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General Linear Methods for $y'' = f(y(t))$

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Abstract In this paper we consider the family of General Linear Methods (GLMs) for the numerical solution of special second order Ordinary Differential Equations (ODEs) of the type $y'' = f(y(t))$, with the aim to provide a unifying approach for the analysis of the properties of consistency, zero-stability and convergence. This class of methods properly includes all the classical methods already considered in the literature (e.g. linear multistep methods, Runge-Kutta-Nyström methods, two-step hybrid methods and two-step Runge-Kutta-Nyström methods) as special cases. We deal with formulation of GLMs and present some general results regarding consistency, zero-stability and convergence. The approach we use is the natural extension of the GLMs theory developed for first order ODEs.

Keywords Second order Ordinary Differential Equations · General Linear Methods · convergence analysis

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1 Introduction

It is the purpose of this paper to introduce a general family of numerical methods suited to numerically integrate initial value problems based on special second order Ordinary Differential Equations (ODEs)

$$\begin{cases} y''(t) = f(y(t)), \quad t \in [t_0, T], \\ y(t_0) = y_0 \in \mathbb{R}^d, \\ y'(t_0) = y'_0 \in \mathbb{R}^d, \end{cases} \tag{1.1}$$

where the function $f : \mathbb{R}^d \to \mathbb{R}^d$ does not explicitly depend on y' and is supposed to be smooth enough to ensure that the corresponding problem (1.1) is Hadamard well-posed. Although the problem (1.1) could be transformed into a system of twice the dimension first order ODEs and solved by standard formulae for first order differential systems, the development of numerical methods for its direct integration is more natural and efficient.

The development of a general family of numerical methods for the solution of first order ODEs is due to John C. Butcher (compare [2,3,11] and the references therein), who provided a unifying theory approaching the basic questions of convergence, linear and nonlinear stability of numerical methods for ODEs. His studies lead to the introduction of the family of General Linear Methods (GLMs), later used not only as a framework for the analysis of accuracy and stability matters: in fact, it is worth observing that the discovery of a GLM theory "opened the possibility of obtaining essentially new methods which were neither Runge-Kutta nor linear multistep methods nor slight variations of these methods" (compare [13]).

For second order ODEs (1.1) many linear and nonlinear methods appeared in the literature (see, for instance, [7–10] and references therein), but a systematic investigation on GLMs has never been in considered till now. In order to transfer to second order ODEs of the same benefits obtained in the case of first order ODEs, the purpose of this paper is the foundation of a theory of GLMs for the numerical solution of (1.1). The paper is organized as follows: the formulation of GLMs for (1.1) is introduced in Section 2; Section 3 is devoted to the introduction of the concept of consistency for GLM; the study of zero-stability for GLMs is carried out in Section 4 where, together with the definition of zero-stability, a criterion to analyze zero-stability is also proved; Section 5 concerns with the convergence analysis of GLMs and a useful characterization for GLMs is provided; Section 6 contains some conclusions and possible further development of this research.

2 Representation of General Linear Methods

In this section, we discuss a general representation formula of GLMs for second order ODEs (1.1), in order to properly embrace a wide number of classical numerical methods for (1.1) and with the initial aim to establish the necessary results for the development of a unifying theory of numerical methods for (1.1). Thus, following the lines drawn in [2,3,11], we consider in this paper the uniform grid

$$I_h = \{t_n = t_0 + nh,\ n = 0, 1, ..., N,\ Nh = T - t_0\},$$

which provides the discrete counterpart of the interval of the definition I of the problem (1.1), considering in our preliminary analysis a fixed stepsize h. We assume as a point of reference of our analysis the family of GLMs for first order ODEs (compare [2,3,11] and references therein), i.e.

$$\begin{cases} Y_i^{[n]} = \sum\limits_{j=1}^{s} a_{ij} h f(Y_j^{[n]}) + \sum\limits_{j=1}^{r} u_{ij} y_j^{[n-1]},\ i = 1, 2, \ldots, s, \\ y_i^{[n]} = \sum\limits_{j=1}^{s} b_{ij} h f(Y_j^{[n]}) + \sum\limits_{j=1}^{r} v_{ij} y_j^{[n-1]},\ \ i = 1, 2, \ldots, r, \end{cases} \tag{2.1}$$

introduced by Burrage and Butcher [1] in 1980. In order to adapt such formulation to second order ODEs (1.1) and achieve the mentioned purpose of gaining a very general class of numerical methods to solve this problem, we inherit the same structure as in (2.1) but also include explicit dependence on the approximations to the first derivative of the solution. Thus, we introduce the abscissa vector $\mathbf{c} = [c_1, c_2, \ldots, c_s]$ and define the following supervectors

$$y^{[n-1]} = \begin{bmatrix} y_1^{[n-1]} \\ y_2^{[n-1]} \\ \vdots \\ y_r^{[n-1]} \end{bmatrix} \in \mathbb{R}^{rd},\ y'^{[n-1]} = \begin{bmatrix} y'^{[n-1]}_1 \\ y'^{[n-1]}_2 \\ \vdots \\ y'^{[n-1]}_{r'} \end{bmatrix} \in \mathbb{R}^{r'd},\ Y^{[n]} = \begin{bmatrix} Y_1^{[n]} \\ Y_2^{[n]} \\ \vdots \\ Y_s^{[n]} \end{bmatrix} \in \mathbb{R}^{sd}.$$

The vector $y^{[n-1]}$ is denoted as *input vector* of the external stages, and contains all the informations we want to transfer advancing from the point t_{n-1} to the point t_n of the grid. It is important to observe that such a vector could also contain not only approximations to the solution of the problem in the grid points inherited from the previous steps, but also other informations computed in the past that we want to use in the integration process. The vector $y'^{[n-1]}$ instead contains previous approximations to the first derivative of the solution computed in previous step points, while the values $Y_j^{[n-1]}$, denoted as *internal stage* values, provide an approximation to the solution in the internal points $t_{n-1} + c_j h$, $j = 1, 2, \ldots, s$.

Our formulation of GLMs for second order ODEs then involves nine coefficient matrices $\mathbf{A} \in \mathbb{R}^{\mathbf{s}\times\mathbf{s}}$, $\mathbf{P} \in \mathbb{R}^{\mathbf{s}\times\mathbf{r}'}$, $\mathbf{U} \in \mathbb{R}^{\mathbf{s}\times\mathbf{r}}$, $\mathbf{C} \in \mathbb{R}^{\mathbf{r}'\times\mathbf{s}}$, $\mathbf{R} \in \mathbb{R}^{\mathbf{r}'\times\mathbf{r}'}$, $\mathbf{W} \in \mathbb{R}^{\mathbf{r}'\times\mathbf{r}}$, $\mathbf{B} \in \mathbb{R}^{\mathbf{r}\times\mathbf{s}}$, $\mathbf{Q} \in \mathbb{R}^{\mathbf{r}\times\mathbf{r}'}$, $\mathbf{V} \in \mathbb{R}^{\mathbf{r}\times\mathbf{r}}$, which are put together in the following partitioned $(s + r' + r) \times (s + r' + r)$ matrix

