coordinate functions of a smooth minimal surface are harmonic and so they have the mean value property (Section 2).

B) We introduce some natural hypothesis on the ends of S (essentially the existence of a limit normal vector in each end and bounded growth of the number of triangles) and we prove that these hypothesis, finite total curvature (cf. Definition 2.6), finite topological type and the property of semi-minimality (which is a conse-

C) We replace the hypothesis on ends by a different hypothesis on the embedding of S; also in this case we obtain the above result (cf. Section 4, 5).

D) We construct a family of examples of minimal polyhedral surfaces with genus 0 and two ends, which have all the properties listed above (cf. Section 7).

2. – Definitions and first properties.

Let K be a piecewise linear geometric realization in some $\mathbb{R}^n$ of a 2-dimensional, locally finite, regular simplicial complex (also denoted by K) [RS].

We assume that K is homeomorphic to a surface obtained by taking away a closed discrete set of points $\{p_i\}_{i\in I}$ from a surface M of genus g without boundary.

If g and $\#(I)$ are finite we say that the surface $M\setminus\{p_i\}_{i\in I}$ has *finite topological type*. Once for all, we fix a homeomorphism between $M\setminus\{p_i\}_{i\in I}$ and K.

Let us recall the definition of ends of K in a suitable way.

We assume that for every $i\in I$ there exists a neighborood $U(p_i)$ of p_i in M such that:

1) the boundary of $U(p_i)\setminus p_i$ in K is made by 1-simplies of K;

2) $(U(p_1)\setminus p_i)\cap(U(p_j)\setminus p_j)=\emptyset$ if $i\neq j$.

For any given $U(p_i)$ with the property above, there exists a sequence of neighborhoods $\{U_n(p_i)\}_{n\in\mathbb{N}}$ with the same properties such that

$$U_{n+1}(p_i)\subset U_n(p_i)\subset\ldots\subset U_1(p_i)=U(p_i)$$

and that for each compact set $C\subset\mathbb{R}^n$, $C\cap U_n(p_i)=\emptyset$ for n large enough.

Let $E_n^i=U_n(p_i)\setminus p_i$ where $n\in\mathbb{N}$, $i\in I$ and let

$$\mathcal{E}=\{E_n^i, n\in\mathbb{N}, i\in I\}.$$

We say that two elements E_n^i and E_m^j are equivalent if and only if $i=j$. This is an equivalence relation on $\mathcal{E}$ ...

we say that a certain property holds for E if it holds for the set $\{E_n\}_{n\in N}$ of its representants for n large enough.

Let v be a vertex of K: the *star* of v is the union of simplies of K which contain v. We denote it by st(v).

A *simplicial immersion* is a continuous map φ: K→$\mathbb{R}$ which sends i-simplices in i-simplices, i = 0, 1, 2, it is linear on each simpex and injective on each star.

A *simplicial embedding* is a globally injective simplicial immersion and if it is such that the inverse image of a compact set is a compact set, it is a *proper* simplicial immersion.

Let φ: K→$\mathbb{R}^3$ be a simplicial immersion; we call *polyhedral surface* the set S = φ(K)

Vertices, stars and ends of S are the image of vertices, stars and ends of K respectively; in particular when we speak about angles we mean the angles in S.

REMARK 2.1. – An orientation on K induces an orientation on S; then it is possible to choose a system of unit normal vectors to all the triangles of S in a compatible way. If ζ belongs to the interior of a triangle of S, we denote by N_ζ the unit normal vector to the triangle that contains ζ.

Henceforth S will always denote a polyhedral surface with a fixed orientation.

REMARK 2.2. – We interpret the 1-skeleton of K as a metric geodesic space, giving length 1 to each edge. Fix a vertex v_0 of K and consider the ball of radius n and center v_0 in this metric; let $B_n^{v_0}$ be the subset of K formed by the simplices which either lie or have their boundary in the ball just defined.

Let φ: K→$\mathbb{R}^3$ be a proper immersion which defines S; then for every n there exist r_1, r_2 such that $B(0,r_1)\cap S\subset\varphi(B_n^{v_0})\subset B(0,r_2)\cap S$ ($B(0,r)$ is the Euclidean ball). We freely use this property in the following.

Let v be a vertex of S, $a^1,\ldots,a^n$ the boundary vertices of st(v) and $\alpha^1,\ldots,\alpha^n$ the angles at v of st(v) in S (by abuse of notation, we will often denote a vector in $\mathbb{R}^3$ and its terminal point with the same symbol).

Now, we give a notion of minimality on a polyhedral surface. The idea of this definition arises from the fact that the coordinate func-

Definition 2.3. – We say that the star st(v) is *minimal* if:

$$\sum_{j=1}^{n}(a^j - v) = 0 .$$

The surface S is minimal if, for each vertex v of S, st(v) is minimal.

If we interpret each vertex of S as a point of unitary mass, this condition means that every vertex of S is the barycenter of its star.

Remark 2.4. – It follows from the definition that:

(i) the minimal star with three boundary vertices is a planar star;

(ii) if st(v) is a non planar minimal star and P is a plane such that $v \in P$, then st(v) cannot lie completely in one of the closed halfspaces determined by P.

We often use this last property instead of minimality so it is convenient to give the following definition.

Definition 2.5. – The non planar star st(v) is called *semi-minimal* if for any plane P passing by v, st(v) cannot lie completely in any of the two closed halfspaces determined by P. The surface S is called semi-minimal if for each vertex v of S st(v) is semi-minimal.

This condition is analogous to Gauss curvature strictly smaller than 0 in the smooth case. A non planar minimal surface (i.e. a minimal surface without planar stars) is in particular a semi-minimal surface.

Definition 2.6. – The real number

$$C(\mathrm{st}(v)) = 2\pi - \sum_{j=1}^{n} \alpha^j$$

is called *curvature* of st(v); if $\{v_i\}_{i \in I}$ is the set of vertices of S, the formal series

$$C(\mathrm{S}) = \sum_{i \in I} C(\mathrm{st}(v_i))$$

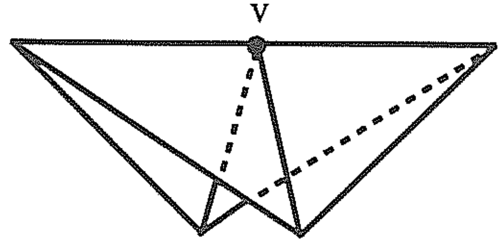

Figure 2.1.

