Trigonometrically fitted two-step hybrid methods for special second order Ordinary Differential Equations

R. D'Ambrosio* M. Ferro† B. Paternoster ‡

Abstract. The purpose of this paper is to derive two-step hybrid methods for second order ordinary differential equations with oscillatory or periodic solutions. We show the constructive technique of methods based on trigonometric and mixed polynomial fitting and consider the linear stability analysis of such methods. We then carry out some numerical experiments underlining the properties of the derived classes of methods.

Key words. Two-step hybrid methods, trigonometrical fitting, second order ordinary differential equations.

1 Introduction

It is the aim of our paper to derive new classes of numerical methods for solving initial value problems based on second order ordinary differential equations (ODEs)

$$\begin{cases} y'' = f(x, y), \\ y'(x_0) = y'_0, \\ y(x_0) = y_0 \end{cases} \tag{1}$$

with f smooth enough in order to ensure the existence and uniqueness of the solution. Although problem (1) can be solved by transforming it into a system of first order ODEs of double dimension, the development of numerical methods for its direct integration seems more natural and efficient. This problem, having periodic or oscillatory solutions, often appears in many applications: celestial mechanics, seismology, molecular dynamics, and so on (see for istance [23, 26] and references therein contained). Classical numerical methods for ODEs relied on polynomials may not be very well-suited to periodic or oscillatory behaviour. In the framework of exponential fitting many numerical methods have been adapted in order to exactly integrate basis of functions other than polynomials, for instance the exponential basis (see [16] and references therein contained), in order to catch the oscillatory behaviour. The parameters of these methods depend on the values of frequencies, which appear in the solution. In order to adapt the collocation technique [14, 18] to an oscillatory behaviour, the collocation function has been chosen as a linear combination of trigonometric functions [20] or of powers and exponential functions [4]. Many

*Dipartimento di Matematica e Informatica, Universitá di Salerno, Fisciano (Sa), 84084 Italy, e-mail: rdambrosio@unisa.it.

†Dipartimento di Matematica e Applicazioni, Universitá di Napoli-"Federico II", 80126 Napoli, Italy, e-mail: maria.ferro@dma.unina.it.

‡Dipartimento di Matematica e Informatica, Universitá di Salerno, Fisciano (Sa), 84084 Italy, e-mail: beapat@unisa.it.

modifications of classical methods have been presented in the literature for problem (1): exponentially-fitted Runge-Kutta methods (see for example [13, 24]), or trigonometrically-fitted Numerov methods [12, 25] and many others (for a more extensive bibliography see [16] and references in the already cited papers).

The methods we show in this context belong to the class of two-step hybrid methods

$$Y_i^{[n]} = (1+c_i)y_n - c_i y_{n-1} + h^2 \sum_{j=1}^{s} a_{ij} f(Y_j^{[n]}), \quad i = 1, ..., s \tag{2}$$

$$y_{n+1} = 2y_n - y_{n-1} + h^2 \sum_{i=1}^{s} b_i f(Y_i^{[n]}). \tag{3}$$

introduced by Coleman in [3], which can also be represented through the Butcher array

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array} \tag{4}$$

with $c = (c_1, c_2, ..., c_s)^T$, $A = (a_{ij})_{i,j=1}^s$, $b = (b_1, b_2, ..., b_s)^T$, where s is the number of stages.

The aim of this paper is to adapt the coefficients of methods (2)-(3) to an oscillatory behaviour, in such a way that it exactly integrates linear combinations of power and trigonometric functions depending on one and two frequencies, which we suppose can be estimated in advance. Frequency-dependent methods within the class (2)-(3) have already been considered in [28], where the coefficients of methods were modified to produce phase-fitted and amplification-fitted methods.

In section 2 we rewrite the hybrid method (2)-(3) as an $\mathcal{A} - method$, following the idea in [1, 19], in order to regard it as a generalized linear multistep method and consider linear operators associated to it, which will play a crucial rule in the development of the new methods. In section 3 we derive the methods, by imposing that the internal and external stages exactly integrate linear combinations of mixed basis functions. In particular, we construct methods with constant coefficients and methods with parameters depending on one or two frequencies. In section 4 we analyze linear stability properties of the derived methods. Finally section 5 provides numerical tests, in order to illustrate features of the methods and to compare our methods with other ones already known in literature. The paper concludes with an appendix, where we report arrays of some methods.

2 Two-step hybrid methods as $\mathcal{A}$-methods: order conditions

In this section we show some preliminary and helpful results we will use in the remainder of this paper in order to carry out the construction of numerical methods belonging to the class of two-step hybrid methods (2)-(3).

In particular, following the approach introduced by Albrecht (cfr. [1, 19]), we rewrite the class of two-step hybrid methods as $\mathcal{A}$-methods. We first define the following vectors in $\mathbb{R}^{s+2}$

$$\begin{aligned} Y_{n+1} &= [Y_1^{[n]}, ..., Y_s^{[n]}, y_n, y_{n+1}]^T, \\ F(x_n, Y_{n+1}; h) &= [f(x_n + c_1 h, Y_1), ..., f(x_n + c_s h, Y_s), f(x_n, y_n), f(x_n, y_{n+1})]^T. \end{aligned}$$

In this way, a two-step hybrid method (2)-(3) can be expressed as an $\mathcal{A}$–method of the form

$$Y_{n+1} = \mathcal{A} Y_n + h^2 \mathcal{B} F(x_n, Y_{n+1}; h), \tag{5}$$

with

$$\mathcal{A} = \begin{bmatrix} 0 & -c & \mathbf{e}+c \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \; \mathcal{B} = \begin{bmatrix} A & 0 & 0 \\ b^T & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{(s+2)\times(s+2)}. \tag{6}$$

$\mathbf{e}$ is the unitary vector of $\mathbb{R}^s$.

This representation is very useful, because it constitutes an $s+2$ linear stages representation, in the sense that each of the s internal stages and the external stages are linear, so we can look at them as a generalized linear multistep formula on a nonequidistant grid. For this reason, we can consider the following $s+1$ linear operators

$$\begin{aligned} L_i[z(x);h] &= z(x+c_ih) - (1+c_i)z(x) + c_iz(x-h) \\ &- h^2\sum_{j=1}^{s} a_{ij}z''(x+c_jh), \quad i=1,...,s \end{aligned} \tag{7}$$

$$\widehat{L}[z(x);h] = z(x+h) - 2z(x) + z(x-h) - h^2\sum_{i=1}^{s} b_iz''(x+c_ih) \tag{8}$$

where $z(x)$ is a smooth enough function. Expanding in power series of h around x we obtain

$$\begin{aligned} L_i[z(x);h] &= C_{i2}h^2z^{(2)}(x) + C_{i3}h^3z^{(3)}(x) + ..., \quad i=1,...,s \\ \widehat{L}[z(x);h] &= \widehat{C}_2h^2z^{(2)}(x) + \widehat{C}_3h^3z^{(3)}(x) + ... \end{aligned}$$

where

$$\begin{aligned} C_{iq} &= \frac{c_i^q}{q!} + \frac{(-1)^q}{q!} - \frac{1}{(q-2)!}\sum_{j=1}^{s} a_{ij}c_j^{q-2}, \quad i=1,...,s, \; q=2,3,... \\ \widehat{C}_q &= \frac{1}{q!} + \frac{(-1)^q}{q!} - \frac{1}{(q-2)!}\sum_{j=1}^{s} b_jc_j^{q-2}, \; q=2,3,... \end{aligned}$$

As a consequence, we can give the following definition.