$$\left[\begin{array}{c|c|c} \mathbf{A} & \mathbf{P} & \mathbf{U} \\ \hline \mathbf{C} & \mathbf{R} & \mathbf{W} \\ \hline \mathbf{B} & \mathbf{Q} & \mathbf{V} \end{array}\right], \tag{2.2}$$

which is denoted as the Butcher tableau of the GLM. Using these notations, a GLM for second order ODEs can then be expressed as follows:

$$\begin{cases} Y^{[n]} = h^2(\mathbf{A}\otimes\mathbf{I})F^{[n]} + h(\mathbf{P}\otimes\mathbf{I})y'^{[n-1]} + (\mathbf{U}\otimes\mathbf{I})y^{[n-1]}, \\ hy'^{[n]} = h^2(\mathbf{C}\otimes\mathbf{I})F^{[n]} + h(\mathbf{R}\otimes\mathbf{I})y'^{[n-1]} + (\mathbf{W}\otimes\mathbf{I})y^{[n-1]}, \\ y^{[n]} = h^2(\mathbf{B}\otimes\mathbf{I})F^{[n]} + h(\mathbf{Q}\otimes\mathbf{I})y'^{[n-1]} + (\mathbf{V}\otimes\mathbf{I})y^{[n-1]}, \end{cases} \tag{2.3}$$

where $\otimes$ denotes the usual Kronecker tensor product, $\mathbf{I}$ is the identity matrix in $\mathbb{R}^{d\times d}$ and $F^{[n]} = [f(Y_1^{[n]}), f(Y_2^{[n]}), \ldots, f(Y_s^{[n]})]^T$. Componentwise,

$$\begin{aligned} Y_i^{[n]} &= h^2\sum_{j=1}^{s} a_{ij} f(Y_j^{[n]}) + h\sum_{j=1}^{r'} p_{ij} y'^{[n-1]}_j + \sum_{j=1}^{r} u_{ij} y_j^{[n-1]} \quad i = 1, ..., s, \\ hy'^{[n]}_i &= h^2\sum_{j=1}^{s} c_{ij} f(Y_j^{[n]}) + h\sum_{j=1}^{r'} r_{ij} y'^{[n-1]}_j + \sum_{j=1}^{r} w_{ij} y_j^{[n-1]} \quad i = 1, ..., r' \\ y_i^{[n]} &= h^2\sum_{j=1}^{s} b_{ij} f(Y_j^{[n]}) + h\sum_{j=1}^{r'} q_{ij} y'^{[n-1]}_j + \sum_{j=1}^{r} v_{ij} y_j^{[n-1]} \quad i = 1, ..., r. \end{aligned} \tag{2.4}$$

We are aware that a more compact representation could be provided by merging, for instance, the approximations of the first derivative into the input vector of the external stages. However, we have decided to explicitly represent the approximations to the first derivative, as it is usually done in the context of numerical methods for second order ODEs: this is typical, for instance, of Runge-Kutta-Nyström methods (see [7]).

We also observe that, if the methods do not explicitly depend on approximations to the first derivative (as it happens in the case of linear multistep methods [7] or Coleman hybrid methods [4]), the weights of the first derivative appearing in each equation of (2.3) are equal to zero, i.e. the matrices $\mathbf{P}, \mathbf{Q}, \mathbf{C}, \mathbf{R}, \mathbf{W}$ are all equal to the zero matrix.

3 Preconsistency and consistency

Following [2,3,11], we first address our attention on the definition of some minimal accuracy requirements for GLMs, i.e. preconsistency and consistency, which guarantee the coherence of GLMs with respect to problems whose solution is a constant or a linear polynomial.

In order to satisfy such minimal accuracy requirements, we assume that there exist three vectors

$$\begin{aligned} \mathbf{q}_0 &= [q_{1,0} \quad q_{2,0} \quad \cdots \quad q_{r,0}]^T, \\ \mathbf{q}_1 &= [q_{1,1} \quad q_{2,1} \quad \cdots \quad q_{r,1}]^T, \\ \mathbf{q}_2 &= [q_{1,2} \quad q_{2,2} \quad \cdots \quad q_{r,2}]^T, \end{aligned}$$

such that the components of the input and the output vectors of the external stages respectively satisfy

$$\begin{aligned} y_i^{[n-1]} &= q_{i,0}y(t_{n-1}) + q_{i,1}hy'(t_{n-1}) + q_{i,2}h^2y''(t_{n-1}) + O(h^3), \\ y_i^{[n]} &= q_{i,0}y(t_n) + q_{i,1}hy'(t_n) + q_{i,2}h^2y''(t_n) + O(h^3), \end{aligned}$$

$i = 1, 2, \ldots, r$ and, moreover, that there exist two vectors

$$\mathbf{q}_1' = [q_{1,1}' \quad q_{2,1}' \quad \ldots \quad q_{r',1}']^T, \quad \mathbf{q}_2' = [q_{1,2}' \quad q_{2,2}' \quad \ldots \quad q_{r',2}']^T,$$

such that the components of the input and the output vectors associated to the first derivative approximations satisfy

$$\begin{aligned} hy_i'^{[n-1]} &= q_{i,1}'hy'(t_{n-1}) + q_{i,2}'h^2y''(t_{n-1}) + O(h^3), \\ hy_i'^{[n]} &= q_{i,1}'hy'(t_n) + q_{i,2}'h^2y''(t_n) + O(h^3), \end{aligned}$$

$i = 1, \ldots, r'$. We finally assume that the components of the stage vector $Y^{[n]}$ satisfy the condition

$$Y_i^{[n]} = y(t_{n-1} + c_ih) + O(h^3), \quad i = 1, \ldots, s.$$

which, by expanding the right hand side in Taylor series around the point t_{n-1}, leads to the condition

$$Y_i^{[n]} = y(t_{n-1}) + c_ihy'(t_{n-1}) + \frac{(c_ih)^2}{2}y''(t_{n-1}) + O(h^3), \quad i = 1, \ldots, s.$$

Substituting these relations in the GLM (2.4) and comparing the powers of h up to h^2 leads to the following definitions.

Definition 3.1 A GLM (2.4) is *preconsistent* if there exist vectors $\mathbf{q}_0$, $\mathbf{q}_1$ and $\mathbf{q}_1'$ such that

$$\begin{gathered} \mathbf{U}\mathbf{q}_0 = e, \quad \mathbf{W}\mathbf{q}_0 = 0, \quad \mathbf{V}\mathbf{q}_0 = \mathbf{q}_0, \\ \mathbf{P}\mathbf{q}_1' + \mathbf{U}\mathbf{q}_1 = \mathbf{c}, \quad \mathbf{R}\mathbf{q}_1' + \mathbf{W}\mathbf{q}_1 = \mathbf{q}_1', \quad \mathbf{Q}\mathbf{q}_1' + \mathbf{V}\mathbf{q}_1 = \mathbf{q}_0 + \mathbf{q}_1. \end{gathered}$$

In the context of GLMs for first order ODEs, Butcher [3] observed that preconsistency is equivalent to the concept of *covariance* of a GLM, which essentially ensures that numerical approximations are appropriately transformed by a shift of origin and a constant value times $\mathbf{q}_0$ persists from step to step.