If the series is convergent we say that S has *finite total curvature* and we write $C(\mathrm{S}) > -\infty$.

Remark 2.7. – In the case of semi-minimal surfaces (and so for minimal surfaces) it is easy to see that the formal series which exprimes the total curvature has all the terms of sign less or equal to zero, so its convergence and sum do not depend on the ordering of terms; furthermore a minimal star has curvature equal to 0 (we say *cylindrical star*, see fig. 2.1) if and only if it is planar star; so for a minimal surface S, $C(\mathrm{S}) = 0$ if and only if S is a plane.

Remark 2.8. – Let S be a semi-minimal, not planar polyhedral surface with finite total curvature. Then curvature of stars in the ends of S tends to be 0, i.e. for each $\varepsilon > 0$ there exists $n(\varepsilon) \in \mathbb{N}$ such that for every $n \geq n(\varepsilon)$ and $v \in S \setminus B_n^{v_0}$ (where v_0 is an arbitrary fixed vertex of S), we have

$$C(\mathrm{st}(v)) \geq -\varepsilon .$$

This is a sort of convergence to planes for the stars which belong to the ends of a semi-minimal surface with finite total curvature. We recall that in the smooth case, a complete minimal surface with finite total curvature has the conformal structure of a compact Riemann surface minus a finite number of points, it has normal vector well defined in the ends and each end is a graph over the complementary of a compact set of the plane orthogonal to the limit vector. Further, if ends are disjoint, this limit vector (and so the plane) is the same for each end of the surface (cf. [JM]).

The convergence we have remarked for semi-minimal polyhedral surfaces does not guarantee that stars in the ends are graph over a plane and that the normal vector is well defined in each end, in fact

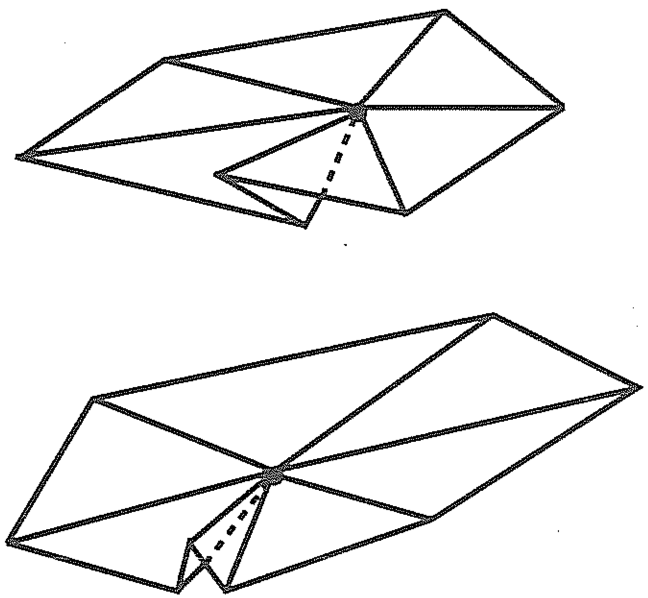

Figure 2.2.

In Section 3, we will define a type of end such that patologies of this kind are excluded. This condition will be a consequence of minimality, finitness of curvature, bounded growth and good triangles (cf. Section 6).

3. – Good ends.

DEFINITION 3.1. – Let E be an end of a polyhedral surface S; we say that E is a *good* end if there exists an unitary vector $\boldsymbol{w} \in \mathbb{R}^3$ such that for each $\varepsilon > 0$ there exists $n(\varepsilon)$ such that for every $n > n(\varepsilon)$ and for each normal vector N to (triangles of) E_n:

$$|\langle N, \boldsymbol{w} \rangle| \geqslant 1 - \varepsilon ,$$

where $\langle\, ,\rangle$ is the standard scalar product in $\mathbb{R}^3$.

If E is a good end, angles between triangles of E tend to be 0 i.e. stars in the ends tend to planar.

If an end E of S is good we say that normal vectors to E tend to a limit vector $\boldsymbol{w}$.

surface S is a graph over the complementary of a compact set of the plane P orthogonal to the limit vector of the normal vectors to E. In fact it is obvious that E is a graph over a subset of P and by the fact that the immersion is proper, the map which describes E as a graph is defined over the complementary of a compact set of the plane P.

THEOREM 3.3. – *Let* S *be a polyhedral, properly embedded, semi-minimal surface, with good ends. Then each end is a graph over the complementary of a compact set of the same plane.*

PROOF. – We restict to an end E of S and let $\zeta = (\zeta_1, \xi_2, \xi_3)$ be its coordinates in $\mathbb{R}^3$; up to a rotation we can assume that E is a graph over a non compact set of the plane $\{z = 0\}$ that does not contain the origin; the limit vector for E is $\boldsymbol{w} = (0,0,1)$, hence for every ε there exists r_ε such that

$$|\langle N_\zeta, \boldsymbol{w} \rangle| \geqslant 1 - \varepsilon \tag{3.1}$$

for every $r > r_\varepsilon$, $\zeta \in \mathrm{E}_n$, $n > n_r$.

For each $\zeta \in E$ there exists a vertical plane π_ζ passing by the origin and containing ζ. The projection of $E \cap \pi_\zeta$ on the plane $\{z = 0\}$ is the union of two half straight-lines, say l_ζ and m_ζ, that do not intersect; hence $E \cap \pi_\zeta$ has two connected components that are graphs over l_ζ and m_ζ of a piecewise linear function $f_\zeta \colon l_\zeta \cup m_\zeta \to \boldsymbol{R}$.

We can assume that ζ is the image of a point of l_ζ and consider the parameter t on l_ζ; for every $t \in l_\zeta$ the absolute value of the angle between l_ζ and the linear tract to which $f_\zeta(t)$ belongs is equal to the absolute value of the angle between the normal to $f_\zeta(t)$ on π_ζ and $\boldsymbol{w}$ which is smaller than the absolute value of the angle θ_t between the normal to the simplex of the surface to which $f_\zeta(t)$ belongs and $\boldsymbol{w}$; so, where it is defined

$$|f'_\zeta(t)| \leqslant |\tan \theta_t| .$$

By (3.1), there exists t_ε such that for every $t > t_\varepsilon$

$$|\cos \theta_t| \geqslant 1 - \varepsilon, \ |\sin \theta_t| \leqslant \sqrt{2\varepsilon} \tag{3.2}$$

and then

$$\sqrt{2\varepsilon}$$

This means that the piecewise linear curves that constitute $E \cap \pi_\zeta$ have horizontal asymptotic directions.