Definition 2.1 *The i^{th} internal stage (2) of a two-step hybrid method has order p_i if*

$$C_{i2}=0, \; C_{i3}=0, \; \dots, \; C_{ip_i+1}=0, \quad C_{ip_i+2}\neq 0, \tag{9}$$

while the external stage (3) has order p if

$$\widehat{C}_2=0, \; \widehat{C}_3=0, \; \dots, \; \widehat{C}_{p+1}=0, \quad \widehat{C}_{p+2}\neq 0. \tag{10}$$

We know that necessary condition for a two-step hybrid method (2)-(3) to have order p is that the external stage must have order p, i.e.

$$b^Tc^{q-2} = \frac{1+(-1)^q}{q(q-1)}, \quad q=2,3,...,p+1, \tag{11}$$

where the vector power is componentwise. In order to look for conditions that are also sufficient, in line to Albercht's approach, we need to look at the global error. We omit the details achieving order conditions, because they are outside the original aim of this paper, and which can be found in [3]. The following table shows the set of order conditions up to 4.

Order	Order conditions
1	$\sum_i b_i = 1$
2	$\sum_i b_i c_i = 0$
3	$\sum_i b_i c_i^2 = \frac{1}{6}$ $\sum_i \sum_j b_i a_{ij} = \frac{1}{12}$
4	$\sum_i b_i c_i^3 = 0$ $\sum_i \sum_j b_i c_i a_{ij} = \frac{1}{12}$ $\sum_i \sum_j b_i a_{ij} c_j = 0$

Table 1: Order conditions for two-step hybrid methods (2)-(3) of order up to 4.

3 Constructive technique of mixed-trigonometrically fitted two-step hybrid methods

In this section we show the construction of some two-step hybrid methods for the numerical solution of second order ODEs, whose solutions depend on one or more frequencies, which at the moment we suppose can be estimated in advance. In particular, we require that both the internal and external stages of the resulting methods exactly integrate linear combinations of the following basis functions:

$$\{1, x, \ldots, x^q, cos(\omega_i x), sin(\omega_i x), \; q, i = 1, 2, \ldots\} \tag{12}$$

depending on the frequencies ω_i, with i, q such that the dimension of the basis is $s+2$, so $2i+q = s+1$. The case of mixed basis follows the idea of using mixed interpolation of type

$$acos(\omega x) + bsin(\omega x) + \sum_{i=0}^{s-1} c_i x^i, \tag{13}$$

presented in [11] and used in [4]. In each case the vector $\mathbf{c}$ is considered to be free. Each c_i, $i = 1, 2, ..., s$ can be chosen in order to improve the stability properties of the methods or, imposing a special set of constraints in the implicit case, in order to achieve superconvergence. In the appendix we list the coefficients of the derived methods.

3.1 Methods with constant coefficients

In order to derive s-stage methods of type (2)-(3) with constant coefficients, we annihilate the linear operators (7)-(8) on the functional basis

$$\{1, x, x^2, ..., x^q\} \tag{14}$$

with $q = s+1$. It trivially happens that

$$\begin{aligned} L_i[1; h] &= \widehat{L}[1; h] = 0, \quad i = 1,\ 2, ...,\ s, \\ L_i[x; h] &= \widehat{L}[x; h] = 0, \quad i = 1,\ 2, ...,\ s, \end{aligned}$$

while, for $2 \leq k \leq q$, it is

$$L_j[x^k; h] \;=\; 0,\ i = 1,\ 2, ...,\ s \quad \Leftrightarrow \quad \frac{c_i^k + (-1)^k c_i}{k(k-1)} = \sum_{j=1}^{s} a_{ij} c_j^{k-2}$$

$$\widehat{L}[x^k;h] \;=\; 0 \quad \Leftrightarrow \quad \frac{1+(-1)^k}{k(k-1)} = \sum_{i=1}^{s} b_i c_i^{k-2}.$$

Therefore, we have obtained the set of conditions

$$\begin{cases} \dfrac{c_i^k + (-1)^k c_i}{k(k-1)} = \displaystyle\sum_{j=1}^{s} a_{ij} c_j^{k-2} \\ \dfrac{1+(-1)^k}{k(k-1)} = \displaystyle\sum_{j=1}^{s} b_j c_j^{k-2}, \end{cases} \tag{15}$$

for $i = 1,\ 2, ...,\ s$ and $2 \leq k \leq q$, which is a system of $s(s+1)$ equations in the unknowns a_{ij}, b_i, for $i,\ j = 1,\ 2, ...,\ s$.

Remark 3.1 *It is easy to verify that the methods obtained by solving the order conditions (15) are equal to the methods described in [10], which are based on collocation through algebraic polynomials, and have been derived by extending the multistep collocation technique described in [15]. Therefore conditions (15) are the order conditions for collocation methods within class (2)-(3).*

3.2 Methods with parameters depending on one frequency

In order to derive numerical methods for second order ODEs whose solution depends on the frequency ω, a priori known, we consider the function basis $\{1, x, \ldots, x^q, cos(\omega x), sin(\omega x),\ q = 1, 2, \ldots\}$.
In particular, to derive two-stage methods, we consider $\{1, cos(\omega x), sin(\omega x)\}$ as function basis and we impose that the numerical method exactly integrates second order ODEs whose solution is a linear combination of the basis functions. In this way we obtain the class of trigonometrically fitted two-step hybrid methods. As it automatically happens that

$$L_j[1;h] = \widehat{L}[1;h] = 0,\ j = 1, 2,$$

we need to impose the following set of conditions

$$L_j[\cos\omega x; h] \;=\; 0, \quad j = 1, 2, \qquad \widehat{L}[\cos\omega x; h] = 0, \tag{16}$$

$$L_j[\sin\omega x; h] \;=\; 0, \quad j = 1, 2, \qquad \widehat{L}[\sin\omega x; h] = 0, \tag{17}$$

which constitutes a 6×6 linear system in the unknowns $a_{11},\ a_{12},\ a_{21},\ a_{22},\ b_1,\ b_2$.

In order to construct methods with 3 or more stages, we impose that the numerical method exactly integrates second order ODEs whose solution is a linear combination of the basis function

$$\{1, x, x^2, ...x^{s-1}, cos(\omega x), sin(\omega x)\}$$

depending on the frequency ω. In this case we obtain *mixed-trigonometrically fitted two-step hybrid methods*. As also $L_j[x;h] = \widehat{L}[x;h] = 0,\ j = 1, ..., s$, it is sufficient to impose that

$$L_j[x^q; h] \;=\; 0, \quad \widehat{L}[x^q; h] = 0, \quad j = 1, 2, ..., s, \quad q = 2, 3..., s-1, \tag{18}$$

$$L_j[\cos\omega x; h] \;=\; 0, \quad \widehat{L}[\cos\omega x; h] = 0, \quad j = 1, 2, ..., s, \tag{19}$$

$$L_j[\sin\omega x; h] \;=\; 0, \quad \widehat{L}[\sin\omega x; h] = 0, \quad j = 1, 2, ..., s. \tag{20}$$