Definition 3.2 A preconsistent GLM (2.4) is *consistent* if exist vectors $\mathbf{q}_2$ and $\mathbf{q}_2'$ such that

$$\mathbf{C}e + \mathbf{R}\mathbf{q}_2' + \mathbf{W}\mathbf{q}_2 = \mathbf{q}_1' + \mathbf{q}_2', \quad \mathbf{B}e + \mathbf{Q}\mathbf{q}_2' + \mathbf{V}\mathbf{q}_2 = \frac{\mathbf{q}_0}{2} + \mathbf{q}_1 + \mathbf{q}_2.$$

Definition 3.3 A consistent GLM (2.4) is *stage-consistent* if

$$\mathbf{A}e + \mathbf{P}\mathbf{q}_2' + \mathbf{U}\mathbf{q}_2 = \frac{\mathbf{c}^2}{2}.$$

4 Zero-stability

Another basic requirement in the context of the numerical integration of ODEs is, together with consistency, also zero-stability. In order to define such minimal stability requirement, we apply the GLM (2.3) to the problem

$$y'' = 0,$$

obtaining the recurrence relation

$$\begin{bmatrix} hy'^{[n]} \\ y^{[n]} \end{bmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{W} \\ \mathbf{Q} & \mathbf{V} \end{bmatrix} \begin{bmatrix} hy'^{[n-1]} \\ y^{[n-1]} \end{bmatrix}.$$

The matrix

$$\mathbf{M}_0 = \begin{bmatrix} \mathbf{R} & \mathbf{W} \\ \mathbf{Q} & \mathbf{V} \end{bmatrix}$$

is denoted as the *zero-stability matrix* of the GLM (2.3). The following definition occurs.

Definition 4.1 A GLM (2.3) is zero-stable if there exist two real constants C and D such that

$$\|\mathbf{M}_0^m\| \leq mC + D, \quad \forall m = 1, 2, \ldots. \tag{4.1}$$

A criterion equivalent to condition (4.1) is given in the following theorem. This result follows the lines drawn by Butcher in [2].

Theorem 4.1
The following statements are equivalent:

(i) $\mathbf{M}_0$ *satisfies the bound (4.1);*
(ii) the roots of the minimal polynomial of the matrix $\mathbf{M}_0$ *lie on or within the unit circle and the multiplicity of the zeros on the unit circle is at most two;*
(iii) there exist a matrix B *similar to* $\mathbf{M}_0$ *such that*

$$\sup_m \{\|B^m\|_\infty, \ m \geq 1\} \leq m + 1.$$

Proof The result holds by proving the implications $(i) \Rightarrow (ii)$, $(ii) \Rightarrow (iii)$, and $(iii) \Rightarrow (i)$. We first prove that $(i) \Rightarrow (ii)$. Suppose that λ is an eigenvalue of $\mathbf{M}_0$ and denote by v the corresponding eigenvector. As a consequence, we obtain

$$\|\mathbf{M}_0^m\|_\infty = \sup_{x \neq 0} \frac{\|\mathbf{M}_0^m x\|_\infty}{\|x\|_\infty} \geq \frac{\|\mathbf{M}_0^m v\|_\infty}{\|v\|_\infty} = \frac{\|\lambda^m v\|_\infty}{\|v\|_\infty} = |\lambda|^m,$$

and, taking into account that $\mathbf{M}_0$ satisfies assumption (i), we obtain $|\lambda| \leq 1$.

Since λ is an element of the spectrum of $\mathbf{M}_0$, it is also a root of its minimal polynomial. We suppose that λ is a repeated zero of the minimal polynomial with multiplicity $\mu(\lambda) = 3$: then there exist three nonzero vectors u, v and w such that

$$\mathbf{M}_0 w = \lambda w + u, \quad \mathbf{M}_0 u = \lambda u + v, \quad \mathbf{M}_0 v = \lambda v.$$

It is easy to prove by induction that

$$\mathbf{M}_0^m w = \lambda^m u + m\lambda^{m-1} u + \frac{m(m-1)}{2}\lambda^{m-2} v, \quad \text{for any} \;\; m \geq 2.$$

As a consequence, the following bound holds:

$$\begin{aligned}
\|\mathbf{M}_0^m\|_\infty &= \sup_{x\neq 0} \frac{\|\mathbf{M}_0^m x\|_\infty}{\|x\|_\infty} \geq \frac{\|\mathbf{M}_0^m w\|_\infty}{\|w\|_\infty} \\
&= \frac{\left\|\lambda^m w + m\lambda^{m-1} u + \frac{m(m-1)}{2}\lambda^{m-2} v\right\|_\infty}{\|w\|_\infty} \\
&= |\lambda|^{m-2}\left(|\lambda|^2 - m|\lambda|\frac{\|u\|_\infty}{\|w\|_\infty} - \frac{m(m-1)}{2}\frac{\|v\|_\infty}{\|w\|_\infty}\right).
\end{aligned}$$

If $|\lambda| = 1$, then

$$\|\mathbf{M}_0^m\|_\infty \geq 1 - m\frac{\|u\|_\infty}{\|w\|_\infty} - \frac{m(m-1)}{2}\frac{\|v\|_\infty}{\|w\|_\infty},$$

and, by setting $C := \frac{\|u\|_\infty}{\|w\|_\infty}$ and $D := \frac{\|v\|_\infty}{\|w\|_\infty}$, we obtain

$$\|\mathbf{M}_0^m\|_\infty \geq 1 + mC + \frac{m(m-1)}{2}D,$$

which means that $\|\mathbf{M}_0^m\|$ cannot be linearly bounded as $m \to \infty$, against the hypothesis (i). In conclusion, if $\mu(\lambda) = 3$, then $|\lambda| < 1$. In correspondence of $\mu(\lambda) = 2$, we have

$$\|\mathbf{M}_0^m\|_\infty \geq |\lambda|^{m-1}\left(m\frac{\|v\|_\infty}{\|u\|_\infty} - |\lambda|\right),$$

which, for $|\lambda| = 1$, leads to the bound

$$\|\mathbf{M}_0^m\|_\infty \geq mC - 1,$$

which can coherently be combined with the assumption (i). We next suppose that (ii) holds: then, we can choose the matrix B as the Jordan canonical form of $\mathbf{M}_0$

$$B = \begin{bmatrix} \mathbf{J}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_2 \end{bmatrix},$$

where the block $\mathbf{J}_1$ assumes the form

$$\mathbf{J}_1 = \begin{bmatrix} \lambda & a \\ 0 & \lambda \end{bmatrix},$$

with $|\lambda| = 1$ and

$$a = \begin{cases} 1, & \text{if } \mu(\lambda) = 2, \\ 0, & \text{if } \mu(\lambda) = 1, \end{cases}$$

while the block $\mathbf{J}_2$ contains the eigenvalues λ_i of modulus less than 1 on the diagonal and $1 - |\lambda_i|$ on the upper co-diagonal. Since B is a block diagonal matrix, we have

$$B^m = \begin{bmatrix} \mathbf{J}_1^m & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_2^m \end{bmatrix},$$

with

$$\mathbf{J}_1^m = \begin{bmatrix} \lambda^m & am\lambda^{m-1} \\ 0 & \lambda^m \end{bmatrix}.$$