For r big enough, $S^2(r) \cap E$ is a union of points and piecewise smooth curves. If we exclude a discrete growing sequence $\{s_n\}_{n\in\mathbb{N}}$ of values of r (where $S^2(r) \cap E$ contains points), $S^2(r) \cap E$ is constituted by only one curve C_r that turns around the z axis. In fact if $S^2(r) \cap E$ has more than one curve or a curve that does not turn around the z axis, there exist a compact set on E with boundary on $S^2(r)$ and this easily gives a contradiction by semi-minimality.

From now on, we exclude the sequence $\{s_n\}_{n\in\mathbb{N}}$ from the values allowed for r. We will prove that C_r tends to the horizontal equator $S^2(r) \cap \{z=0\}$ as $r \to \infty$; this is equivalent to prove that for each fixed ε

$$\left|\left\langle \frac{\zeta}{|\zeta|}, w \right\rangle\right| \leq \varepsilon$$

for every $\zeta \in S^2(r) \cap E$, for r big enough.

Let $\zeta = (r, f_\zeta(r))$ then

$$\left|\left\langle \frac{\zeta}{|\zeta|}, w \right\rangle\right| \leq \frac{|f_\zeta(r)|}{r}.$$

It follows easily from (3.2) that for every $\varepsilon' > 0$ there exists $r_{\varepsilon'}$ such that

$$|f_\zeta(r)| \leq (c_\zeta + r)\sqrt{2\varepsilon'}$$

for $r > r_{\varepsilon'}$, where c_ζ is a constant depending on the direction determined by $\pi_\zeta \cap \{z=0\}$; hence $c = \sup_{\zeta\in E} c_\zeta = \sup_{\zeta\in S^1} c_\zeta < \infty$ and c depends on E only.

Thus, if $\varepsilon' = \varepsilon^2/8$ and $r > \max\{c, r_{\varepsilon'}\}$, we have

$$\left|\left\langle \frac{\zeta}{|\zeta|}, w \right\rangle\right| \leq \left(\frac{c+r}{r}\right)\sqrt{2\varepsilon'} \leq \varepsilon.$$

If two ends E, E' of S have different normal limit vectors, the curves $C_r = S^2(r) \cap E$ and $C'_r = S^2(r) \cap E$ must intersect for r big enough (as they are asymptotic to different equators) and this is in contrast with the fact that S is embedded. ■

4. – The Gauss-Bonnet formula and the topological structure.

There is a classical analogous of Gauss-Bonnet theorem for compact polyhedral surfaces, proved in [B] and [P], that is the following.

Assume that S is a compact polyhedral surface and let $\chi(S)$ be its Euler characteristic. If S has not boundary then

$$C(S) = 2\pi\chi(S); \tag{4.1}$$

if S has a boundary made up by a finite number of piecewise linear curves, then

$$C(S) + \sum_{j=1}^{r}\sum_{i=1}^{p}(\pi - \alpha_i^j) = 2\pi\chi(S) \tag{4.2}$$

where if C^j is the j-th boundary curve, each α_i^j $i = 1, \ldots, p$ is the sum of the angles of the surface at the i-th vertex of C^j.

Consider a polyhedral surface S such that its intersection with $S^2(r)$ is made by Jordan piecewise smooth curves C_r^j, $j = 1, \ldots, k$.

We define the total curvature of the boundary curve C_r^j as follows.

$$C_{\text{tot}}(C_r^j) = \sum_{1=1}^{p_j^r}[\theta_{r,i}^j + \beta_{r,i}^j]$$

where $\beta_{r,i}^j$ is the length of the arc of circle which determines the i-th tract of the curve $C_r^j = \bigcup_{i=1}^{p_j^r} C_{r,i}^j$ and $\theta_{r,i}^j$ is defined as follows: let $\gamma_{r,i-1}^j$ and $\gamma_{r,i}^j$ be the angles (internal to S_r) that the arcs $C_{r,i-1}^j$ and $C_{r,i}^j$ (i.e. their tangents) form with the edge common to the two triangles of S which determine $C_{r,i-1}^j$, $C_{r,i}^j$; then $\theta_{r,i}^j = |\pi - \gamma_{r,i-1}^j - \gamma_{r,i}^j|$ (see Fig. 4.1).

Here is an easy generalization of polyhedral Gauss-Bonnet theorem in the case of a polyhedral surface whose boundary is made by arcs of circle.

LEMMA 4.1. – *If* $S_r = S \cap B^2(r)$ *in the notations above, we have:*

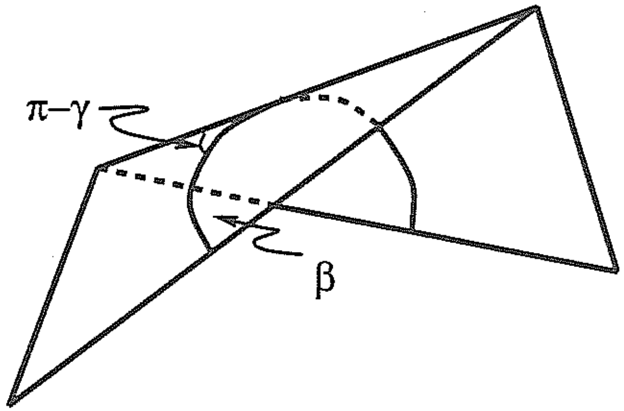

Figure 4.1.

PROOF. – We want to obtain (4.3) from (4.2) by approximating the boundary arcs of circle in a suitable way.

Fix a boundary curve C_r^j and one of its tracts. For simplicity of notation we omit indices and denote the fixed tract by l, the radius of the corresponding circle by R and the subtended arc by β. Approximate l by the sequence of linear tracts $\{L_h\}$ defined as follows:

L_1 is the segment which joins the extremities of l; let l_1^2, l_2^2 be the two arcs which result by dividing l in two equal parts and let L_2 be the union of the segments which joint the extremities of these arcs; in general let $l_1^h, \ldots, l_{2^{h-1}}^h$ be the arcs obtained dividing by half the arcs $l_1^{h-1}, \ldots, l_{2^{h-2}}^{h-1}$ and let L_h be the union of the segment that joint their extremities. L_h is a curve made by 2^{h-1} linear tracts, whose vertices divide the arc l in 2^{h-1} equal parts.