It arises a system of $s(s+1)$ conditions in the unknowns $a_{ij},\ b_i,\ i,j = 1,2,...,s$. This system is equivalent to the following set of conditions:

$$\begin{aligned}
\sum_{i=1}^{s} b_i c_i^{q-2} &= \frac{1+(-1)^q}{q(q-1)}, \quad q = 2,...s,\\
\sum_{j=1}^{s} a_{ij} c_j^{q-2} &= c_i^q \frac{1+(-1)^q c_i}{q(q-1)}, \quad i = 1,2,...s, \quad q = 2,...s,\\
\sum_{i=1}^{s} b_i \cos(c_i\theta) &= 2\frac{(1-\cos\theta)}{\theta^2},\\
\sum_{j=1}^{s} a_{ij} \cos(c_j\theta) &= -\frac{\cos(c_i\theta)+1+c_i+c_i\cos\theta}{\theta^2} \quad i = 1,2,...s,\\
\sum_{i=1}^{s} b_i \sin(c_i\theta) &= 0,\\
\sum_{j=1}^{s} a_{ij} \sin(c_j\theta) &= \frac{c_i\sin(\theta)-\sin(c_i\theta)}{\theta^2} \quad i = 1,2,...s,
\end{aligned}$$

where $\theta = \omega h$. In both cases, the coefficients of the resulting methods are subjected to heavy numerical cancellation, so it is necessary to represent them through their expansion in power series of θ, as it is shown in the Appendix.

Remark 3.2 *It is possible to prove that, for $\theta \to 0$, the coefficients of the resulting trigonometrically–fitted method tend to the coefficient of the corresponding collocation two–step hybrid methods, derived as shown in* 3.1 *or in* [*10*]. *Therefore the two and three stage methods, above described, are the trigonometric based collocation methods within class (*2*)-(*3*).*

3.3 Methods with parameters depending on two frequencies

We now deal with the case of second order ODEs whose solution depends on two frequencies ω_1 and ω_2, both estimated in advance. We require that the methods must exactly solve the problem when its solution is linear combination of the basis functions $\{1, x, \ldots, x^{s-3}, cos(\omega_1 x), sin(\omega_1 x), cos(\omega_2 x), sin(\omega_2 x)\}$, with $s \geq 4$. In order to derive such methods, we impose the following set of conditions

$$\begin{aligned}
L_j[\cos\omega_1 x; h] &= 0, \quad \widehat{L}[\cos\omega_1 x; h] = 0, \quad j = 1,\ldots,s,\\
L_j[\sin\omega_1 x; h] &= 0, \quad \widehat{L}[\sin\omega_1 x; h] = 0, \quad j = 1,\ldots,s,\\
L_j[\cos\omega_2 x; h] &= 0, \quad \widehat{L}[\cos\omega_2 x; h] = 0, \quad j = 1,\ldots,s,\\
L_j[\sin\omega_2 x; h] &= 0, \quad \widehat{L}[\sin\omega_2 x; h] = 0, \quad j = 1,\ldots,s.
\end{aligned} \tag{21}$$

Then if we are interested in methods with 4 stages, we only have to solve the system (21) in the unknowns $a_{ij}, b_i, \quad i,j = 1,...,4$. It has now become clear that, if we annihilate also

$$L_j[x^q; h] = 0, \quad \widehat{L}[x^q; h] = 0 \quad j = 1,2,...,s, \quad q = 2,3...,s-3, \tag{22}$$

more stages are necessary. The methods derived by solving (21), whose coefficients are listed in Appendix, have parameters depending on $\theta_1 = \omega_1 h$ and $\theta_2 = \omega_2 h$.

Remark 3.3 *Also in this case, for $\theta_1 \to 0$ and $\theta_2 \to 0$, the coefficients of the derived methods tend to the coefficients of the two–step collocation hybrid methods, derived in* 3.1 *and [10]. Therefore the four stage method derived by (21) is the two frequencies trigonometric based collocation methods within class (2)-(3).*

4 Linear stability analysis

We handle the linear stability analysis [21, 26, 27] of the obtained methods. We consider both the cases of methods with constant coefficients and with coefficients depending on one or two frequencies.

4.1 Methods with constant coefficients

Following [26], we apply the class of methods (2)-(3), to the test problem

$$y'' = -\lambda^2 y, \qquad \lambda \in \mathbf{R}$$

obtaining

$$Y_i^{[n]} = -c_i y_{n-1} + (1+c_i) y_n - \nu^2 \sum_{j=1}^{s} a_{ij} Y_j^{[n]}, \tag{23}$$

$$y_{n+1} = -y_{n-1} + 2y_n - \nu^2 \sum_{j=1}^{s} b_j Y_j^{[n]}, \tag{24}$$

where $\nu^2 = \lambda^2 h^2$. In tensor notation,

$$Y^{[n]} = -c y_{n-1} + (e+c) y_n - \nu^2 A Y^{[n]}, \tag{25}$$

$$y_{n+1} = -y_{n-1} + 2y_n - \nu^2 b^T Y^{[n]}, \tag{26}$$

where $Y^{[n]} = (Y_i^{[n]})_{i=1}^{s}$. The following expression for the stage values holds:

$$Y^{[n]} = Q[-c y_{n-1} + (e+c) y_n], \tag{27}$$

where $Q = [I + \nu^2 A]^{-1}$ and I is the identity matrix of dimension s. If we substitute this expression in (26), the following recurrence relation arises:

$$y_{n+1} = [-1 + \nu^2 b^T Q c] y_{n-1} + [2 - \nu^2 b^T Q(e+c)] y_n, \tag{28}$$

that is

$$\begin{bmatrix} y_{n+1} \\ y_n \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_n \\ y_{n-1} \end{bmatrix}, \tag{29}$$

where

$$M_{11} = 2 - \nu^2 b^T Q(e+c), \qquad M_{12} = -1 + \nu^2 b^T Q c.$$

The characteristic polynomial, associated to the recursion (28), is called *stability polynomial.* The so-called *stability or amplification matrix* [26, 27] takes the form

$$M(\nu^2) = \begin{bmatrix} M_{11} & M_{12} \\ 1 & 0 \end{bmatrix}. \tag{30}$$

Let us denote the spectral radius of $(M(\nu^2))$ with $\rho(M(\nu^2))$. From [26, 27], we consider the following definitions.

Definition 4.1 $(0, \beta^2)$ *is a stability interval for the method (2)-(3) if* $\forall \nu^2 \in (0, \beta^2)$ *the spectral radius of* $M(\nu^2)$ *is such that*

$$\rho(M(\nu^2)) < 1. \tag{31}$$

The condition (31) is equivalent to the fact that the roots of stability polynomial are in modulus less than 1, $\forall \nu^2 \in (0, \beta^2)$. Setting $S(\nu^2) = trace(M^2(\nu^2))$ and $P(\nu^2) = det(M^2(\nu^2))$, condition (31) is equivalent to

$$P(\nu^2) < 1, \quad |S(\nu^2)| < P(\nu^2) + 1, \quad \nu \in (0, \beta^2) \tag{32}$$

Definition 4.2 *The method (2)-(3) is* **A***–stable if* $(0, \beta^2) = (0, +\infty)$.