It follows that

$$\|B^m\|_\infty = \max \left\{ \left\| \begin{pmatrix} \lambda^m & am\lambda^{m-1} \\ 0 & \lambda^m \end{pmatrix} \right\|_\infty, \quad \|J_2^m\|_\infty \right\} \leq m + 1.$$

Finally, if (iii) is true, since B is similar to $\mathbf{M}_0$, then there exists a matrix P such that $B = P^{-1}\mathbf{M}_0 P$. As a consequence,

$$\|\mathbf{M}_0^m\|_\infty = \left\|PB^mP^{-1}\right\|_\infty \leq m + 1,$$

i.e. $\mathbf{M}_0$ satisfies the zero-stability bound (4.1). □

Remark 1 Let us note that condition (ii) in Theorem 4.1 is peculiar in the numerical solution of second order ODEs (1.1). In fact, the notion of zero-stability for GLMs solving first order ODEs (compare [2,3,11]) implies that the minimal polynomial of its zero-stability matrix can possess at most one root of modulus one, while all the others have modulus less than one. Instead, in the case of second order ODEs, two roots of the minimal polynomial of the zero-stability matrix lying on the unit circle are allowed, taking into account also the case of complex conjugate roots of modulus one, as might happen in second order ODES (1.1) in the oscillatory case. This is made clear in [7], where the authors prove the necessity for convergence of such a zero-stability condition in the context of linear multistep methods.

5 Convergence

In this section we focus our attention on the convergence analysis of GLMs (2.4), first extending the ideas introduced by Butcher [3] in order to formulate a rigorous definition of convergence for a GLM (2.4). In force of the nature of GLMs, a starting procedure is needed in order to determine the missing starting values $y^{[0]}$ and $y'^{[0]}$ to be used as input for the first step of the integration process: in the context of convergence analysis, we only need to assume that there exist a starting procedure

$$S_h : \mathbb{R}^{2d} \to \mathbb{R}^{dr},$$

associating, for any value of the stepsize h, a starting vector $y^{[0]} = S_h(y_0, y_0')$ such that

$$\lim_{h\to 0} \frac{S_h(y_0, y_0') - (\mathbf{q_0} \otimes I)y(t_0)}{h} = (\mathbf{q_1} \otimes I)y'(t_0), \tag{5.1}$$

and, moreover, the initial vector $y'^{[0]}$ is provided in order to ensure that

$$\lim_{h\to 0} y'^{[0]} = (\mathbf{q_1'} \otimes I)y'(t_0). \tag{5.2}$$

We now introduce the following definition.

Definition 5.1 A preconsistent GLM (2.4) is convergent if, for any well-posed initial value problem (1.1), there exist a starting procedure S_h satisfying (5.1) such that the sequence of vectors $y^{[n]}$, computed using n steps with stepsize $h = (\bar{t} - t_0)/n$ and using $y^{[0]} = S_h(y_0, y_0')$, converges to $\mathbf{q}_0 y(\bar{t})$, and the sequence of vectors $y'^{[n]}$, computed using n steps with the same stepsize h starting from $y'^{[0]}$ satisfying (5.2), converges to $\mathbf{q'}_1 y'(\bar{t})$, for any $\bar{t} \in [t_0, T]$.

Proving the convergence of a numerical method is generally a nontrivial task: however, the following results create a very close connection among the concepts of convergence, consistency and zero-stability and allow to prove the convergence of a numerical scheme by checking some algebraic conditions involving the coefficients of the method: indeed, we prove that a GLM (2.4) is convergent if and only if it is consistent and zero-stable. This powerful result has already been proved in the context of GLMs for first order ODEs [3,11]. We now extend it to the case of GLMs (2.4) for second order ODEs, by first proving the sufficiency of consistency and zero-stability, while their necessity is object of Theorem 5.2.

Theorem 5.1 *A GLM (2.4) is convergent if it is consistent and zero-stable.*

Proof We introduce the vectors

$$\hat{y}^{[n-1]} = \begin{bmatrix} \hat{y}_1^{[n-1]} \\ \hat{y}_2^{[n-1]} \\ \vdots \\ \hat{y}_r^{[n-1]} \end{bmatrix}, \hat{y}^{[n]} = \begin{bmatrix} \hat{y}_1^{[n]} \\ \hat{y}_2^{[n]} \\ \vdots \\ \hat{y}_r^{[n]} \end{bmatrix}, \hat{y}'^{[n-1]} = \begin{bmatrix} \hat{y}'^{[n-1]}_1 \\ \hat{y}'^{[n-1]}_2 \\ \vdots \\ \hat{y}'^{[n-1]}_{r'} \end{bmatrix}, \hat{y}'^{[n]} = \begin{bmatrix} \hat{y}'^{[n]}_1 \\ \hat{y}'^{[n]}_2 \\ \vdots \\ \hat{y}'^{[n]}_{r'} \end{bmatrix},$$

defined by

$$\begin{aligned} \hat{y}_i^{[n-1]} &= q_{i0} y(t_{n-1}) + q_{i1} h y'(t_{n-1}) + q_{i2} h^2 y''(t_{n-1}), \\ \hat{y}_i^{[n]} &= q_{i0} y(t_n) + q_{i1} h y'(t_n) + q_{i2} h^2 y''(t_n), \\ h\hat{y}'^{[n-1]}_i &= q'_{i1} h y'(t_{n-1}) + q'_{i2} h^2 y''(t_{n-1}), \\ h\hat{y}'^{[n]}_i &= q'_{i1} h y'(t_n) + q'_{i2} h^2 y''(t_n), \end{aligned}$$

where $q_{i,0}$, $q_{i,1}$ and $q'_{i,1}$ are the components of the preconsistency vectors $\mathbf{q}_0$, $\mathbf{q}_1$ and $\mathbf{q}'_1$, while $q_{i,2}$ and $q'_{i,2}$ are the components of the consistency vectors $\mathbf{q}_2$ and $\mathbf{q}'_2$. We next denote by

$$\xi_i(h), \quad \eta_i(h), \quad \zeta_i(h),$$

the residua arising after replacing in (2.4) $y_i^{[n-1]}$, $y_i^{[n]}$, ${y'}_i^{[n-1]}$, ${y'}_i^{[n]}$ by $\hat{y}_i^{[n-1]}$, $\hat{y}_i^{[n]}$, $\hat{y}'^{[n-1]}_i$, $\hat{y}'^{[n]}_i$ respectively and, moreover, $Y_i^{[n]}$ by $y(t_{n-1}+c_ih)$. The mentioned replacements lead to the following equations:

$$\begin{aligned} y(t_{n-1}+c_ih) = h^2 \sum_{j=1}^{s} a_{ij} y''(t_{n-1}+c_jh) \\ + h \sum_{j=1}^{r'} p_{ij}(q'_{j1}y'(t_{n-1}) + hq'_{j2}y''(t_{n-1})) \\ + \sum_{j=1}^{r} u_{ij}(q_{j0}y(t_{n-1}) + q_{j1}hy'(t_{n-1}) + q_{j2}h^2y''(t_{n-1})) \\ + \xi_i(h), \qquad i = 1, 2, \dots, s, \end{aligned} \tag{5.3}$$

$$\begin{aligned} q'_{i1}hy'(t_n) + q'_{i2}h^2y''(t_n) = h^2 \sum_{j=1}^{s} c_{ij} y''(t_{n-1}+c_jh) \\ + h \sum_{j=1}^{r'} r_{ij}(q'_{j1}y'(t_{n-1}) + hq'_{j2}y''(t_{n-1})) \\ + \sum_{j=1}^{r} w_{ij}(q_{j0}y(t_{n-1}) + q_{j1}hy'(t_{n-1}) + q_{j2}h^2y''(t_{n-1})) \\ + \zeta_i(h), \qquad i = 1, 2, \dots, r', \end{aligned} \tag{5.4}$$

$$\begin{aligned} q_{i0}y(t_n) + q_{i1}hy'(t_n) + q_{i2}h^2y''(t_n) = h^2 \sum_{j=1}^{s} b_{ij} y''(t_{n-1}+c_jh) \\ + h \sum_{j=1}^{r'} a_{ij}(q'_{j1}y'(t_{n-1}) + hq'_{j2}y''(t_{n-1})) \\ + \sum_{j=1}^{r} v_{ij}(q_{j0}y(t_{n-1}) + q_{j1}hy'(t_{n-1}) + q_{j2}h^2y''(t_{n-1})) \\ + \eta_i(h), \qquad i = 1, 2, \dots, r. \end{aligned} \tag{5.5}$$

By expanding $y(t_{n-1}+c_ih)$, $y''(t_{n-1}+c_jh)$ in Taylor series around t_{n-1}, replacing the obtained expansions in (5.3) and using the hypothesis of preconsistency of the method, we obtain that

$$\xi_i(h) = O(h^2), \quad i = 1, 2, \dots, s.$$

In particular, we observe that if the method (2.4) is also stage consistent, we have

$$\xi_i(h) = O(h^3), \quad i = 1, 2, \dots, s.$$

Proceeding in analogous way for (5.4) and (5.5), using in these cases the pre-consistency and consistency conditions, we obtain

$$\zeta_i(h) = O(h^3), \quad i = 1, 2, \dots, r',$$

and

$$\eta_i(h) = O(h^3), \quad i = 1, 2, \dots, r.$$

Subtracting the equations for $y_i^{[n]}$ and $\hat{y}_i^{[n]}$, we obtain

$$\begin{aligned} y_i^{[n]} - \hat{y}_i^{[n]} &= h^2 \sum_{j=1}^{s} b_{ij} \Big(f(Y_j^{[n]}) - f(y(t_{n-1} + c_j h)) \Big) + h \sum_{j=1}^{r'} q_{ij} (y'^{[n-1]}_j - \hat{y}'^{[n-1]}_j) \\ &+ \sum_{j=1}^{r} v_{ij} (y_j^{[n-1]} - \hat{y}_j^{[n-1]}) - \eta_i(h) \end{aligned}$$

or, equivalently, in tensor form

$$\begin{aligned} y^{[n]} - \hat{y}^{[n]} = h^2 (\mathbf{B} \otimes I) \Big(F(Y^{[n]}) - F(y(t_{n-1} + \mathbf{c}h)) \Big) + h(\mathbf{Q} \otimes I) \cdot \\ \cdot \ (y'^{[n-1]} - \hat{y}'^{[n-1]}) + (\mathbf{V} \otimes I)(y^{[n-1]} - \hat{y}^{[n-1]}) - \eta(h). \end{aligned} \tag{5.6}$$

By means of analogous arguments, we obtain the following representation of the difference between $hy'^{[n]}$ and $h\hat{y}'^{[n]}$:

$$\begin{aligned} h(y'^{[n]} - \hat{y}'^{[n]}) = h^2 (\mathbf{C} \otimes I) \Big(F(Y^{[n]}) - F(y(t_{n-1} + \mathbf{c}h)) \Big) + h(\mathbf{R} \otimes I) \cdot \\ \cdot \ (y'^{[n-1]} - \hat{y}'^{[n-1]}) + (\mathbf{W} \otimes I)(y^{[n-1]} - \hat{y}^{[n-1]}) - \zeta(h). \end{aligned} \tag{5.7}$$

In order to provide a more compact version of formulae (5.6) and (5.7), we introduce the notations

$$\begin{aligned} u_n &= y^{[n]} - \hat{y}^{[n]}, \\ v_n &= h(y'^{[n]} - \hat{y}'^{[n]}), \\ w_n &= h^2 (\mathbf{B} \otimes I) \Big(F(Y^{[n]}) - F(y(t_{n-1} + \mathbf{c}h)) \Big) - \eta(h), \\ z_n &= h^2 (\mathbf{B} \otimes I) \Big(F(Y^{[n]}) - F(y(t_{n-1} + \mathbf{c}h)) \Big) - \zeta(h). \end{aligned}$$

With these notations, formulae (5.6) and (5.7) respectively assume the form

$$u_n = w_n + (\mathbf{Q} \otimes I) v_{n-1} + (\mathbf{V} \otimes I) u_{n-1}, \tag{5.8}$$
$$v_n = z_n + (\mathbf{R} \otimes I) v_{n-1} + (\mathbf{W} \otimes I) u_{n-1}. \tag{5.9}$$

Observe that, by applying the Lipschitz continuity of the function F, the following bound for w_n arises

$$\|w_n\| \leq h^2 L\|B\|\|Y^{[n]} - y(t_{n-1} + \mathbf{c}h)\| + \|\eta(h)\|, \tag{5.10}$$

where L is the Lipschitz constant of F. In order to establish a bound for $\|Y^{[n]} - y(t_{n-1} + \mathbf{c}h)\|$, we use the following representation to the difference inside the norm:

$$\begin{aligned} Y^{[n]} - y(t_{n-1} + \mathbf{c}h) = h^2(\mathbf{A} \otimes I)\Big(F(Y^{[n]}) - F(y(t_{n-1} + \mathbf{c}h))\Big) + (\mathbf{P} \otimes I)v_{n-1} \\ + (\mathbf{U} \otimes I)u_{n-1} - \xi(h). \end{aligned}$$

As a consequence, the following bound holds:

$$\begin{aligned} \|Y^{[n]} - y(t_{n-1} + \mathbf{c}h)\| \leq h^2 L\|\mathbf{A}\|\|Y^{[n]} - y(t_{n-1} + \mathbf{c}h)\| + \|\mathbf{P}\|\|v_{n-1}\| \\ + \|\mathbf{U}\|\|u_{n-1}\| + \|\xi(h)\|. \end{aligned}$$