Let a and b be the two extremities of l, and let e_a and e_b be the two 1-simplices of the surface S (and then of S_r) which contain a and b; let γ_a and γ_b be the angles that the tangent to l form with e_a and e_b in a and b respectively, let γ_a^h and γ_b^h be the angles that l_1^h and $l_{2^{h-1}}^h$ make with e_a and e_b respectively, then:

$$\gamma_a^h \to \gamma_a, \quad \gamma_b^h \to \gamma_b \quad \text{as } h \to \infty .$$

By computing the contributions K_h of each L_h in (4.2) we obtain:

$$K_1 = 0, \quad K_2 = \frac{\beta}{2}, \quad K_3 = \frac{3}{4}\beta, \ldots, K_h = (1 - 2^{1-h})\beta .$$

Now, writing (4.2) for surfaces $\tilde{S}_r^h$ obtained from S_r by approximation of the boundary curves in the way described above, we obtain:

$$C(\tilde{S}_r^h) + \sum_{j=1}^{k} \sum_{i=1}^{p_r^j} [(\theta_{r,i}^j)_h + (1 - 2^{1-h})\beta_{r,i}^j] = 2\pi\chi(\tilde{S}_r^h) \tag{4.4}$$

where k_r is the number of the boundary curves, p_r^j is the number of arcs of circle in the j-th boundary curve of S_r, and $\beta_{r,i}^j$ are the equivalent to the previous $|\pi - \gamma_a - \gamma_a'|$ (where γ_a and γ_a' are on the two triangles of S which have a in common) and β respectively.

Observe that $\chi(\tilde{S}_r^h) = \chi(S_r)$; further $C(\tilde{S}_r^h) = C(S_r)$ because the total curvature involves complete stars only and they are the same in $\tilde{S}_r^h$ and in S_r). Then, if we take the limit of the two sides of (4.4) as $h \to \infty$ we obtain

$$C(S_r) + \sum_{j=1}^{k} \sum_{i=1}^{p_r^j} [\theta_{r,i}^j + \beta_{r,i}^j] = 2\pi\chi(S_r) . \quad \blacksquare$$

We introduce a further hypothesis on the ends of S and we prove a formula of Gauss-Bonnet type for S (the same formula in the smooth case is established in [JM]); we remark that the examples that we discuss satisfy this hypothesis (see Section 7).

DEFINITION 4.2. – Let E be an end of a polyhedral embedded surface S and L a positive constant; we say that E has *L-bounded growth* if for each r

$$\#\{\text{triangles } T \text{ of } E \,|\, T \cap S^2(r) \neq \emptyset\} \leq L .$$

THEOREM 4.3. – *Let* S *be a polyhedral, properly embedded, semiminimal surface of finite topological type, with k good ends with L-bounded growth. Then:*

$$C(S) = 2\pi(\chi(S) - k) .$$

PROOF. – By Theorem 3.3 we may assume that all the ends of S are graphs over the plane $\{z = 0\}$ and we can choose r big enough such that $S^2(r) \cap S = \bigcup_{j=1}^{k} C_r^j$, where C_r^j are disjoint Jordan curves, each union of a finite number of arcs of circle. As in Theorem 3.3 we

r big enough, S_r is a compact surface with boundary components $C_r^1, \dots, C_r^k$. By the generalized Gauss-Bonnet formula applied to S_r, we have

$$C(S_r) + \sum_{j=1}^{k} C_{\text{tot}}(C_r^j) = 2\pi\chi(S_r).$$

Observe that for r big enough, $\chi(S_r) = \chi(S)$ and that $C(S_r) \to C(S)$ as $r \to \infty$. Then it is enough to prove that for each $j = 1, \dots, k$

$$(4.5) \qquad C_{\text{tot}}(C_r^j) = \sum_{i=1}^{p_r^j} [\theta_{r,i}^j + \beta_{r,i}^j] \to 2\pi \quad \text{as } r \to \infty .$$

We fix a $j \in \{1, \dots, k\}$ and we omit the index j in all the symbols involved with the curve C_r^j.

We claim that

$$(4.6) \qquad \sum_{i=1}^{p_r} \theta_{r,i} \to 0 \sum_{i=1}^{p_r} \beta_{r,i} \to 2\pi \quad \text{as } r \to \infty .$$

Let $\Theta_r = \max_{i=1,\dots,p_r} \theta_{r,i}$ and $\theta_r = \min_{i=1,\dots,p_r} \theta_{r,i}$, then Θ_r, $\theta_r \to 0$ as $r \to \infty$ because each pair of triangles which determine $\theta_{r,i}$ tend to be on the same plane (good ends). Then for each ε there exists r_ε such that $|\theta_r|$, $|\Theta_r| \leq \varepsilon L^{-1}$ for every $r > r_\varepsilon$; as $p_r \leq L$ we have

$$-\varepsilon \leq L\theta_r \leq \sum_{i=1}^{p_r} \theta_{r,i} \leq L\Theta_r \leq \varepsilon$$

and this gives the former limit in (4.6).

For the latter limit we proceed in the following way.

The curve $c_r = C_r / r$ is the radial projection of C_r on S^2 the angles $\beta_{r,i}$ are invariant for this projection, hence we can compute the limit by considering the curve c_r. We parametrize c_r as follows. Let $\alpha: [0, 2\pi] \to S^1$ be a parametrization by arc length of $S^1 \subset \{z = 0\}$; for every $t \in [0, 2\pi]$ consider the maximum circle $k_{r,t}$ of S^2 passing by $\alpha(t)$ and by the point $(0,0,1)$ and let $\beta_r(t)$ be the point common to $k_{r,t}$ and c_r which is nearest to $\alpha(t)$. Each point $\beta_r(t)$ is uniquely determined, so that $\beta_r: [0, 2\pi] \to S^2$ is a parametrization of c_r; it is piecewise differentiable because it is made by a finite ($\leq L$) number of arcs of circle.

tion $\varphi_r: [0, 2\pi] \to \mathbb{R}$ such that

$$\beta_r(t) = (\cos t \sin \varphi_r(t), \sin t \sin \varphi_r(t), \cos \varphi_r(t))$$

and $|\beta_r'| = \sqrt{\sin^2 \varphi_r(t) + \varphi_r'(t)^2}$, where φ_r' is defined.