In order to reach A–stability, it must be $\rho(M(\nu^2)) < 1$, for any value of ν^2, where $\rho(M(\nu^2))$ is the spectral radius of the stability matrix, i.e. both the eigenvalues λ_1, λ_2 of $M(\nu^2)$ must satisfy the condition $|\lambda_1| < 1$, $|\lambda_2| < 1$. If the eigenvalues of the stability matrix (30) (or in equivalent way, the roots of the stability polynomial) are on the unit circle, then the interval of stability becomes an interval of periodicity, according to the following definition.

Definition 4.3 $(0, H_0^2)$ *is a periodicity interval if* $\forall \nu^2 \in (0, H_0^2)$ *the eigenvalues* $r_1(\nu^2), r_2(\nu^2)$ *of the stability matrix (30) are complex conjugate and* $|r_{1,2}(\nu^2) = 1|$.

The interval of periodicity is defined by [26]

$$(0, H_0^2) := \{\nu^2 | \nu^2 > 0, P(\nu^2) \equiv 1, |S(\nu^2)| < 2\}. \tag{33}$$

Definition 4.4 *The method (2), (3) is* **P***–stable if its periodicity interval is* $(0, +\infty)$.

4.2 Methods depending on one frequency

We now analyse the stability properties of mixed trigonometrically fitted methods depending on one frequency. In [5] the authors discussed the modifications introduced in the linear stability analysis, when the parameters depend on one fitted frequency ω. As a consequence of the presence of the fitted frequency ω, the interval of stability now becomes a two-dimensional region for the one parameter family of methods.
In this analysis, we denote the stability matrix as $M(\nu^2, \theta)$ and $R(\nu^2, \theta) = \frac{1}{2} trace(M(\nu^2, \theta))$, $P(\nu^2, \theta) = det(M(\nu^2, \theta))$, because it depends not only on $\nu^2 = \lambda^2 h^2$ but also on $\theta = \omega h$.
The eigenvalues of the stability matrix $M(\nu^2, \theta)$ satisfy the following equation

$$\xi^2 - 2R(\nu^2, \theta)\xi + P(\nu^2, \theta) = 0. \tag{34}$$

It is known in literature (see [4, 5, 18]) that methods such that

$$|P(\nu^2, \theta)| \equiv 1, \tag{35}$$

i.e. the roots of (34) lie on the unit circle, are of particular interest. For example, Runge–Kutta Nyström methods based on polynomial approximations with symmetric abscissas c_i in $[0,1]$, have an interval of periodicity, but they are not P–stable, if collocation based. If (35) holds, the study of periodicity can be developed just looking at the so–called *stability function* $R(\nu^2,\theta)$, in agreement with the following definition [5].

Definition 4.5 *For a trigonometrically fitted method of the type (2)-(3) satisfying $|P(\nu^2,\theta)| \equiv 1$, we define the primary interval of periodicity as the largest interval $(0,h_0)$ such that $|R(\nu^2,\theta)| < 1$, for all steplengths $h \in (0,h_0)$. If, when h_0 is finite, $|R(\nu^2,\theta)| < 1$ also for $\gamma < h\delta$, where $\gamma > h_0$, then the interval (γ,δ) is a secondary interval of periodicity.*

Exponentially fitted linear multistep methods in [5] verify (35), but only for few methods in the literature condition (35) holds. In our analysis, we found that for two-stage methods (2)-(3) the values of the abscissas such that $|P(\nu^2,\theta)| \equiv 1$ are only $c_1 = 0, c_2 = 1$. We relax the definition 2 of region of stability in [5], in order to consider also methods for which $P(\nu^2,\theta) < 1$, in the following way [12]:

Definition 4.6 *A region of stability Ω is a region of the (ν,θ) plane, such that $\forall(\nu^2,\theta) \in \Omega$*

$$P(\nu^2,\theta) < 1, \quad |R(\nu^2,\theta)| < \frac{1}{2}(P(\nu^2,\theta)+1). \tag{36}$$

Any closed curve defined by $P(\nu^2,\theta) \equiv 1$ and $|R(\nu^2,\theta)| = \frac{1}{2}(P(\nu^2,\theta)+1)$ is a stability boundary.

Fig. 1 and fig. 2 show some examples of stability regions for one–frequency depending methods, in the cases $s = 2$ and $s = 3$.

4.3 Methods depending on two frequencies

We now consider the linear stability analysis of methods depending on two frequencies. As stated before, for methods with constant coefficient, the stability region is an interval on the real axis, while methods depending on one frequency have bidimensional stability region. In the case of methods depending on the values of two frequencies, ω_1,ω_2, opportunely adapting the approach that Coleman and Ixaru in [5] introduced for one frequency depending methods, the stability region becomes tridimensional. We now denote the stability matrix of the methods as $M(\nu^2,\theta_1,\theta_2)$, with $\nu^2 = \lambda^2 h^2, \theta_1 = \omega_1 h, \theta_2 = \omega_2 h$. Its eigenvalues satisfy the following equation

$$\xi^2 - 2R(\nu^2,\theta_1,\theta_2)\xi + P(\nu^2,\theta_1,\theta_2) = 0, \tag{37}$$

where $R(\nu^2,\theta_1,\theta_2) = \frac{1}{2}\text{trace}(M(\nu^2,\theta_1,\theta_2))$ and $P(\nu^2,\theta_1,\theta_2) = \det(M(\nu^2,\theta_1,\theta_2))$ are rational functions of ν^2. The definition of stability region for two frequencies depending methods can now be adapted as follows [12]:

Definition 4.7 *A three dimensional region Ω of the $(\nu^2,\theta_1,\theta_2)$ space is said to be the region of stability of the corresponding two-frequencies depending method if $\forall(\nu^2,\theta_1,\theta_2) \in \Omega$*

$$P(\nu^2,\theta_1,\theta_2) < 1, \quad |R(\nu^2,\theta_1,\theta_2)| < \frac{1}{2}(P(\nu^2,\theta_1,\theta_2)+1). \tag{38}$$

Any closed curve defined by

$$P(\nu^2,\theta_1,\theta_2) \equiv 1, \quad |R(\nu^2,\theta_1,\theta_2)| = \frac{1}{2}(P(\nu^2,\theta_1,\theta_2)+1). \tag{39}$$

is a stability boundary for the method.