Assuming that $h < h_0$ and $h_0 L\|A\| < 1$, we obtain

$$\begin{aligned} \|Y^{[n]} - y(t_{n-1} + \mathbf{c}h)\| \leq \frac{\|P\|}{1 - h_0^2 L\|A\|}\|v_{n-1}\| \\ + \frac{\|U\|}{1 - h_0^2 L\|A\|}\|u_{n-1}\| + \frac{\|\xi(h)\|}{1 - h_0^2 L\|A\|}. \end{aligned} \tag{5.11}$$

Substituting in (5.10), we obtain

$$\|w_n\| \leq h^2(D\|v_{n-1}\| + E\|u_{n-1}\|) + h^2\delta(h), \tag{5.12}$$

where

$$D = \frac{L\|B\|\|P\|}{1 - h_0^2 L\|A\|}, \qquad E = \frac{L\|B\|\|U\|}{1 - h_0^2 L\|A\|}, \qquad \delta(h) = \frac{L\|B\|\|\xi(h)\|}{1 - h_0^2 L\|A\|} + \|\eta(h)\|.$$

In analogous way, we obtain the following bound for z_n:

$$\|z_n\| \leq h^2(\bar{D}\|v_{n-1}\| + \bar{E}\|u_{n-1}\|) + h^2\bar{\delta}(h), \tag{5.13}$$

where

$$\bar{D} = \frac{L\|C\|\|P\|}{1 - h_0^2 L\|A\|}, \qquad \bar{E} = \frac{L\|C\|\|U\|}{1 - h_0^2 L\|A\|}, \qquad \bar{\delta}(h) = \frac{L\|C\|\|\xi(h)\|}{1 - h_0^2 L\|A\|}.$$

We put together the two bounds (5.12) and (5.13) obtaining, in vector form,

$$\|e_n\| \leq h^2\|\Lambda\| \cdot \|d_{n-1}\| + h^2\|\sigma\|, \tag{5.14}$$

where

$$e_n = \begin{bmatrix} w_n \\ z_n \end{bmatrix}, \qquad \Lambda = \begin{bmatrix} D & E \\ \bar{D} & \bar{E} \end{bmatrix}, \qquad \sigma = \begin{bmatrix} \delta(h) \\ \bar{\delta}(h) \end{bmatrix}, \qquad d_{n-1} = \begin{bmatrix} u_{n-1} \\ v_{n-1} \end{bmatrix}.$$

Proceeding in analogous way for Equations (5.8) and (5.9), we obtain

$$d_n = \mathbf{M}_0 d_{n-1} + e_n. \tag{5.15}$$

Applying Equation (5.15) n times, we obtain

$$d_n = \mathbf{M}_0^n d_0 + \sum_{j=1}^{n} \mathbf{M}_0^{n-j} e_j, \quad n \geq 0$$

and, passing through the norm, we obtain the bound

$$\|d_n\| \leq \|\mathbf{M}_0^n\| \cdot \|d_0\| + \sum_{j=1}^{n} \|\mathbf{M}_0^{n-j}\| \|e_j\|.$$

Since the hypothesis of zero-stability holds, there exist $C_1, D_1 \in \mathbb{R}$ such that $\|\mathbf{M}_0^n\| \leq nC_1 + D_1$. By using this bound and the estimation (5.14), we obtain

$$\|d_n\| \leq (n + C_1 + D_1)\|d_0\| + \sum_{j=1}^{n} \left((n-j)C_1 + D_1\right)\left(C_2\|d_{j-1}\| + D_2\right),$$

where $C_2 = h^2\|\Lambda\|$ and $D_2 = h^2\|\sigma\|$. This bound, after some calculations, can be rewritten as

$$\|d_n\| \leq \alpha(n) + \sum_{j=2}^{n} \beta_j(n)\|d_j\|, \tag{5.16}$$

where

$$\alpha(n) = (nC_1 + (n-1)C_1C_2 + D_1 + C_2D_1)\|d_0\| + \left(\frac{n(n-1)}{2}C_1 + nD_1\right)D_2,$$

$$\beta_j(n) = \left((n-j-1)C_1C_2 + C_2D_1\right).$$

We set $j = i_1$ and apply the corresponding inequality (5.16) for $\|d_{i_1}\|$, i.e.

$$\|d_{i_1}\| \leq \alpha(n) + \sum_{i_2=2}^{i_1} \beta_{i_2}(n)\|d_{i_2}\|.$$

Replacing this inequality in (5.16) leads to

$$\|d_n\| \leq \alpha(n) + \sum_{i_1=2}^{n} \beta_j(n)\alpha(j) + \sum_{i_1=2}^{n}\sum_{i_2=2}^{i_1} \beta_{i_1}(n)\beta_{i_2}(i_1)\|d_{i_2}\|.$$

By iterating this process, we obtain

$$\|d_n\| \leq \alpha(n) + \sum_{i_1=2}^{n}\sum_{i_2=2}^{i_1}\sum_{i_3=2}^{i_2}\cdots\sum_{i_N=2}^{2}\prod_{j=1}^{N} \beta_{i_j}(i_{j-1})\alpha(i_j), \tag{5.17}$$

under the assumption that $i_0 = n$. We observe that the right hand side of the inequality (5.17) is expressed as the summation of $\alpha(n)$, which can be bounded by $D_1\|d_0\|$ as n tends to infinity, plus a series whose principal term behaves as $O(1/n^2)$ and, therefore, it converges. Then, the following bound holds

$$\|d_n\| \leq \left(D_1 + C_1^2\|\Lambda\|(\bar{t} - t_0)\right)\|d_0\| + O(h^2), \tag{5.18}$$

which completes the proof. □

We now prove that consistency and zero-stability are also implied by convergence: thus, Theorems 5.1 and 5.2 provide a necessary and sufficient condition for convergence of GLMs (2.4).

Theorem 5.2 *A convergent GLM* (2.4) *is zero-stable and consistent.*

Proof Following the lines drawn in [3,11], we first prove that convergence implies zero-stability. We suppose, by contradiction, that the method is not zero-stable: then, for any $C, D \in \mathbb{R}$, the sequence $\{\|\mathbf{M}_0^n\|,\ n > 0\}$ is never upper bounded by the linear term $nC + D$. Thus, since $\|\mathbf{M}_0^n\| = \max_{\|w\|=1} \|\mathbf{M}_0^n w\|$, there exists a sequence of vectors w_n, $n > 0$, having unitary norm, such that $\|\mathbf{M}_0^n w_n\| > nC + D$. We next consider the problem

$$y''(t) = 0, \quad y'(0) = 0, \quad y(0) = 0,$$

with $t \geq 0$, whose exact solution is $y(t) = 0$, and perform n steps of stepsize $h = 1/n$ up to the final point $\bar{t} = 1$, assuming as initial input vector

$$d^{[0]} = \begin{bmatrix} hy'^{[0]} \\ y^{[0]} \end{bmatrix} = \frac{w_n}{\max\limits_{1\leq i\leq n} \|\mathbf{M}_0^i w_i\|}.$$