We have that

$$(4.7) \qquad \sum_{i=1}^{p_r} \beta_{r,i} = \int_0^{2\pi} |\beta_r'(t)| \, dt$$

where the integral is well defined because β_r is in $L^1([0, 2\pi])$.

The 3rd component of β_r tends to 0 as $r \to \infty$ i.e. for each ε there exists r_ε such that for every $r > r_\varepsilon$ and $t \in [0, 2\pi]$

$$(4.8) \qquad |\cos \varphi_r(t)| \leq \varepsilon$$

As ends are good, the tangent vector to β_r (where it is defined) tends to have 0 as third component i.e. for each ε there exists $\bar{r}_\varepsilon$ such that for every $r > \bar{r}_\varepsilon$

$$(4.9) \qquad |\sin \varphi_r(t) \varphi_r'(t)| \leq \varepsilon$$

where φ_r' is defined.

Inequalities (4.8) and (4.9) imply that $|\beta_r'| \to 1$ in $L^1([0, 2\pi])$ then the integral in (4.7) tends to 2π and this gives the latter limit in (4.6). ■

With a further hypothesis of uniform convergence of the normal vectors to (the faces of) the ends of the surface (that is obviously satisfied when the surface has a finite number of ends) we will deduce finite topological type from finite total curvature.

More precisely, let S be a polyhedral semi-minimal surface with finite total curvature and good ends and let $\boldsymbol{w}$ be the limit normal vector for the ends of S.

DEFINITION 4.4. – We say that S has *uniformly good ends* if there exists a dense subset D of $\mathbb{R}$ such that for each ε there exists $r(\varepsilon)$ such that for every $r \in [r(\varepsilon), \infty) \cap D$

$$|\langle N_\zeta, \boldsymbol{w} \rangle| \geq 1 - \varepsilon$$

for every $\zeta \in S \cap S^2(r)$ such that N_ζ is defined.

This means that the normal vectors to each end tend to $\boldsymbol{w}$ uniformly. Now we can state the following theorem.

minimal surface with finite total curvature and uniformly good ends with bounded growth; then S *has finite topological type.*

PROOF. – We want to use the generalized Gauss-Bonnet formula for the surfaces

$$S_r = B(0,r) \cap S$$

whose curvature tends to the curvature of S as $r \to \infty$.

Let g, k and g_r, k_r be the genus and the number of boundary components of S and S_r respectively; $g_r \to g$ and $k_r \to k$ as $r \to \infty$.

As the ends of S are good, for $r \in D$ big enough, we have that $S^2(r) \cap S = \bigcup_{j=1}^{k_r} C_r^j$, where C_r^j are closed curves or points.

For each S_r, we have equality (4.3) and by uniformity $C_{\text{tot}}(C_r^j) \to 2\pi$ as $r \to \infty$ independently on j. Hence, for every $j = 1, \dots, k_r$ and $\varepsilon > 0$ there exists $\bar{r}$ such that for every $r > \bar{r}$

$$2\pi - \varepsilon \leqslant C_{\text{tot}}(C_r^j) \leqslant 2\pi + \varepsilon .$$

Then:

$$2\pi\chi(S_r) - k_r(2\pi + \varepsilon) \leqslant C(S_r) \leqslant 2\pi\chi(S_r) - k_r(2\pi - \varepsilon) .$$

Theorem 4.5 follows easily by this inequality and $C(S) > -\infty$.

5. – The conformal structure.

Let S be a polyhedral surface; we define a conformal structure on S by constructing an atlas $\{U_\lambda, \phi_\lambda\}_{\lambda \in \Lambda}$ for S in the following way.

If $P \in S$ is an interior point of a triangle T, then $U_\lambda(P) = \text{int}(T)$ and ϕ_λ maps int(T) identically on an open set of the complex plane.

If $P \in S$ is an interior point of an edge l and T_l, T_l' are the triangles whose boundaries contain l, then $U_\lambda(P) = \text{int}(T_l) \cup \text{int}(T_l')$ and ϕ_λ maps l on an interval $[a, b]$ of the real axis of the complex plane and int(T_l), int(T_l') identically on two open sets such that intersection of closures of the sets $\phi_\lambda(\text{int}(T_l))$ and $\phi_\lambda(\text{int}(T_l'))$ is equal to $[a, b]$.

If $P \in S$ is a vertex and γ is the sum of the angles of st(P) incident in P, then $U_\lambda(P) = \text{st}(P)$ and ϕ_λ is the composition of

map defined by $z \to z^{2\gamma^{-1}\pi}$. The transiction functions of the atlas $\{U_\lambda, \phi_\lambda\}_{\lambda \in \Lambda}$ are conformal.

We remark that this conformal structure depends on the immersion in $\mathbb{R}^3$; for some remarks about conformal structures on minimal polyhedral surfaces cf. [PP].

THEOREM 5.1. – *Let* S *be a polyhedral, properly embedded, semi-minimal surface with finite total curvature, good ends with bounded growth and finite topological type. Then* S *is conformally equivalent to a compact Riemann surface minus a finite number of points.*

PROOF. – By hypothesis, S is homeomorphic to a compact Riemann surface of genus $g < \infty$ minus $k < \infty$ points and each end of S is a graph over the complementary of a compact set of the same plane, that we can assume to be $\{z = 0\}$, so it is homeomorphic to a disc minus one point.

This implies that each end of S is conformally equivalent to a disc minus one point or to an annulus. To prove that only the first case is realized, we make use of some properties of quasiconformal maps in the plane (cf. [A], [LV]).

Let E be an end of S, let $\{T_{n,m}\}_{m \in \mathbb{N}}$ be the triangles of E_n, $\vartheta_{n,m}$ the angle between the plane $\{z = 0\}$ and the plane $\pi_{n,m}$ to which $T_{n,m}$ belongs. Let $\vartheta_n = \sup_m \vartheta_{n,m}$; as E is a good end, we have $\lim_{n \to \infty} \vartheta_n = 0$.

The restriction of the orthogonal projection pr_z on $\{z = 0\}$ to a triangle $T_{n,m}$ is an affine map, then it is $K_{n,m}$-quasiconformal; we compute $K_{n,m}$.