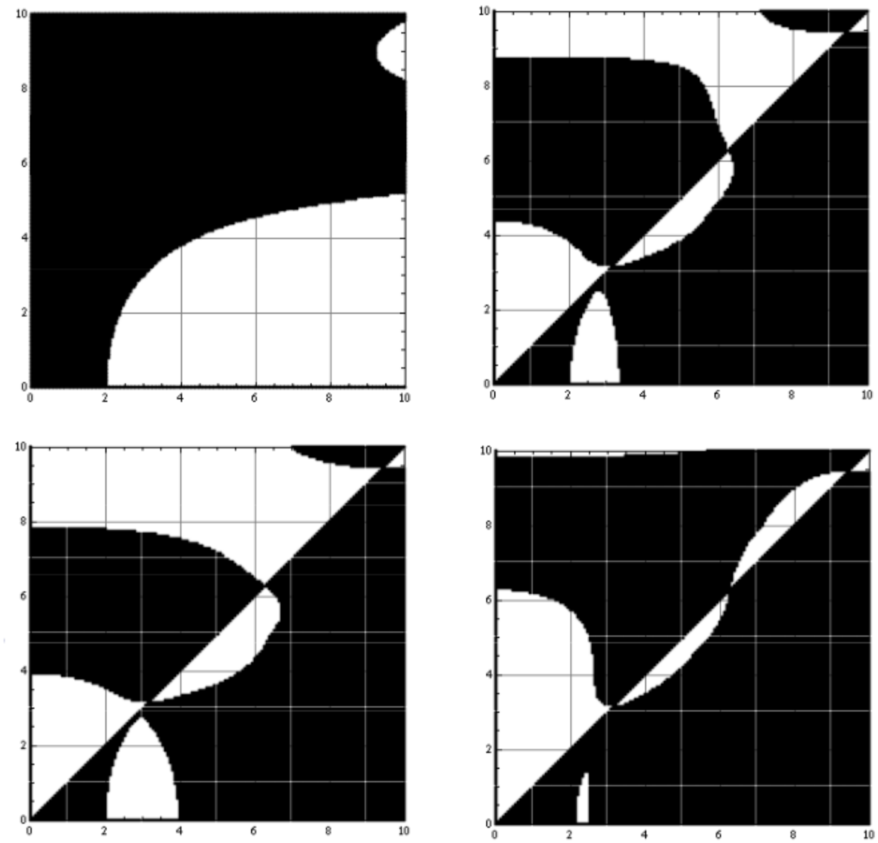

Figure 1: Regions of stability in the (ν, θ)–plane for the two-step methods for $s = 2$ with nodes $(0, 1)$, $(\frac{1}{7}, \frac{6}{7})$, $(\frac{1}{10}, \frac{9}{10})$, $(\frac{3}{4}, 1)$ respectively.

Fig. 3 shows an example of three dimensional stability region, while fig. 4 shows the projection of three dimensional regions on a particular plane.

5 Numerical experiments

We now show some numerical results we have obtained applying our families of solvers to some linear and nonlinear problems depending on one or two frequencies, in order to test the accuracy of the derived methods and also to compare them with ones already considered in literature for second order ODEs.
Test 1. We consider the following test equation

$$\begin{cases} y''(x) = -25y(x), \quad x \in [0, 2\pi], \\ y'(0) = y'_0, \\ y(0) = 1 \end{cases} \tag{40}$$

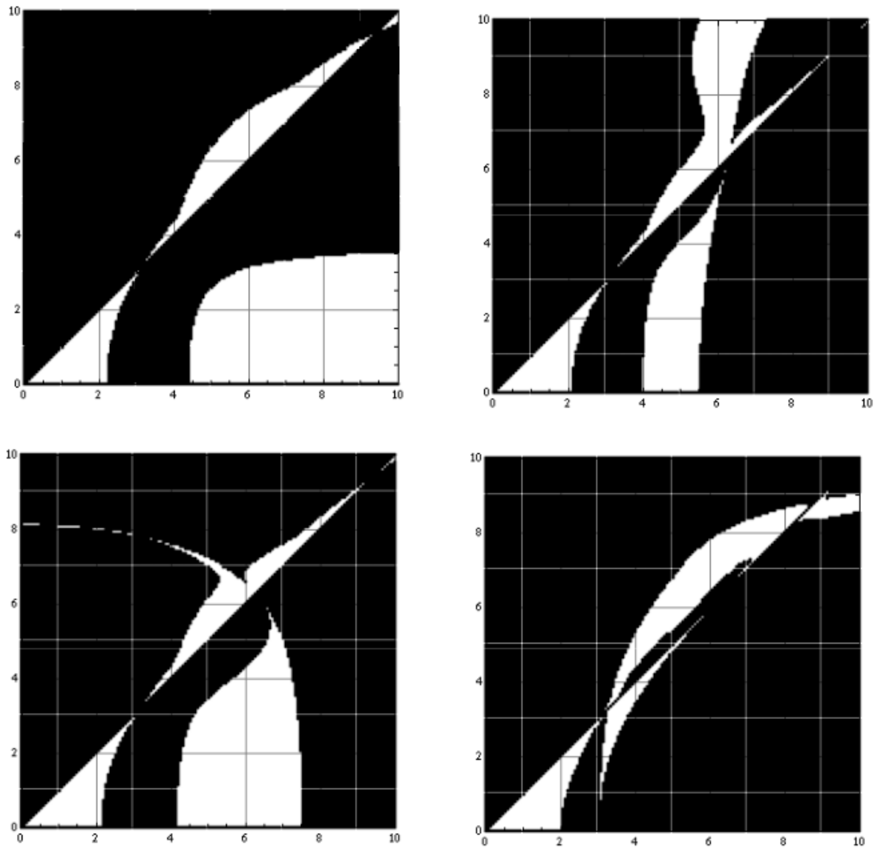

Figure 2: Regions of stability in the (ν, θ)–plane for the two-step methods for $s = 3$ with nodes $(0, \frac{1}{2}, 1)$, $(\frac{1}{4}, \frac{1}{2}, \frac{3}{4})$, $(\frac{1}{9}, \frac{1}{2}, \frac{8}{9})$, $(\frac{1}{2}, \frac{3}{4}, 1)$ respectively.

whose exact solution is $y(x) = \cos(5x)$, so it depends on the frequency $\omega = 5$. We solve this problem using the following solvers:

- COLEM: two–step hybrid method, [3],

$$\begin{array}{c|cc} \frac{1}{\sqrt{6}} & \frac{1+\sqrt{6}}{12} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{\sqrt{6}}{12} & \frac{1}{12} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array} \tag{41}$$

- TRIGFIT1: trigonometrically fitted two–step hybrid method, with 2 stages and order 2, $c = [0, 1]$, derived in subsection 3.2,

- TRIGFIT2: trigonometrically fitted two–step hybrid method, with 2 stages and order 2, $c = [0, 3/4]$, derived in subsection 3.2,

- TRIGFIT3: trigonometrically fitted two–step hybrid method, with 2 stages and order 2, $c = [3/4, 1]$, derived in subsection 3.2,

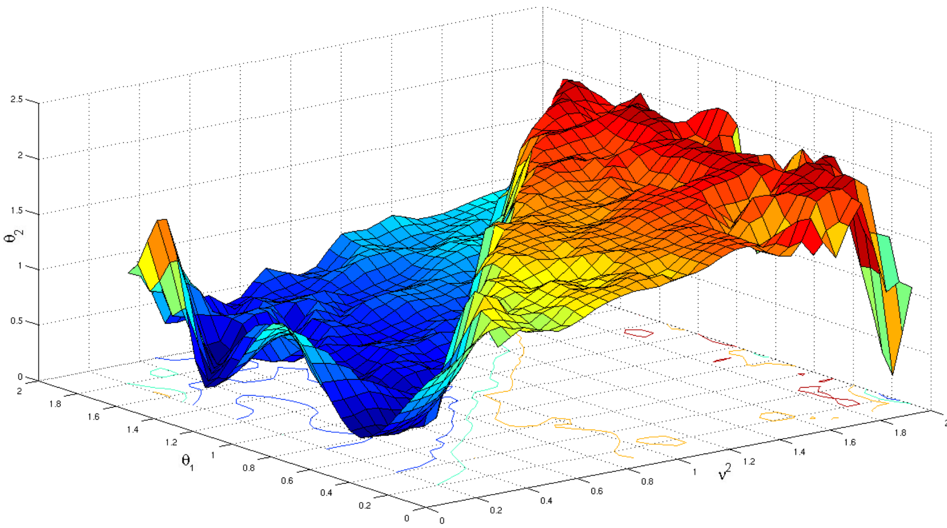

Figure 3: Region of stability in the $(\nu^2, \theta_1, \theta_2)$–plane for the two-step methods for $m = 4$ with nodes $(0, \frac{1}{3}, \frac{2}{3}, 1)$.