We observe that such a starting procedure fulfills requirements (5.1) and (5.2). The approximations provided after n steps is then given by

$$d^{[n]} = \begin{bmatrix} hy'^{[n]} \\ y^{[n]} \end{bmatrix} = \mathbf{M}_0^n d^{[0]} = \frac{\mathbf{M}_0^n w_n}{\max\limits_{1\leq i\leq n} \|\mathbf{M}_0^i w_i\|},$$

whose norm is equal to

$$\|d^{[n]}\| = \frac{\|\mathbf{M}_0^n w_n\|}{\max\limits_{1\leq i\leq n} \|\mathbf{M}_0^i w_i\|}.$$

Due to the unboundedness of the sequence $\|\mathbf{M}_0^n w_n\|$, there exists a monotonically increasing sequence $\|\mathbf{M}_0^{n_j} w_{n_j}\|$ tending to infinity, when j tends to infinity, and such that $\max_{1\leq i\leq n_j} \|\mathbf{M}_0^i w_i\| = \mathbf{M}_0^{n_j} w_{n_j}$. Thus,

$$\|d^{[n]}\| = \frac{\|\mathbf{M}_0^{n_j} w_{n_j}\|}{\|\mathbf{M}_0^{n_j} w_{n_j}\|} = 1,$$

which contradicts the convergence of the method.

We now prove that convergence implies consistency. To achieve this purpose, we consider the initial value problem

$$y''(t) = 1, \quad y'(0) = 0, \quad y(0) = 0,$$

$t \geq 0$, whose exact solution is $y(t) = t^2/2$, and perform n step of steplength $h = 1/n$ up to $\bar{t} = 1$. Thus, after n steps, the vector $d^{[n]}$ of the numerical approximations is given by

$$d^{[n]} = h^2 D + \mathbf{M}_0 d^{[n-1]},$$

where $D = [Ce\ Be]^T$. By recursion, we get to the following representation of the vector $d^{[n]}$:

$$d^{[n]} = h^2(I + \mathbf{M}_0 + \dots \mathbf{M}_0^{n-1})D,$$

where I stands for the identity matrix of dimension $r' + r$. Due to the preconsistency of the method, which leads to

$$h^2(I + \mathbf{M}_0 + \dots \mathbf{M}_0^{n-1}) \begin{bmatrix} \mathbf{q}_1' \\ \frac{\mathbf{q}_0}{2} + \mathbf{q}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{q}_1' \\ \frac{\mathbf{q}_0}{2} + \mathbf{q}_1 \end{bmatrix},$$

we get

$$d^{[n]} - \begin{bmatrix} \mathbf{q}_1' \\ \frac{\mathbf{q}_0}{2} + \mathbf{q}_1 \end{bmatrix} = h^2(I + \mathbf{M}_0 + \dots \mathbf{M}_0^{n-1}) \left(D - \begin{bmatrix} \mathbf{q}_1' \\ \frac{\mathbf{q}_0}{2} + \mathbf{q}_1 \end{bmatrix} \right).$$

We have already proved that convergence implies zero-stability: according to point (*iii*) of Theorem 4.1, there exists a nonsingular matrix P such that

$$\mathbf{M}_0 = P^{-1} \begin{bmatrix} \mathbf{J}_1 & 0 \\ 0 & \mathbf{J}_2 \end{bmatrix} P,$$

where the blocks $\mathbf{J}_1$ and $\mathbf{J}_2$ are those defined in the proof of Theorem 4.1. Then,

$$d^{[n]} - \begin{bmatrix} \mathbf{q}_1' \\ \frac{\mathbf{q}_0}{2} + \mathbf{q}_1 \end{bmatrix} = P^{-1}\mathbf{J}P \left(D - \begin{bmatrix} \mathbf{q}_1' \\ \frac{\mathbf{q}_0}{2} + \mathbf{q}_1 \end{bmatrix} \right),$$

with

$$\mathbf{J} = \begin{bmatrix} h^2(I - \mathbf{J}_1)^{-1}(I - \mathbf{J}_1^n) & 0 \\ 0 & h^2(I - \mathbf{J}_2)^{-1}(I - \mathbf{J}_2^n) \end{bmatrix}.$$

It is worth observing that the above identity still holds true when n goes to infinity. Thus, following the idea of [3, 11], we can conclude that

$$P \left(d^{[n]} - \begin{bmatrix} \mathbf{q}_1' \\ \frac{\mathbf{q}_0}{2} + \mathbf{q}_1 \end{bmatrix} \right) = \begin{bmatrix} (I - \mathbf{J}_1)\widetilde{\mathbf{q}}_2' \\ (I - \mathbf{J}_2)\widetilde{\mathbf{q}}_2 \end{bmatrix},$$

for some $\widetilde{\mathbf{q}}_2' \in \mathbb{R}^{r'}$ and $\widetilde{\mathbf{q}}_2 \in \mathbb{R}^r$. We next define the vector

$$\begin{bmatrix} \mathbf{q}_2' \\ \mathbf{q}_2 \end{bmatrix} = P^{-1} \begin{bmatrix} I - \mathbf{J}_1 & 0 \\ 0 & I - \mathbf{J}_2 \end{bmatrix}^{-1} \begin{bmatrix} (I - \mathbf{J}_1)\widetilde{\mathbf{q}}_2' \\ (I - \mathbf{J}_2)\widetilde{\mathbf{q}}_2 \end{bmatrix}.$$

Then,

$$P \left(D - \begin{bmatrix} \mathbf{q}_1' \\ \frac{\mathbf{q}_0}{2} + \mathbf{q}_1 \end{bmatrix} \right) = \begin{bmatrix} I - \mathbf{J}_1 & 0 \\ 0 & I - \mathbf{J}_2 \end{bmatrix} P \begin{bmatrix} \mathbf{q}_2' \\ \mathbf{q}_2 \end{bmatrix} = P(I - \mathbf{M}_0) \begin{bmatrix} \mathbf{q}_2' \\ \mathbf{q}_2 \end{bmatrix},$$

which leads to

$$D - \begin{bmatrix} \mathbf{q}_1' \\ \frac{\mathbf{q}_0}{2} + \mathbf{q}_1 \end{bmatrix} = (I - \mathbf{M}_0) \begin{bmatrix} \mathbf{q}_2' \\ \mathbf{q}_2 \end{bmatrix}.$$

The last equation provides the conditions of consistency introduced in Definition 3.2. □

6 Order conditions

The derivation of order conditions for GLMs solving first order ODEs has been successfully and elegantly treated by Butcher [3], via rooted trees and B-series arguments. However, in the case of high stage order methods, a different approach to derive order conditions can be used. This approach has been discussed by Butcher himself in the context of diagonally implicit multistage integration methods (see [11]), in the cases $q = p$ and $q = p - 1$, where p is the order of the method and q is its stage order. We use this approach to derive order conditions of GLMs for second order ODEs (1.1). As initial case of study, we assume that the order p of the GLM is equal to its stage order q: this choice allows the methods to have a uniform order of convergence and, as a consequence, they would not suffer from order reduction (see [2] as regards first order ODEs) in the integration of stiff differential systems.