If we choose appropriate coordinates on the plane $\pi_{n,m}$ and on the plane $\{z = 0\}$, $pr_{z|T_{n,m}}$ is defined (up to translation) by

$$f_{T_{n,m}}(y_1, y_2) = (y_1, y_2 \cos \vartheta_{T_{n,m}}) .$$

Then (cf. [A])

$$K_{n,m} = \frac{|1 + \cos \vartheta_{n,m}| + |1 - \cos \vartheta_{n,m}|}{|1 + \cos \vartheta_{n,m}| - |1 - \cos \vartheta_{n,m}|}$$

and if we fix ε, there exists $n(\varepsilon)$ such that for every $n > n(\varepsilon)$

$$K_{\cdots} = \frac{1}{\cdots} \leqslant \frac{1}{\cdots} = K .$$

Then for every $n>n(\varepsilon)$ and $m\in\mathbb{N}$, $pr_{z|T_{n,m}}$ is a K-quasiconformal map with K independent on n, m.

It follows that pr_z: $\mathrm{E}\to pr_z(\mathrm{E})$ is a piecewise affine homeomorphism and it is K- quasiconformal on the complement of a set of zero measure (the 1-skelton of E), then it is a K-quasiconformal map.

Hence, each end of S is conformally equivalent to a disc minus one point.

Theorem 5.1 is proved. ■

We remark that in Theorem 5.1, we can substitute the hypothesis of finite topological type with uniformly good ends (by Theorem 4.5).

6. – Good triangles and homogeneous ends.

In this Section, S is a polyhedral surface defined by a proper embedding $\varphi:\mathrm{K}\to\mathbb{R}^3$ such that each end of S has L-bounded growth.

We will discuss a class of polyhedral minimal surfaces which falls to be in the hypothesis of Section 4.

DEFINITION 6.1. – We say that triangles of S are *good* if there exists a constant $h=h(\varphi)>0$ such that for each triangle T of S, the ratio between the radius R_T of the circumscribed circle and the radius r_T of the inscribed circles is such that:

$$1<\frac{R_T}{r_T}<h\,.$$

REMARK 6.2. – Let T a good triangle, a_1, a_2, a_3 its edges and α_1, α_2, α_3 the angles opposite to these edges respectively (we often denote an edge and its length with the same symbol). By elementary trigonometry, there exist two constants h_1, $h_2>0$, which only depend on the constant h, such that

$$h_1<\alpha_1,\alpha_2,\alpha_3<h_2$$

and

$$\sin h_1<\frac{a_i}{a_j}<\frac{1}{\sin h_1}\qquad \forall i,j=1,2,3\,.$$

REMARK 6.3. – Let S be a polyhedral semi-minimal surface with

Semi-minimality and $C(\mathrm{S})>-\infty$ imply that for each $\varepsilon>0$ there exists $n(\varepsilon)$ such that for every $n>n(\varepsilon)$ and for each vertex $v\in\mathrm{S}\cap(\mathbb{R}^3\setminus B(0,n))$, we have

$$2\pi<\alpha_1+\ldots+\alpha_M<2\pi+\varepsilon$$

where $\alpha_1,\ldots,\alpha_M$ are the angles at v of st(v). With notations of Remark 6.2 we have $Mh_1<\alpha_1+\ldots+\alpha_M<2\pi+\varepsilon$, hence $M<(2\pi+\varepsilon)/h_1$; so

$$M<\left[\frac{2\pi}{h_1}\right]$$

because ε is arbitrarly small; this means that the valency of the vertices of S is bounded.

Let E be an end of S and $\{v_n\}$ a sequence of vertices of E; if $v_n\in E\setminus B(0,n)$ for each n, we say that the sequence tends to infinity.

LEMMA 6.4. – *Let* S *be a polyhedral properly embedded minimal surface, with* $C(\mathrm{S})>-\infty$ *and good triangles and let* E *be an end of* S*. For each sequence of vertices* $\{v_n\}\subset E$ *which tends to infinity, there exists an unitary vector* $\boldsymbol{v}\in\mathbb{R}^3$ *and a subsequence of* $\{v_n\}$ *such that the normal vectors to the stars of the subsequence tend to* $\boldsymbol{v}$.

PROOF. – Up to subsequences we can assume that the valency of v_n is M for each n (by abuse of notation we will often denote sequence and subsequence by the same symbol).

Let l_j^n, T_j^n, N_j^n and α_j^n, $j=1,\ldots,M$ be the edges and the triangles of st(v_n), their normal unitary vectors and their angles at v_n respectively.

Translate each st(v_n) in the origin and divide each edge by $\sum_{j=1}^{M}$ area (T_j^n); denote by st$_0(v_n)$ the obtain star.

Area $(T_j^n)=l_j^n\,l_{j+1}^n\sin\alpha_j^n$, $j=1,\ldots,M$ and as triangles are good, there exists a positive constant h such that for each n, j, $\sin\alpha_j^n>\sin h$; then the edges of st$_0(v_n)$ have bounded length.

Identify st$_0(v_n)$ with the set of its edges i.e. an M-uple of vectors in $\mathbb{R}^3$, say $(u_1^n,\ldots,u_M^n)\in(\mathbb{R}^3)^M$.

As each u_j^n has bounded, the set $\{(u_j^n,\ldots,u_M^n)\}$ is relatively compact in the topology of the M-uples of $\mathbb{R}^3$. Thus, there exists a subsequence that converges to an M-uple $(u_1,\ldots,u_M)\in(\mathbb{R}^3)^M$

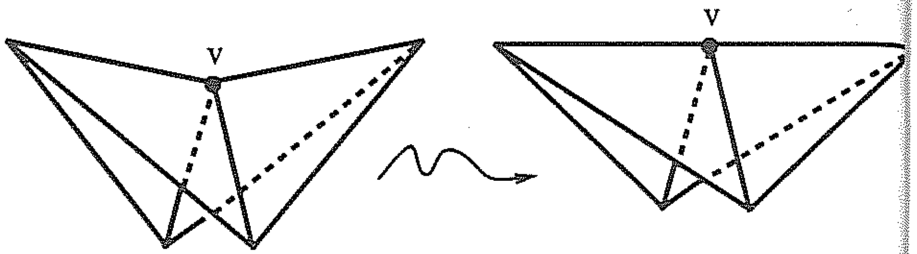

Figure 6.1.

equal to 0, hence it is a planar star. Thus, there exists an unitary vector N such that $N_1^n, \dots, N_M^n$ tend to N as n tends to infinity. ∎

We remark that the «limit» of a semi-minimal star is not necessarily semi-minimal: for example the cilindrical star (not semi-minimal) can be obtained as a limit of a sequence of semi-minimal stars as in fig. 6.1. So in this lemma, minimality is essential.