- POL: two–step hybrid method [10], with 2 stages and order 2;
- MTRIGFIT: mixed–trigonometrically fitted two–step hybrid method, with 3 stages and order 3,$c = [1/3, 1/2, 1]$, derived in subsection 3.2,
- TRIGFIT4S: trigonometrically fitted two–step hybrid method, with 4 stages and order 4, $c = [0, 1/3, 2/3, 1]$, derived in subsection 3.3,

The tables shows the global error in the final point of the integration interval, while *cd* is the number of the correct digits. Table 2 compares the new methods with classical ones, having constant coeffi-

Method	$h = \pi/64$	cd	$h = \pi/128$	cd
COLEM	0.07313	1.1358	0.0042	2.3698
TRIGFIT1	4.21885e-15	14.3748	4.44089e-16	15.3525
TRIGFIT2	4.44089e-15	14.3525	4.44090e-16	15.3530
TRIGFIT3	2.88658e-15	14.539	8.88178e-16	15.0515
POL	0.00286835	2.54236	0.0006948	3.158135
MTRIGFIT	2.22045e-15	14.6536	1.11466e-13	13.9529

Table 2: Numerical results for the problem (40).

cients. Trigonometrically fitted methods would solve this kind of problem exactly, of course, in exact arithmetic. The errors are the effect of the accumulation of round off errors in finite precision calculation.

Test 2. The Prothero-Robinson problem

$$y'' + v^2[y - \cos(10x)]^3 = -100y, \quad x \in [0, 20\pi], \tag{42}$$

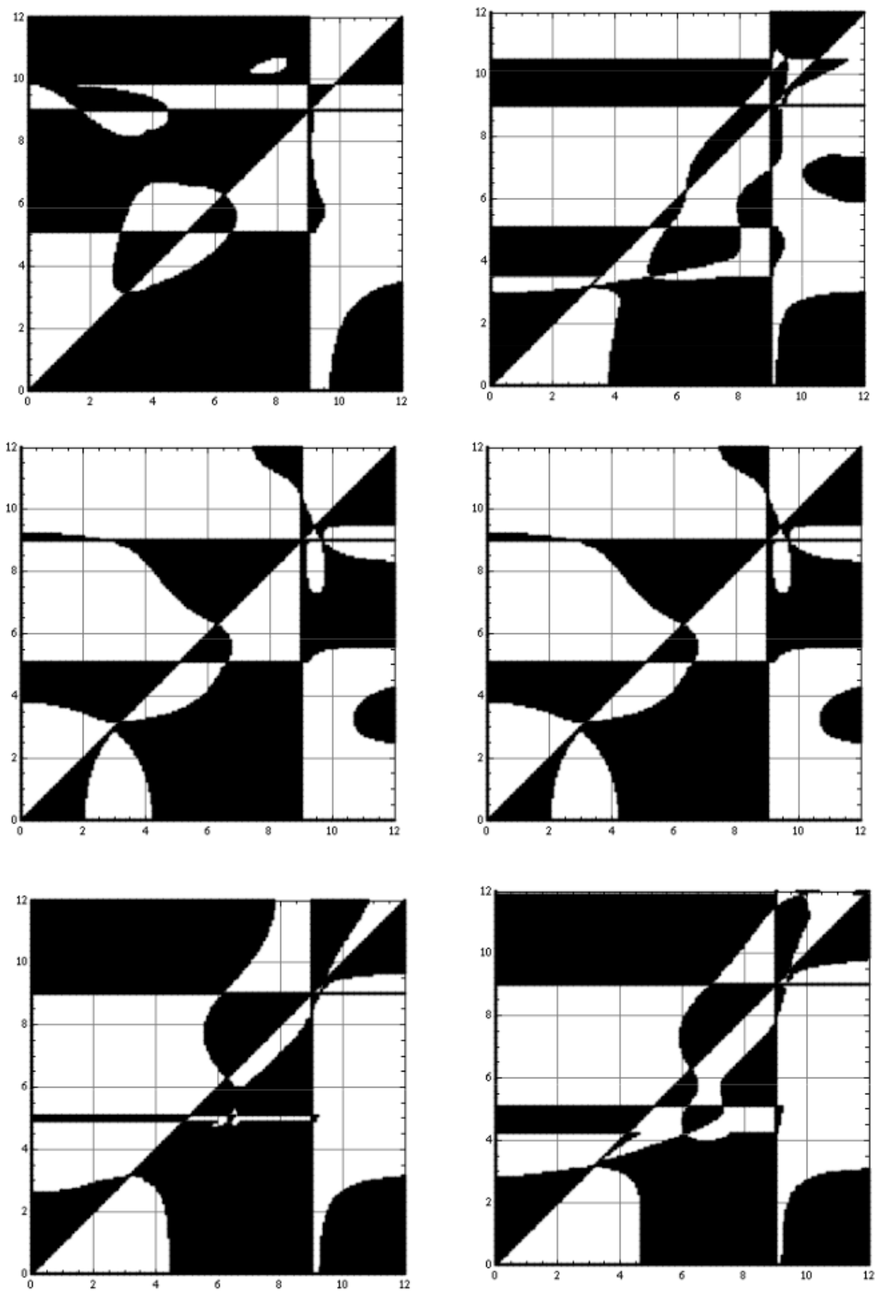

Figure 4: Regions of stability in the (ν, θ_1)–plane for the two-step methods for $m = 4$ with nodes $(0, \frac{1}{3}, \frac{2}{3}, 1)$, $(0, \frac{1}{10}, \frac{9}{10}, 1)$, $(0, \frac{2}{5}, \frac{3}{5}, 1)$, $(0, \frac{9}{20}, \frac{11}{20}, 1)$, $(\frac{1}{4}, \frac{3}{4})$, $(\frac{1}{5}, \frac{4}{5})$ respectively.

with $v >> 0$, $y(0) = 1$, $y'(0) = 0$, whose exact solution is $y(x) = \cos(10x)$, is an example of nonlinear equation, depending on the frequency $\omega = 10$.

Method	$h = \pi/8$	cd	$h = \pi/16$	cd	$h = \pi/32$	cd
TRIGFIT1	4.4409e-16	15.35	8.8818e-15	14.05	6.2172e-15	14.21
TRIGFIT2	1.7745e-15	14.75	4.2188e-15	14.37	6.8834e-15	14.16
TRIGFIT3	1.9984e-15	14.6993	4.21885e-15	14.3748	6.88338e-15	14.1622
MTRIGFIT	9.4058e-13	12.066	1.48992e-13	12.8268	2.18714e-14	13.6601

Table 3: Numerical results for the problem (42).

Numerical results show that trigonometrically fitted methods and mixed-trigonometrically fitted ones are both *exact* also for this nonlinear problem. Small differences in numerical errors are due to the round-off errors.