We first assume that the components of the input and output vectors respectively satisfy

$$y_i^{[n-1]} = \sum_{k=0}^{p} q_{ik} h^k y^{(k)}(t_{n-1}) + O(h^{p+1}), \tag{6.1}$$

$$y_i^{[n]} = \sum_{k=0}^{p} q_{ik} h^k y^{(k)}(t_n) + O(h^{p+1}), \tag{6.2}$$

for some real parameters q_{ik}, $i = 1, 2, \ldots, r$, $k = 0, 1, \ldots, p$. We will next denote p as the *order* of the method. We then assume that the components of the internal stages $Y_i^{[n]}$ are approximations of order q to the solution of (1.1) at the internal points $t_{n-1} + c_i h$, i.e.

$$Y_i^{[n]} = y(t_{n-1} + c_i h) + O(h^{q+1}), \quad i = 1, 2, \ldots, s. \tag{6.3}$$

We will next denote q as the *stage order* of the method. We also request that the components of the input and output vectors of the derivatives respectively satisfy

$$h{y'}_i^{[n-1]} = \sum_{k=1}^{p} q'_{ik} h^k y^{(k)}(t_{n-1}) + O(h^{p+1}), \tag{6.4}$$

$$h{y'}_i^{[n]} = \sum_{k=1}^{p} q'_{ik} h^k y^{(k)}(t_n) + O(h^{p+1}), \tag{6.5}$$

for some real parameters q'_{ik}, $i = 1, 2, \ldots, r'$, $k = 1, 2, \ldots, p$. We introduce the following notation

$$e^{\mathbf{c}z} = [e^{c_1 z}, e^{c_2 z}, \ldots, e^{c_s z}],$$

and define the vectors

$$\mathbf{w} = \mathbf{w}(z) = \sum_{k=0}^{p} \mathbf{q}_k z^k, \quad \text{and} \quad \mathbf{w}' = \mathbf{w}'(z) = \sum_{k=1}^{p} \mathbf{q}'_k z^k.$$

We aim to obtain algebraic conditions ensuring that a GLM (2.4) has order $p = q$. The following theorem holds.

Theorem 6.1 *Assume that $y^{[n-1]}$ and $y'^{[n-1]}$ satisfy respectively (6.1) and (6.4). Then the GLM (2.4) of order p and stage order $q = p$ satisfies (6.2), (6.3) and (6.5) if and only if*

$$e^{\mathbf{c}z} = z^2\mathbf{A}e^{\mathbf{c}z} + \mathbf{P}\mathbf{w}'(z) + \mathbf{U}\mathbf{w}(z) + O(z^{p+1}), \tag{6.6}$$

$$e^{z}\mathbf{w}'(z) = z^2\mathbf{C}e^{\mathbf{c}z} + \mathbf{R}\mathbf{w}'(z) + \mathbf{W}\mathbf{w}(z) + O(z^{p+1}), \tag{6.7}$$

$$e^{z}\mathbf{w}(z) = z^2\mathbf{B}e^{\mathbf{c}z} + \mathbf{Q}\mathbf{w}'(z) + \mathbf{V}\mathbf{w}(z) + O(z^{p+1}). \tag{6.8}$$

Proof Since $Y_i^{[n]} = y(t_{n-1} + c_i h) + O(h^{q+1})$, $i = 1, 2, \ldots, s$, it follows that

$$\begin{aligned} h^2 f(Y_i^{[n]}) &= h^2 y''(t_{n-1} + c_i h) + O(h^{p+3}) \\ &= \sum_{k=2}^{p} \frac{c_i^{k-2}}{(k-2)!} h^k y^{(k)}(t_{n-1}) + O(h^{p+1}). \end{aligned}$$

Expanding in Taylor series around t_{n-1}, Equation (6.2) can be written in the form

$$y_i^{[n]} = \sum_{k=0}^{p} \left(\sum_{l=0}^{k} \frac{1}{l!} q_{i,k-l} \right) h^k y^{(k)}(t_{n-1}) + O(h^{p+1}). \tag{6.9}$$

We substitute the relations (6.1), (6.2), (6.3), (6.4) and (6.5) in the GLM formulation (2.4). Then, by equating to zero the coefficients of $h^k y^{(k)}(t_{n-1})/k!$, $k = 0, 1, \ldots, p$, multiplying them by $z^k/k!$, and summing them over k from 0 to p, we obtain

$$e^{c_i z} - z^2 \sum_{j=1}^{s} a_{ij} e^{c_j z} - \sum_{j=1}^{r'} p_{ij} w'_j - \sum_{j=1}^{r} u_{ij} w_j = O(z^{p+1}), \quad i = 1, 2, \ldots, s,$$

$$e^{z} w'_i - z^2 \sum_{j=1}^{s} c_{ij} e^{c_j z} - \sum_{j=1}^{r'} r_{ij} w'_j - \sum_{j=1}^{r} w_{ij} w_j = O(z^{p+1}), \quad i = 1, 2, \ldots, r',$$

$$e^{z} w_i - z^2 \sum_{j=1}^{s} b_{ij} e^{c_j z} - \sum_{j=1}^{r'} q_{ij} w'_j - \sum_{j=1}^{r} v_{ij} w_j = O(z^{p+1}), \quad i = 1, 2, \ldots, r.$$

These relations are equivalent to (6.6), (6.7) and (6.8).

□

It follows from the proof of Theorem 6.1 that the conditions (6.6), (6.7) and (6.8) are respectively equivalent to

$$\mathbf{c}^k - k(k-1)\mathbf{A}\mathbf{c}^{k-2} - k!\mathbf{P}\mathbf{q}'_k - k!\mathbf{U}\mathbf{q}_k = 0, \tag{6.10}$$

$$\sum_{l=0}^{k} \frac{k!}{l!} \mathbf{q}'_{k-l} - k(k-1)\mathbf{C}\mathbf{c}^{k-2} - k!\mathbf{R}\mathbf{q}'_k - k!\mathbf{W}\mathbf{q}_k = 0, \tag{6.11}$$

$$\sum_{l=0}^{k} \frac{k!}{l!} \mathbf{q}_{k-l} - k(k-1)\mathbf{B}\mathbf{c}^{k-2} - k!\mathbf{Q}\mathbf{q}'_k - k!\mathbf{V}\mathbf{q}_k = 0, \tag{6.12}$$

for $k = 2 \ldots, p+1$. These equalities constitute the system of order condition that a GLM has to satisfy in order to achieve order p equal to the stage order q.

7 Conclusions and future works

In this paper we have addressed our attention on the development of a unifying framework for the numerical solution of special second order ODEs (1.1), by considering the family of General Linear Methods (2.4) for this problem. Although the techniques used in the paper are suitable generalizations of the ones developed for first order ODEs in [2,3,11], we think this is the work that had to be done in order to introduce a unifying theory of numerical methods for (1.1). We have presented the formulation of GLMs and the main results regarding consistency, zero-stability and convergence. These general results are being exploited and, together with the derivation of order conditions, are allowing to derive and analyze new numerical methods for (1.1), easily achieving their convergence properties.

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