Lemma 6.4 allows us to consider only one of the sequences $\{N_j^n\}$, to determine the limit vector $\boldsymbol{v}$, that we call the limit normal vector to the sequence of stars. Once for all, we choose the sequence $\{N_1^n\}$ and we denote it by $\{N^{v_n}\}$.

We introduce a further assumption on the ends of S, in order to obtain a property which imitates the analyticity of the Gauss Map of the smooth case.

DEFINITION 6.5. – Let E be an end of S and K a positive constant; we say that E is *K-omogeneous* if for each sequence $\{v_n\}$ of vertices of E which tends to infinity and such that $N^{v_n} \to \boldsymbol{v}$ there exists a $\{v_n'\}$ such that $v_n' \in B_{v_0}^{n+K} \setminus B_{v_0}^n$ and $N^{v_n'} \to \boldsymbol{v}$. This means that for each accumulation point of the set of unitary normal vectors to E, there is a sequence of unitary normal vectors which tends to it, which has an element in each $B_{v_0}^{n+K} \setminus B_{v_0}^n$.

LEMMA 6.6. – *Let* E *be a K-homogeneus minimal embedded end which has L-bounded growth and good triangles, then for each sequence $\{v_n\} \in$ E which tends to infinity the sequence $\{N^{v_n}\}$ converges.*

PROOF. – It is enough to prove that there is only one accumulation point for the set of normal unitary vectors to E: then we apply

By contradiction, suppose there exist two different accumulation points, say $\boldsymbol{v}$ and $\boldsymbol{w}$; as E is a K-homogeneus end we can find two sequences $\{v_n\}$ and $\{w_n\}$ such that $v_n, w_n \in B_{v_0}^{n+K} \setminus B_{v_0}^n$ and $N^{v_n} \to \boldsymbol{v}$, $N^{w_n} \to \boldsymbol{w}$.

L-bounded growth implies that there exist $a_n^1, \dots, a_N^{H(n)}$, $H(n) \leq KL$ vertices of a path made of edges in $B_{v_0}^{n+K} \setminus B_{v_0}^n$ which joins v_n and w_n; without loss of generality we can assume $H(n) = LK$ for each $n \in \mathbb{N}$.

For each n, $\mathrm{st}(v_n)$ and $\mathrm{st}(a_n^1)$ have a triangle in common at least, then by Lemma 6.5 there exists a subsequence of $\{a_n^1\}$ such that $\{N^{a_n^1}\}$ converges to $\boldsymbol{v}$.

By a diagonal process, we obtain $\boldsymbol{v} = \boldsymbol{w}$. Absurd. ∎

COROLLARY 6.7. – *Let* S *be a polyhedral embedded minimal surface, with finite total curvature, good triangles, bounded growth and homogeneous ends. Then* S *has uniformly good ends.*

PROOF. – By previous lemmas, such a surface has good ends. By Theorem 3.3 each sequence of unitary normal vectors of stars which tends to infinity either converges to $\boldsymbol{v}$ (or $-\boldsymbol{v}$) or swings between these two vectors. If ends are not uniformly good, there exists a sequence of vertices which tends to infinity such that the associated sequence of normal vectors remains at a bounded distance from $\boldsymbol{v}$ and $-\boldsymbol{v}$: contradiction. ∎

Then we have the analogous of Theorem 4.3, 4.5 and 5.1.

THEOREM 6.8. – *Let* S *be a minimal surface of finite total curvature, with good triangles, bounded growth and homogeneous ends. Then it has finite topological type and*

$$C(S) = 2\pi(\chi(S) - k).$$

THEOREM 6.9. – *Let* S *be a minimal surface with finite total curvature, good triangles, bounded growth and homogeneous ends. Then* S *is conformally equivalent to a Riemann surface minus a finite numbers of points.*

7. – An example.

In this Section we construct a family depending on two parame-

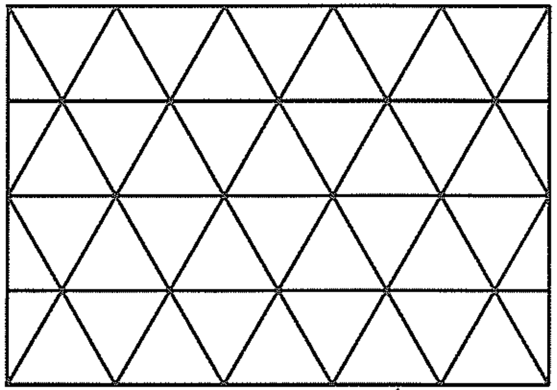

Figure 7.1.

ends; they are a polyhedral realization of the classical catenoid (cf. [O]).

First of all we describe a symplicial complex by one of its realization in $\mathbb{R}^3$. In the plane with coordinates x, y, consider the strip $[0, 2\pi] \times \mathbb{R}$; divide each segment $\{y = n, n \in \mathbb{N}\} \cap [0, 2\pi] \times \mathbb{R}$ in m equal portions and consider the triangulation with isosceles triangles described in fig. 7.1.

Then, identify the lines $x = 0$ and $x = 2\pi$ such that the point $(0, n)$ coincides with the point $(2\pi, n)$ for each n.

Denote by $K(m)$ the symplicial complex obtained in this way; we want to find a simplicial minimal immersion $\varphi_m : K(m) \to \mathbb{R}^3$ such that:

(i) $\varphi_m(\mathrm{K}(m))$ is symmetric with respect to the plane $\{z = 0\}$;

(ii) $\varphi_m(\mathrm{K}(m))$ intersects the plane $\{z = 0\}$ in the regular m-gon inscribed in the unitary circle around the origin;

(iii) $\varphi_m(\mathrm{K}(m))$ intersects each horizontal plane in a regular m-gon inscribed in a circle centred on the z axis.

The projection of such surface on the horizontal plane will be of the kind indicated in fig. 7.2 (case m = 6).