Test 3. We test our methods also on a well known example of stiff system, from [18, 21]

$$y''(t) = \begin{pmatrix} \mu - 2 & 2\mu - 2 \\ 1 - \mu & 1 - 2\mu \end{pmatrix} y(t), \quad y(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \; y'(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad t \in [0, 20\pi], \tag{43}$$

where μ is an arbitrary parameter. The exact solution is $y_1(t) = 2\cos t$, $y_2(t) = -\cos t$, i.e. it is independent on μ. When $\mu = 2500$, then (43) is Kramarz's system [18], which is often used in numerical experiments on stiffness in second order ODEs.

The eigenvalues of the coefficient matrix of the system (43) are -1 and $-\mu$, so that the analytical solution of the system exhibits the two frequencies 1 and $\sqrt{\mu}$, but the initial conditions eliminate the high frequency component, which corresponds to $\sqrt{\mu}$ when $\mu >> 1$. Notwithstanding this, its presence in the general solution of the system dictates strong restrictions on the choice of the stepsize, so that the system exhibits the phenomenon of *periodic stiffness* [23].

Method	$h = \pi/2$	cd	$h = \pi/4$	cd
TRIGFIT3	2.33916e-10	9.63094	1.95534e-9	8.70878
MTRIGFIT	1.44983e-8	7.83868	5.01173e-8	7.30001
TRIGFIT4S	3.80591e-10	9.41954	9.65342e-8	7.01532

Table 4: Numerical results for the problem 43.

The behaviour of our class of solvers is still similar to the one shown in the previous cases. The choice of the stepsize is such that the methods result stable, and it is possible to integrate this problem with a large stepsize. On the contrary, methods with constant coefficients are stable only for small values of the stepsize. Anyway methods Table 4 show that here we are in the presence of an unstability effect. A natural conclusion is that the advantage of using versions based on trigonometrically fitted coefficients is that they allow obtaining highly accurate results at values of the stepsize which are still big, well before that the unstability effect becomes severe.

6 Conclusions

In this paper, we present new trigonometrically fitted hybrid methods with parameters depending on one and two frequencies, by modifying the classical hybrid method presented in [3], and analyze the linear stability properties. We think that the used technique can be extended to adapt the coefficients of general linear methods [17] to an oscillatory behaviour, especially in the context of collocation methods, by modifying the choice of the collocation functions, considering not only the polynomial case, as done in [7, 8, 9, 10] but also trigonometric and mixed basis [4, 20, 22]. At the moment we have supposed that the value of the frequencies are a priori known, but it is worth mentioning that in the literature some techniques are in development to remove this restriction, and predict proper values of the frequencies in such a way that the local truncation error can be minimized (see for instance [6, 16]).

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Appendix: some arrays of some methods.

1. **Two-stage methods depending on one frequency.**
   The solution of the system (16) for $s = 2$ is

$$
\begin{aligned}
a_{11} &= -\frac{(\sin((c_1-c_2)\theta) + (1+c_1)\sin(c_2\theta) - c_1\sin((1+c_2)\theta))}{\theta^2\sin((c_1-c_2)\theta)}, \\
a_{12} &= \frac{((1+c_1)\sin(c_1\theta) - c_1\sin((1+c_1)\theta))}{\theta^2\sin((c_1-c_2)\theta)} \\
a_{21} &= \frac{(-((1+c_2)\sin(c_2\theta)) + c_2\sin((1+c_2)\theta))}{\theta^2\sin((c_1-c_2)\theta)} \\
a_{22} &= \frac{((1+c_2)\sin(c_1\theta) - c_2\sin((1+c_1)\theta) - \sin((c_1-c_2)\theta))}{\theta^2\sin((c_1-c_2)\theta)} \\
b_1 &= \frac{2(-1+\cos(\theta))\sin(c_2\theta)}{\theta^2\sin((c_1-c_2)\theta)} \\
b_2 &= \frac{-2(-1+\cos(\theta))\sin(c_1\theta)}{\theta^2\sin((c_1-c_2)\theta)}
\end{aligned}
$$

where $\theta = \omega h$. The coefficients expressed in this form are not of practical utility, because they are subject to heavy numerical cancellation: this is the reason why we handle their Taylor series expansion. For brevity, we give only the Taylor series expansion of the coefficients of the two stage method having $c = [3/4, 1]$

$$
\begin{aligned}
a_{11} &= \frac{91}{32} - \frac{4375}{6144}\theta^2 + \frac{198451}{2949120}\theta^4 - \frac{263429}{75497472}\theta^6 + \frac{62606173}{543581798400}\theta^8 - \frac{4225415771}{1607262661509120}\theta^{10} + O(\theta^{12}) \\
a_{12} &= -\frac{35}{16} + \frac{287}{768}\theta^2 - \frac{475}{18432}\theta^4 + \frac{2921}{2949120}\theta^6 - \frac{5149}{212336640}\theta^8 + \frac{1606607}{3923981107200}\theta^{10} + O(\theta^{12}) \\
a_{21} &= 4 - \frac{23}{24}\theta^2 + \frac{2071}{23040}\theta^4 - \frac{24019}{5160960}\theta^6 + \frac{4565341}{29727129600}\theta^8 - \frac{110038253}{31391848857600}\theta^{10} + O(\theta^{12}) \\
a_{22} &= -3 + \frac{1}{2}\theta^2 - \frac{11}{320}\theta^4 + \frac{71}{53760}\theta^6 - \frac{2503}{77414400}\theta^8 + \frac{3719}{6812467200}\theta^{10} + O(\theta^{12}) \\
b_1 &= 4 - \frac{23}{24}\theta^2 + \frac{2071}{23040}\theta^4 - \frac{24019}{5160960}\theta^6 + \frac{4565341}{29727129600}\theta^8 - \frac{110038253}{31391848857600}\theta^{10} + O(\theta^{12}) \\
b_2 &= -3 + \frac{1}{2}\theta^2 - \frac{11}{320}\theta^4 + \frac{71}{53760}\theta^6 - \frac{2503}{77414400}\theta^8 + \frac{3719}{6812467200}\theta^{10} + O(\theta^{12}).
\end{aligned}
$$

This representation of the coefficients is very expressive, because it allows us to easily consider what follows. First of all we can notice that, for $\theta \to 0$, these coefficients tend to the ones of the corresponding polynomial collocation method. Moreover, we can also easily derive the order of the resulting methods:

$$
\begin{aligned}
b_1 + b_2 &= 1 + O(\theta^2) \\
b_1c_1 + b_2c_2 &= O(\theta^2) \\
b_1c_1^2 + b_2c_2^2 &= -c_1c_2 + O(\theta^2).
\end{aligned}
$$

Therefore, the method has algebraic order 2. The stability region in the (ν, θ)-plane of the methods corresponding to some values of c_1 and c_2 are drawn in fig. 1.