Let us construct the immersion. We call step 1 the completion of the stars of the vertices lying on the unitary circle of the plane $\{z = 0\}$ and, by recurrence, step i the completion of the stars of the vertices lying on the regular m-gon which is the image of the line

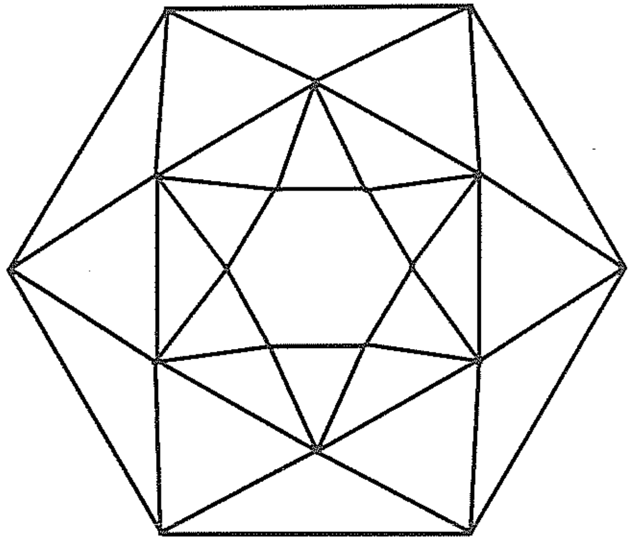

Figure 7.2.

tal m-gon are equal, then to obtain minimality we must impose only one condition along the horizontal component and one condition on the vertical one at each step; further we can restrict to the halfspace $\{z \geq 0\}$.

Denote by $r_i(m)$ the distance from the origin of the projection of the vertices on $\{z = 0\}$ at step i; hence minimality on the horizontal component means

$$\begin{cases} r_0(m) = 1, \\ r_1(m) = b(m), \\ r_{i+1}(m) = 2b(m)\, r_i(m) - r_{i-1}(m), \qquad i \geq 1, \end{cases}$$

where $b(m) = ((\sin \pi / m)^2 + 1)(\cos \pi / m)^{-1}$. We obtain the following expression for $r_i(m)$

$$r_i(m) = \frac{1}{2}(b(m) + \sqrt{b(m)^2 - 1})^i + (b(m) + \sqrt{b(m)^2 - 1})^{-i}, \qquad i \geq 0.$$

We remark that:

(i) $b(m) \searrow 1$ as $m \to \infty$;

Now we determine the vertical components.

Denote by β_1 the dihedral angle between the horizontal plane and the triangles of step 1 which have an edge on $\{z=0\}$; by simmetry we can chose $\beta_1 \in (0, \pi/2)$ arbitrarily. Denote by $\beta_i(m,\beta_1)$ the dihedral angle between the horizontal plane and the triangles of step i which have an edge on the (i-1)-th m-gon. By minimality we obtain

$$\tan\beta_{i+1}(m,\beta_1) = 2\left(\sin\frac{\pi}{m}\right)^2\left[\cos\frac{\pi}{m}\left(r_i(m) - r_{i-1}(m)\cos\frac{\pi}{m}\right)\right]^{-1}\tan\beta_1,$$

hence the vertical skip is

$$h(m,\beta_1) = 2\left(\sin\frac{\pi}{m}\right)^2\left(\cos\frac{\pi}{m}\right)^{-1}\tan\beta_1 .$$

We remark that:

(i) the vertical skip is independent on i;

(ii) for each m and β_1 we have

$$\lim_{i\to\infty} \tan\beta_i(m,\beta_1) = 0 ,$$

$$\lim_{i\to\infty} ih(m,\beta_1) = \infty ,$$

hence the ends are good and the surface is not contained in a slab.

We denote by $S(m,\beta_1)$ the polyhedral minimal surface obtained by this immersion; for each $m \geq 6$ and for each $\beta_1 \in (0,\pi/2))$, $S(m,\beta_1)$ is a properly embedded polyhedral minimal surface of genus 0 and two good ends.

Let us look at the angles of $S(m,\beta_1)$.

At each step i there are only two different kinds of isosceles triangles: those with the base on an edge of the (i-1)-th m-gon and those with the base on an edge of the i-th m-gon. It is clear that we can restrict ourselves to the study of the first kind of triangles, which form a dyhedral angle β_i with the horizontal plane; if we denote by δ_i the angle at the base we have that:

$$r_i(m) - r_{i-1}(m)\left(\cos\frac{\pi}{m}\right) \quad \overline{}$$

and then

$$\sup_i (\tan\delta_i) \leq \left(2\sin\frac{\pi}{m} + \sqrt{3+\left(\sin\frac{\pi}{m}\right)^2}\right)\left(\cos\frac{\pi}{m}\right)^{-1} + 2\tan\frac{\pi}{m}\tan\beta_1,$$

$$\inf_i (\tan\delta_i) \geq 4\left(\sin\frac{\pi}{m}\right)^2 .$$

These inequalities imply that for each m and for each β_1 the surface $S(m,\beta_1)$ has good triangles.

It is natural to look at the connection between this family of surfaces and the smooth normalized catenoid C parametrized by:

$$\begin{cases} x = \cosh v \cos u , \\ y = \cosh v \sin u , \\ z = v , \end{cases}$$

where $v \in \mathbb{R}$ and $u \in [0, 2\pi)$.

THEOREM 7.1. – *For each $m \geq 6$ there exists $\beta_1(m)$ such that all the vertices of $S(m,\beta_1(m))$ lye on the catenoid* C.

PROOF. – We want to evaluate the difference between the 3rd coordinate $z_i(m,\beta_1)$ of $S(m,\beta_1)$ at step i and $z_i(C) = \operatorname{arcosh}(r_i(m))$ which is the 3rd coordinate of the catenoid C in correspondence with the circle of radius $r_i(m)$.

If we denote by $\Delta_i(m,\beta_1)$ this difference, we have:

$$\Delta_i(m,\beta_1) = \operatorname{arcosh}(r_i(m)) - ih(m,\beta_1) =$$

$$i\left(\ln(b(m) + \sqrt{b(m)^2 - 1}) - 2\left(\sin\frac{\pi}{m}\right)^2\left(\cos\frac{\pi}{m}\right)^{-1}\tan\beta_1\right).$$

Thus, we have that $\Delta_i(m,\beta_1) = 0 \ \forall i \geq 0$ if and only if

$$\beta_1 = \beta_1(m) = \arctan\left(\frac{1}{2}\ln(b(m) + \sqrt{b(m)^2 - 1})\cos\frac{\pi}{m}\left(\sin\frac{\pi}{m}\right)^{-2}\right).$$

We remark that with this choice of β_1 we have then

$$\lim_{n\to\infty} h(m,\beta_1(m))=0, \qquad \lim_{m\to\infty} \tan(\beta_1(m))=\infty \ ,$$

so all is coherent. ∎

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