2. **Three stage mixed trigonometrically fitted methods depending on one frequency.** Solving the system of equations for $m = 3$, we derive the coefficients of three stage methods depending on one frequency. We omit their expression because it is huge and it has no practical utility because of the heavy numerical cancellation it is subject to; anyway the expression of the coefficients can be required to the authors. In our numerical experiment we have used the Taylor expansion of the coefficients. We consider the method corresponding to $(c_1, c_2, c_3) = (\frac{1}{2}, \frac{3}{4}, 1)$:

$$
\begin{aligned}
a_{11} &= \frac{7}{2} - \frac{1081\theta^2}{2560} + \frac{251761\theta^4}{10321920} - \frac{1384021\theta^6}{1651507200} + O(\theta^8)\\
a_{12} &= -\frac{21}{4} + \frac{2293\theta^2}{3840} - \frac{163169\theta^4}{5160960} + \frac{279607\theta^6}{275251200} + O(\theta^8)\\
a_{13} &= \frac{17}{8} - \frac{1343\theta^2}{7680} + \frac{248559\theta^4}{3440640} - \frac{293621\theta^6}{1651507200} + O(\theta^8)\\
a_{21} &= \frac{693}{128} - \frac{781169\theta^2}{122880} + \frac{4318851\theta^4}{11796480} - \frac{31637221\theta^6}{2516582400} + O(\theta^8)\\
a_{22} &= -\frac{511}{64} + \frac{18403\theta^2}{20480} - \frac{279851\theta^4}{5898240} + \frac{57522283\theta^6}{3774873600} + O(\theta^8)\\
a_{23} &= \frac{413}{128} - \frac{322249\theta^2}{122880} + \frac{426617\theta^4}{3932160} - \frac{20134403\theta^6}{7549747200} + O(\theta^8)\\
a_{31} &= \frac{22}{3} - \frac{17\theta^2}{20} + \frac{2953\theta^4}{60480} - \frac{129767\theta^6}{77414400} + O(\theta^8)\\
a_{32} &= -\frac{32}{3} + \frac{6\theta^2}{5} - \frac{3827\theta^4}{60480} + \frac{78647\theta^6}{38707200} + O(\theta^8)\\
a_{33} &= \frac{13}{3} - \frac{7\theta^2}{20} + \frac{437\theta^4}{30240} - \frac{275527\theta^6}{77414400} + O(\theta^8)\\
b_1 &= \frac{22}{3} - \frac{17\theta^2}{20} + \frac{2953\theta^4}{60480} - \frac{129767\theta^6}{77414400} + O(\theta^8)\\
b_2 &= -\frac{32}{3} + \frac{6\theta^2}{5} - \frac{3827\theta^4}{60480} + \frac{78647\theta^6}{38707200} + O(\theta^8)\\
b_3 &= \frac{13}{3} - \frac{7\theta^2}{20} + \frac{437\theta^4}{30240} - \frac{275527\theta^6}{77414400} + O(\theta^8)
\end{aligned}
$$

We can verify the order of this method, applying the set of order conditions [3]

$$
\begin{aligned}
\sum_{i=1}^{3} b_i &= 1 + O(\theta^2) \\
\sum_{i=1}^{3} b_i c_i &= O(\theta^2) \\
\sum_{i=1}^{3} b_i c_i^2 &= \frac{1}{6} + O(\theta^2) \\
\sum_{i=1}^{3}\sum_{j=1}^{3} b_i a_{ij} &= \frac{1}{12} + O(\theta^2) \\
\sum_{i=1}^{3} b_i c_i^3 &\neq O(\theta^2).
\end{aligned}
$$

Therefore, the method has algebraic order 3. The stability region is drawn in fig. 2.

3. **Four stage methods depending on two frequencies.**
We now consider four stage methods of order 4 depending on two frequencies. The following method comes out setting $(c_1, c_2, c_3, c_4) = (0, \frac{1}{3}, \frac{2}{3}, 1)$. We report some terms of the Taylor series expansion of its coefficients.

$$
\begin{aligned}
a_{11} &= 0 \\
a_{12} &= 0 \\
a_{13} &= 0 \\
a_{14} &= 0 \\
a_{21} &= \frac{13676040 - 658854\theta_2^2}{16533720} - \frac{\theta_1^2(355781160 - 6480270\theta_2^2)}{8928208800} + O(\theta_1^4) + O(\theta_2^4) \\
a_{22} &= -\frac{71}{54} + \frac{833\theta_2^2}{9720} + \frac{\theta_1^2(510095880 - 35167230\theta_2^2)}{5952139200} + O(\theta_1^4) + O(\theta_2^4) \\
a_{23} &= \frac{26}{27} - \frac{7\theta_2^2}{135} + \theta_1^2 \frac{\frac{-(7}{135)+(533\theta_2^2)}}{367416} + O(\theta_1^4) + O(\theta_2^4) \\
a_{24} &= \frac{-8368920 + 198450\theta_2^2}{33067440} + \frac{\theta_1^2(21432600 - 103626\theta_2^2)}{3571283520} + O(\theta_1^4) + O(\theta_2^4) \\
a_{31} &= \frac{539}{324} - \frac{929\theta_2^2}{11664} + \theta_1^2 \frac{\frac{-(929}{11664)+(27443\theta_2^2)}}{18895680} + O(\theta_1^4) + O(\theta_2^4) \\
a_{32} &= -\frac{137}{54} + \frac{37\theta_2^2}{216} + \frac{\theta_1^2(203915880 - 14061762\theta_2^2)}{1190427840} + O(\theta_1^4) + O(\theta_2^4) \\
a_{33} &= 42661080 - 2285010\theta_2^2 - \frac{\theta_1^2(246781080 - 6909354\theta_2^2)}{2380855680} + O(\theta_1^4) + O(\theta_2^4) \\
a_{34} &= -8368920 + 198450\theta_2^2 + \frac{\theta_1^2(21432600 - 103626\theta_2^2)}{1785641760} + O(\theta_1^4) + O(\theta_2^4) \\
a_{41} &= \frac{5}{2} - \frac{43\theta_2^2}{360} + \frac{\theta_1^2(-8777160 + 160110\theta_2^2)}{73483200} + O(\theta_1^4) + O(\theta_2^4) \\
a_{42} &= -\frac{15}{4} + \frac{37\theta_2^2}{144} + \frac{\theta_1^2(7552440 - 520722\theta_2^2)}{29393280} + O(\theta_1^4) + O(\theta_2^4)
\end{aligned}
$$

$$
\begin{aligned}
a_{43} &= 3 - \frac{7\theta_2^2}{45} + \frac{\theta_1^2(-5715360 + 160110\theta_2^2)}{36741600} + O(\theta_1^4) + O(\theta_2^4) \\
a_{44} &= -\frac{3}{4} + \frac{13\theta_2^2}{720} + \frac{\theta_1^2(2653560 - 12690\theta_2^2)}{146966400} + O(\theta_1^4) + O(\theta_2^4) \\
b_1 &= \frac{5}{2} - \frac{43\theta_2^2}{360} + \frac{\theta_1^2(-8777160 + 160110\theta_2^2)}{73483200} + O(\theta_1^4) + O(\theta_2^4) \\
b_2 &= -\frac{15}{4} + \frac{37\theta_2^2}{144} + \frac{\theta_1^2(7552440 - 520722\theta_2^2)}{29393280} + O(\theta_1^4) + O(\theta_2^4) \\
b_3 &= \frac{3 - (7\theta_2^2)}{45} + \frac{\theta_1^2(-5715360 + 160110\theta_2^2)}{36741600} + O(\theta_1^4) + O(\theta_2^4) \\
b_4 &= -\frac{3}{4} + \frac{13\theta_2^2}{720} + \frac{\theta_1^2(2653560 - 12690\theta_2^2)}{146966400} + O(\theta_1^4) + O(\theta_2^4)
\end{aligned}
$$

Figures 3 and 4 show the stability region of this method and other methods, obtained in correspondence of different values of the abscissa.